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Professional Forecasters: How to Understand and Exploit Them Through a DSGE Model

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Professional Forecasters: How to Understand and Exploit Them Through a DSGE Model*

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Abstract

This paper derives a link between the forecasts of professional forecasters and a DSGE model. We show that the forecasts of a professional forecaster can be incorporated to the state space representation of the model by allowing the measurement error of the forecast and the structural shocks to be correlated. The parameters capturing this correlation are reduced form parameters that allow to address two issues i) How the forecasts of the professional forecaster can be exploited as a source of information for the estimation of the model and ii) How to characterize the deviations of the professional forecaster from an ideal complete information forecaster in terms of the shocks and the structure of the economy.

Contents

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1 Introduction

The agents in the economy are continuously forming and revising expectations, most of them are thinking about the probability of finding a job in the next month or how much their salaries will rise or the evolution of the interest rate of their debt. While some others, cause of the nature of their business, devote their time and effort to form well informed expectations about macroeconomic aggregates; such as CPI inflation or GDP growth rate. Agents of the latter kind sometimes publish their expectations in terms of forecasts of economic variables[1] and there exists also surveys that collect these forecasts[2] such as the FED's and the ECB's surveys of professional forecasters.

These surveys have been used to characterize, from a merely statistical standpoint, the forecast accuracy and the forecast error of the professional forecasters (PF) (see Bowles, Friz, Genre, Kenny, Meyler, and Rautanen (2007) and Stark (2010)) and also as a source of information to construct atheorical forecasting models (see Genre, Kenny, Meyler, and Timmermann (2010)). In this paper we depart from those previous studies and derive a methodology that allows to expose this data set to "Rational Expectations Econometrics" which Sargent (1989) refers to as:

> "Rational expectations econometrics" aims to interpret economic time series in terms of objects that are meaningfull to economists, namely, parameters describing preferences, technologies, information sets, endowments, and equilibrium concepts or coordination mechanisms.

Using a Dynamic and Stochastic General Equilibrium models (DSGE) we address simultaneously two issues i) How the forecasts of the PF can be exploited as a source of information for the estimation of the model and ii) Characterize the deviations of the PF from an ideal complete information forecaster in terms of the shocks and the structure of the economy. For both issues we stand as an econometrician with a DSGE model for the economy and a set of observable variables that include the forecasts from the PF.

The first issue is addressed formulating two alternative specifications for the PF. The first type of PF differs with the econometrician in the information set and the second might differ in the model of the economy and in the information set. We show for both specifications how to incorporate the forecasts of the PF as observable variables in the model and the implied log-likelihood function. It turns out that to incorporate the PF forecasts a specific structure of the measurement error must be specified with the main feature that the structural shocks of the model and the measurement error are correlated.

The second issue focuses on the reduced form parameters that capture the correlation between the measurement error of the PF forecasts and the structural

[1] Although not every published forecast could be considered as some revealed expectations because of the different nature that may have the loss function of the forecaster.

[2] Some of the respondants of the surveys does not publish their forecast and their id is not revealed when they answer the survey.

shocks of the model. This parameters are shown to characterize which shocks are the main sources of deviation of the PF relative to an ideal complete information forecaster.

After describing a general setup with the model and filtering equations, we present the likelihood function and the reduced form correlation parameters that emerge in its derivation for each of the PF specifications. We provide concrete illustration of how this reduced form parameters capture the difference between the PF and the ideal complete information forecaster.

2 General Setup

Here we set some notation for the economic model, the filtering equations and the log-likelihood function. From this general setup the econometrician and the first specification of the PF are modelled.

There is an economic model with rational expectations whose equilibrium can be represented as a covariance stationary stochastic process. Specifically, the model equilibrium can be represented as

$$x_{t+1} = Tx_t + \varepsilon_t \tag{1}$$

where x_t is a $n \times 1$ vector of variables, the matrix T is a function of the deep parameters of the model and ε_t is a $n \times 1$ vector of structural shocks whose expected value and covariance matrices are characterized by:

$$\begin{aligned} E\{\varepsilon_t \varepsilon_s'\} &= \begin{cases} \mathbf{Q} & \text{for } t = s \\ \mathbf{0} & \text{for } t \neq s \end{cases} \\ E\{\varepsilon_t\} &= \mathbf{0} \end{aligned} \tag{2}$$

where $E\{.\}$ stands as the expectational operator. The economic model is completely represented by (1) and (2).

Related to those variables of the model there is a set of observable variables $\{y_0, y_1, \ldots, y_t, \ldots, y_\tau\}$, where y_t is a $k \times 1$ vector. These relationship is represented by:

$$y_t = Cx_t + \nu_t$$

Where C is a $k \times n$ matrix that captures the linear projection of y_t over x_t. The $k \times 1$ vector ν_t is conformed by stochastic variables that model the movements of y_t not explained by Cx_t. ν_t is commonly known in these context as the measurement errors as each element y_t is intended to "measure" some linear combination of x_t, and v_t stands as the deviation of y_t from that linear combination. The nature of the measurement errors v_t is determined by the following covariance matrices and expected value:

$$
\begin{aligned}
E\{\nu_t \nu_s'\} &= \begin{cases} \mathbf{R} & \text{for } t = s \\ \mathbf{0} & \text{for } t \neq s \end{cases} \\
E\{\nu_t\} &= \mathbf{0}
\end{aligned}
$$

Furthermore, in this general setup we assume that the structural shocks and the measurement errors are orthogonal at any point in time, so we have

$$
E\{\varepsilon_t \nu_s'\} = \mathbf{0} \text{ for all } t, s \tag{3}
$$

Following a time-domain approach we can write the state-space representation of the model as:

$$
\begin{aligned}
x_{t+1} &= T x_t + \varepsilon_t \\
y_t &= C x_t + \nu_t
\end{aligned} \tag{4}
$$

Where the first equation in (4) is the transition equation and the later corresponds to the measurement equation. This specification resembles to the "classical model of measurements initially collected by an agency" presented in Sargent (1989). Following Sargent (1989) we have that the filtered variables can be obtained recursively by (see Appendix A):

$$
\begin{aligned}
\hat{x}_t &= E(x_t | y_t, y_{t-1}, \dots y_0, \hat{x}_0) \\
&= T \hat{x}_{t-1} + K u_t
\end{aligned} \tag{5}
$$

where K is the gain matrix of the Kalman filter and u_t is the one-step ahead forecast error, or more formally

$$
u_t = y_t - E\left\{ y_t | y_{t-1}, y_{t-2}, \dots \right\} \tag{6}
$$

and we make the following definitions:

$$
\begin{aligned}
S &= E\left\{ \left(\hat{x}_t - x_t\right) \left(\hat{x}_t - x_t\right)' \right\} \\
V &= E\left\{ u_t u_t' \right\}
\end{aligned}
$$

then the Gaussian log-likelihood function for the sample $\{y_0, y_1, \dots, y_t, \dots, y_\tau\}$, conditioned on $\hat{x}_0$ is[3]

$$
L = -\tau \ \ln(2\pi) - 0.5 \ \ln|V| - 0.5 \sum_{t=0}^{\tau-1} u_t' V u_t
$$

[3] Appendix A presents the derivation of (5) and the Gaussian log-likelihood function.

3 A Professional Forecaster With a Different Information Set

The information set of the PF and the econometrician might be different, one possible explanation for this is private information either of the PF or the econometrician. We are interested, from the standpoint of the econometrician, to learn about the private information that may have the PF. In terms of the model we want to know which shocks can identify the PF so the econometrician can use his forecasts as an information variable for estimation and forecasting. Also, we want to explain the PF differences with an ideal complete information forecaster in terms of the shocks of the model that are poorly identified by the PF[4].

3.1 The Professional Forecaster

There is a PF who performs optimal forecasts[5] using the economic model mentioned and a data set $(y_0^f, y_1^f, \ldots, y_t^f, \ldots, y_\tau^f)$ where y_t^f is a $k \times 1$ vector of data related to the model variables by

$$\begin{aligned} y_t^f &= C^f x_t + \nu_t^f \\ E\left\{\nu_t^f \left(\nu_t^f\right)'\right\} &= \mathbf{R} \\ E\left\{\nu_t^f\right\} &= \mathbf{0} \end{aligned}$$

Then from (5) the optimal filtering of the PF is:

$$\begin{aligned} \hat{x}_t^f &= E(x_t | y_t^f, y_{t-1}^f, \ldots y_0^f, \hat{x}_0) \\ &= T\hat{x}_{t-1}^f + K^f u_t^f \\ S^f &= E\left\{\left(\hat{x}_t^f - x_t\right)\left(\hat{x}_t^f - x_t\right)'\right\} \end{aligned} \tag{7}$$

where K^f is the gain matrix of the PF. The one step ahead forecast is then

[4]Another possible reason that might generate different information sets between the PF and the econometrician is rational inattention. In the case of the PF he might neglect part of the information that the econometrician have (or viceversa) not because is private but because it is costly to obtain or process it and the gains of including this information are not big enough. In this case we could think of those shocks poorly identified by the PF as possibly shocks less important to quantify for the PF. "El aleman ese 2009" shows how might optimally decide not to identificate if the shock that arrive is a idiosincratic shock or an aggregated shock.

[5]In the sense that minimizes the expected value of the squared forecast error.

$$
\begin{aligned}
\hat{x}^f_{t+1|t} &= E(x_{t+1}|y^f_t, y^f_{t-1}, \ldots y^f_0, \hat{x}_0) \\
&= E(Tx_t + \varepsilon_t|y^f_t, y^f_{t-1}, \ldots y^f_0, \hat{x}_0) \\
&= T\,E(x_t|y^f_t, y^f_{t-1}, \ldots y^f_0, \hat{x}_0) + E(\varepsilon_t|y^f_t, y^f_{t-1}, \ldots y^f_0, \hat{x}_0) \\
&= T\hat{x}^f_t
\end{aligned} \tag{8}
$$

where $E(\varepsilon_t|y^f_t, y^f_{t-1}, \ldots y^f_0, \hat{x}_0) = 0$ follows from (2) and (3).

The PF publishes the one-step ahead forecast of some variables each period. Define $\tilde{y}_t$ as the subset of $\hat{x}^f_{t+1|t}$ that is observable for the econometrician and published at time t, then

$$
\tilde{y}_t \quad = I_s \quad \hat{x}^f_{t+1|t} \tag{9}
$$

where I_s is a selection matrix conformed by the rows of the identity matrix that correspond to a observable variable i.e the row j of the identity matrix is one of the rows of I_s if the entrie j of $\hat{x}^f_{t+1|t}$ is published. Then, from (8)(9) we have that $\tilde{y}_t$ can be written in terms of the filtered values of the PF as

$$
\tilde{y}_t = I_s T \hat{x}^f_t \tag{10}
$$

3.2 Incorporating the forecasts from the PF

Suppose initially (for ease of exposition) that the econometrician only observes $\{\tilde{y}_0, \tilde{y}_1, \ldots, \tilde{y}_t, \ldots, \tilde{y}_\tau\}$. From (10) and (7) we can conform a state-space representation with $\tilde{y}_t$ as the observable and $\hat{x}^f_t$ as the unobservable states. The transition and measurement equation of this representation are respectively

$$
\begin{aligned}
\hat{x}^f_{t+1} &= T\hat{x}^f_t + K^f u^f_{t+1} \\
\tilde{y}_t &= I_s T \hat{x}^f_t
\end{aligned} \tag{11}
$$

The system (11) is in terms of the innovations u^f_t, and the unobservable states $\hat{x}^f_t$ that are the filtered values of the PF. On the other hand we know the law of motion of the variables in the model by (1), so another possible state-space representation with the data $\tilde{y}_t$ as the observable and redifining the unobservables states as x_t can be written as follows

$$
\begin{aligned}
x_{t+1} &= Tx_t + \varepsilon_t \\
\tilde{y}_t &= I_s T x_t + v_t
\end{aligned} \tag{12}
$$

Now a measurement error $v_t = I_s T\left(\hat{x}_t^f - x_t\right)$ emerges. To understand the nature of this measurement error note that if the PF has complete information[6], we have that

$$\hat{x}_t^f = E\{x_t | x_t\} = x_t$$

and then $v_t = 0$. So in this case the measurement error associated with the forecast of the PF reflects the difference between the forecast of the PF $T_{(i\in B)}\hat{x}_t^f$ and the forecast of a complete information forecaster $T_{(i\in B)}x_t$, this can be stated as

$$v_t = E\left\{x_{t+1} | y_t^f, y_{t-1}^f, \ldots y_0^f, \hat{x}_0\right\} - E\{x_{t+1}|x_t\}$$

thus v_t contains the signal extraction uncertainty of the PF.

Defining $e_t = \varepsilon_{t-1}$ we have the contemporaneous form of the state-space representation

$$\begin{aligned} x_t &= Tx_{t-1} + e_t \\ \tilde{y}_t &= I_s T x_t + v_t \end{aligned}$$

In this case we have that e_t and v_t are correlated, the covariance matrix is:

$$\begin{aligned} \Theta &= E\{v_t e_t'\} \\ \Theta &= I_s T\left(K^f C - I\right) \mathbf{Q} \end{aligned} \tag{13}$$

and the variance matrix of the measurement error is:

$$R = E\{v_t v_t'\} = I_s T S^f (I_s T)'$$

v_t is not the standard measurement error because it is autocorrelated. Formally we have that

$$\begin{aligned} E\{v_t v_{t-j}'\} &= E\left\{I_s T\left(\hat{x}_t^f - x_t\right)\left(\hat{x}_{t-j}^f - x_{t-j}\right)'(I_s T)'\right\} \\ &= (I_s T)\left(\prod_{i=1}^{j}\left(I - K^f I_s T\right) T\right) S^f (I_s T)' \end{aligned} \tag{14}$$

The next proposition clarifies the nature of v_t. It resume in a compact form the information presented in (13) and (14) and the relationship of v_t with ν_t^f.

[6] In the sense that knows perfectly the current state of the economy x_t but is uncertain about the shocks that can arrive $(\varepsilon_t, \varepsilon_{t+1}, \varepsilon_{t+2}, \ldots)$.

Proposition. *The stochastic process* $\{v_t\}_{t=1,...,\infty}$ *can be written as a vector autoregresive (VAR) process of the form:*

$$v_t = \Phi v_{t-1} + \Gamma e_t + \Omega \nu_t^f$$

where the matrices Φ, Γ *and* Ω *correspond to:*

$$\begin{aligned} \Phi &= I_s T \left(\left(I - K^f I_s T \right) T \right) \left((I_s T)' I_s T \right)^{-1} (I_s T)' \\ \Gamma &= I_s T \left(K^f C - I \right) = \Theta Q^{-1} \\ \Omega &= I_s T K^f \end{aligned} \tag{15}$$

Proof. The measurement error v_t in equation (12) correspond to:

$$v_t = I_s T \left(\hat{x}_t^f - x_t \right)$$

and replacing $\hat{x}_t^f$ using (11) and x_t using (12) we have:

$$\begin{aligned} v_t &= I_s T \left(T \hat{x}_{t-1}^f + K^f u_t^f \right) - I_s T \left(T x_{t-1} + e_t \right) \\ &= I_s T T \left(\hat{x}_{t-1}^f - x_{t-1} \right) + I_s T K^f u_t^f - I_s T e_t \end{aligned}$$

and replacing the one-step ahead forecast error u_t^f by it´s definition (see (6))

$$\begin{aligned} v_t &= I_s T T \left(\hat{x}_{t-1}^f - x_{t-1} \right) + I_s T K^f \left(y_t^f - C^f T \hat{x}_{t-1}^f \right) - I_s T e_t \\ &= I_s T T \left(\hat{x}_{t-1}^f - x_{t-1} \right) + I_s T K^f \left(C^f x_t + \nu_t^f - C^f T \hat{x}_{t-1}^f \right) - I_s T e_t \end{aligned}$$

and using (12) to solve out for $x_t$and arranging terms we have:

$$v_t = I_s T T \left(\hat{x}_{t-1}^f - x_{t-1} \right) + I_s T (K^f C^f K^f \left(y_t^f - C^f T \hat{x}_{t-1}^f \right) - I_s T e_t$$

□

The matrices in (15) fully characterize the nature of the deviations of the PF from the ideal forecaster, and each of their entries are reduced form parameters that are functions of the deep parameters of the model and the PF parameters, specifically the PF gain matrix. The matrix Γ measures the magnitude of the effect that has each structural shock in v_t and, as v_t arises because of the lack of information of the PF, the entries in Γ reflect the uncertainty of the PF over the corresponding shock weighted by the importance of it on the variable forecasted. On the other hand, Φ measures how the deviations v_t affect v_{t+1}, or in other terms, it captures the persistence structure of the deviations of the PF from the ideal forecaster. In (15) we can see that the persistence depends on the structure of the economy T and the learning process of the PF K^f. If the economy has low persistence and the PF learns fast, the persistence of v_t will tend to zero. Finally Ω captures how the measurement error of the data used by the PF is translated to v_t.

For practical purposes as we generally don´t know which data used the PF, and consequently the size and elements of ν_t^f are not known, we can rewrite 15 in terms of the reduced form vector $\psi_t = \Omega\nu_t^f$ which covariance matrix would reflect the data uncertainty of each of the forecasts. Therefore we have that v_t can be written as $v_t = \Phi v_{t-1} + \Gamma e_t + \psi_t$.

An Extended Data Base

Now we will extend our initial formulation to allow for a more general set of information for the econometrician. Collecting our results so far we have

$$\begin{aligned} x_t &= T x_{t-1} + e_t \\ v_t &= \Phi v_{t-1} + \Gamma e_t + \Omega \nu_t^f \\ \tilde{y}_t &= T_{(i\in B)} x_t + v_t \end{aligned}$$

We define y_t as the data released at time t which is composed by:

$$y_t = \begin{pmatrix} d_t \\ \tilde{y}_t \end{pmatrix} \tag{16}$$

where d_t is data related to the variables of the model and $\tilde{y}_t$ is the vector of the one-step ahead forecasts of the PF. Now the measurement equation is

$$y_t = \begin{pmatrix} N \\ T_{(i\in B)} \end{pmatrix} x_t + \begin{pmatrix} \mu_t \\ v_t \end{pmatrix}$$

where N is a matrix that captures the relation between the variables in the model and the data contained in d_t. μ_t is a vector of the measurement errors associated with d_t. Then, a complete formulation of the state space representation is

$$\begin{aligned} x_t &= T x_{t-1} + e_t \\ v_t &= \Phi v_{t-1} + \Gamma e_t + \Omega \nu_t^f \\ y_t &= \begin{pmatrix} N \\ T_{(i\in B)} \end{pmatrix} x_t + \begin{pmatrix} \mu_t \\ v_t \end{pmatrix} \\ E\left\{e_t e_t'\right\} &= Q \qquad E\left\{e_t\right\} = \mathbf{0} \\ E\left\{\mu_t \mu_t'\right\} &= H \qquad E\left\{\mu_t\right\} = \mathbf{0} \\ E\left\{\nu_t^f \left(\nu_t^f\right)'\right\} &= \mathbf{R} \qquad E\left\{\nu_t^f\right\} = \mathbf{0} \\ E\left\{\mu_t e_s'\right\} &= \mathbf{0} \text{ for all } t, s \\ E\left\{\nu_t^f e_s'\right\} &= \mathbf{0} \text{ for all } t, s \end{aligned} \tag{17}$$

3.3 The Log-Likelihood function neglecting Φ

The more recent innovations might be the main drivers of the measurement errors of the PF forecasts (i.e the discrepancy between the PF and the ideal complete information forecaster). If this is the case v_t will be mainly explained by Γe_t and the term Φv_{t-1} could be neglected, then the state space representation can be restated as

$$
\begin{aligned}
x_t &= T x_{t-1} + \begin{pmatrix} \mathbf{0} & \mathbf{0} & I \end{pmatrix} \begin{pmatrix} \mu_t \\ \nu_t^f \\ e_t \end{pmatrix} \\
y_t &= \begin{pmatrix} N \\ T_{(i \in B)} \end{pmatrix} x_t + \begin{pmatrix} I & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \Omega & \Gamma \end{pmatrix} \begin{pmatrix} \mu_t \\ \nu_t^f \\ e_t \end{pmatrix} \\
E\left\{ \begin{pmatrix} \mu_t \\ \nu_t^f \\ e_t \end{pmatrix} \begin{pmatrix} \mu_t' & \left(\nu_t^f\right)' & e_t' \end{pmatrix} \right\} &= \begin{pmatrix} H & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & Q \end{pmatrix} = \begin{pmatrix} h\,h' & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & r\,r' & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & q\,q' \end{pmatrix} \\
E\left\{ \begin{pmatrix} \mu_t \\ \nu_t^f \\ e_t \end{pmatrix} \right\} &= \mathbf{0}
\end{aligned}
$$

where h, r and q are obtained from the Cholesky decomposition of H, $\mathbf{R}$ and Q respectively. In this specification is evident the correlation between the measurement errors and the structural shocks. Furthermore, writting the state space representation in terms of the orthogonalized shocks (ζ_t) we have

$$
\begin{aligned}
x_t &= T x_{t-1} + \begin{pmatrix} \mathbf{0} & r & q \end{pmatrix} \zeta_t \\
y_t &= \begin{pmatrix} N \\ T_{(i \in B)} \end{pmatrix} x_t + \begin{pmatrix} h & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \Omega r & \Gamma q \end{pmatrix} \zeta_t \\
E\left\{ \zeta_t \zeta_t' \right\} &= I \quad E\zeta_t = \mathbf{0}
\end{aligned}
$$

or in a compact form as

$$
\begin{aligned}
x_t &= \mathbf{T} x_{t-1} + \mathbf{H} \zeta_t \\
y_t &= \mathbf{Z} x_t + \mathbf{G} \zeta_t \\
E\left\{ \zeta_t \zeta_t' \right\} &= I \quad E\zeta_t = \mathbf{0}
\end{aligned}
$$

This particular state-space form and the respective Kalman filter and smoother recursions can be found in Koopman and Harvey (2003). From there the filtered variables can be obtained by:

$$
\begin{aligned}
\hat{x}_t &= E(x_t|y_t, y_{t-1}, \ldots y_0, \hat{x}_0) \\
&= T\hat{x}_t + Ka_{t+1} \\
E\left\{(\hat{x}_t - x_t)(\hat{x}_t - x_t)'\right\} &= \mathbf{S}
\end{aligned}
$$

where K is the gain matrix of the Kalman filter and a_t is the one-step ahead forecast error, or more formally

$$
\begin{aligned}
K &= \left(\mathbf{TS}(\mathbf{ZT})' + \mathbf{H}(\mathbf{G} + \mathbf{ZH})'\right)\mathbf{V}^{-1} \\
a_t &= y_t - E\left\{y_t|y_{t-1}, y_{t-2}, \ldots\right\} \\
E\left\{a_t a_t'\right\} &= \mathbf{V} = \left(\mathbf{ZTS}(\mathbf{ZT})' + (\mathbf{G} + \mathbf{ZH})(\mathbf{G} + \mathbf{ZH})'\right)
\end{aligned}
$$

then the Gaussian log-likelihood function for the sample $\{y_0, y_1, \ldots, y_t, \ldots, y_\tau\}$, conditioned on $\hat{x}_0$ is

$$
L = -\tau\ \ln(2\pi) - 0.5\ \ln|\mathbf{V}| - 0.5\sum_{t=0}^{T-1} a_t'\mathbf{V}a_t
$$

With this log-likelihood function the reduced form parameters contained in Γ and Ω (and the deep parameters too) can be estimated by maximum likelihood or with Bayesian techniques considering the possible characteristics of the gain matrix of the PF to construct the priors. The reduced form approach is very usefull in this scenario for the parameters in Γ and Ω because typically K^f is not observable but we might have some prior knowledge about it.

3.4 The Log-Likelihood function, general form

To obtain the Likelihood function of (17) allowing the matrix Φ to be different from a null matrix we restate the state space representation (17) as follows

$$
\begin{aligned}
\begin{pmatrix} x_t \\ v_t \end{pmatrix} = s_t &= \begin{pmatrix} T & \mathbf{0} \\ \mathbf{0} & \Phi \end{pmatrix} s_{t-1} + \begin{pmatrix} I & \mathbf{0} \\ \Gamma & \Omega \end{pmatrix} \begin{pmatrix} e_t \\ \nu_t^f \end{pmatrix} \\
y_t &= \begin{pmatrix} N & \mathbf{0} \\ T_{(i\in B)} & I \end{pmatrix} s_t + \begin{pmatrix} I \\ \mathbf{0} \end{pmatrix} \mu_t \\
E\left\{\begin{pmatrix} e_t \\ \nu_t^f \end{pmatrix}\begin{pmatrix} e_t' & \left(\nu_t^f\right)' \end{pmatrix}\right\} &= \begin{pmatrix} Q & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{pmatrix} \\
E\left\{\begin{pmatrix} e_t \\ \nu_t^f \end{pmatrix}\right\} &= \mathbf{0} \\
E\left\{\mu_t\mu_t'\right\} &= H \quad E\left\{\mu_t\right\} = \mathbf{0} \\
E\left\{\mu_t\begin{pmatrix} e_s' & \left(\nu_t^f\right)' \end{pmatrix}\right\} &= \mathbf{0} \quad \forall t, s
\end{aligned}
\tag{18}
$$

or in a compact form

$$\begin{aligned}
s_t &= \mathbf{T}s_{t-1} + \mathbf{L}\omega_t \\
y_t &= \mathbf{Z}s_t + \mathbf{B}\mu_t \\
E\{\omega_t\omega_t'\} &= \begin{pmatrix} Q & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{pmatrix} \quad E\{\omega_t\} = \mathbf{0} \\
E\{\mu_t\mu_t'\} &= H \quad E\{\mu_t\} = \mathbf{0} \\
E\{\mu_t e_s\} &= \mathbf{0} \text{ for all } t, s
\end{aligned} \tag{19}$$

with this specification the filtered variables can be obtained by:

$$\begin{aligned}
\hat{s}_t &= E(s_t|y_t, y_{t-1}, \ldots y_0, \hat{x}_0) \\
&= T\hat{s}_t + Ka_{t+1} \\
\mathbf{S} &= E\left\{(\hat{s}_t - s_t)(\hat{s}_t - s_t)'\right\}
\end{aligned}$$

where K is the gain matrix of the Kalman filter and a_t is the one-step ahead forecast error, or more formally

$$\begin{aligned}
a_t &= y_t - E\{y_t|y_{t-1}, y_{t-2}, \ldots\} \\
\mathbf{V} &= E\{a_t a_t'\}
\end{aligned}$$

then the Gaussian log-likelihood function for the sample $\{y_0, y_1, \ldots, y_t, \ldots, y_\tau\}$, conditioned on $\hat{s}_0$ is

$$L = -\tau \ \ln(2\pi) - 0.5 \ \ln|\mathbf{V}| - 0.5\sum_{t=0}^{T-1} a_t'\mathbf{V}a_t$$

with this log-likelihood function the reduced form parameters contained in Γ and Φ can be estimated by maximum likelihood or with Bayesian techniques. Again, the explicit form of Γ, Ω and Φ is an important feature for setting the priors for the estimation. Incorporating Φ allow us to think about the speed of learning of the PF.

4 A PF with a different forecasting model

Until this point the PF constructs his optimal forecasts using the same economic model as the econometrician, perhaps a strong assumption. This section extends the derivation for the case in which we just assume that the forecast function can be approximated by a linear function of the data considered by the PF. In this case we have:

$$x^f_{t+1|t} = F\, y^f_t$$

where F is a matrix that contains the set of weights that the PF assigns to each piece of data contained on y_t^f. This specification does not necessarily impose the restriction that the PF only considers the latest released data because y_t^f might include lags of some variables. This data is itself related to the variables of the model by[7]:

$$y_t^f = C^f x_t + \nu_t^f$$

Where ν_t is the vector of measurement errors. Then we have

$$x_{t+1|t}^f = F\, C^f x_t + F\nu_t^f$$

4.1 Incorporating the forecasts from the PF

Starting with the case where the only observable variables are the one-step ahead forecasts of some variables, we have as before

$$\begin{aligned} \tilde{y}_t &= x_{t+1|t}^f \\ &= F\, C^f x_t + F\nu_t^f \end{aligned} \tag{20}$$

with $E\left\{\nu_t^f \left(\nu_t^f\right)'\right\} = \bar{H}$. (20) can be written in terms of the expectations of the agents in the model in the form:

$$\begin{aligned} \tilde{y}_t &= T_{(i\in B)} x_t + m_t + v_t^f \\ m_t &= (F\, C^f - T_{(i\in B)}) x_t \\ v_t^f &= F\nu_t^f \\ \bar{\mathbf{H}} &= E\left\{v_t^f \left(v_t^f\right)'\right\} = F\bar{H}F' \end{aligned}$$

$m_t + v_t^f$ can be interpreted as a model mismatch error. The model mismatch error characterizes the difference between the forecast from the PF and the complete information forecast at time t, it can be written as:

$$m_t + \nu_t^f = x_{(i\in B)t+1|t}^f - E\left\{x_{(i\in B)t+1} | x_t, x_{t-1}, ...\right\}$$

The model mismatch term emerges in two cases i) if the forecaster has a different model of the economy or ii) If the forecaster has no complete information. The latter case has been covered in the third section, this section extends the formulation to incorporate also the first case. The shortcoming of this approach is that our results rely on terms such as F which are not "structural" strictly speaking. Nevertheless it allows us to show that the reduced form parameters

[7]Again, here we could extend vector x_t to include lags of some relevent model variables in case some of the data is lagged.

obtained in the previous section also emerges (and not any other) in this more general setup.

Analogous to (**??**) the stochastic process $\{m_t\}_{t=1,\ldots,\infty}$ can be represented in the form

$$\begin{aligned} m_t &= \bar{\Phi} m_{t-1} + \bar{\Gamma} e_t \\ \bar{\Phi} &= \left(F\,C - T_{(i\in B)}\right) T \left[\left(F\,C - T_{(i\in B)}\right)' \left(F\,C - T_{(i\in B)}\right)\right]^{-1} \left(F\,C - T_{(i\in B)}\right)' \\ \bar{\Gamma} &= \left(F\,C - T_{(i\in B)}\right) \end{aligned} \tag{21}$$

(21) shows that the magnitude and sign of the model mismatch term depends on the type of shocks that the economy is receiving each moment. The PF, depending on the shocks present in the economy, might have its forecast near or far from the optimal complete information forecast.

Collecting our results the state-space representation of the model is:

$$\begin{aligned} x_t &= T x_{t-1} + e_t \\ m_t &= \bar{\Phi} m_{t-1} + \bar{\Gamma} e_t \\ \tilde{y}_t &= T_{(i\in B)} x_t + m_t + v_t \\ E\left\{e_t e_t'\right\} &= Q \qquad E\left\{e_t\right\} = \mathbf{0} \\ E\left\{v_t v_t'\right\} &= \mathbf{H} \qquad E\left\{v_t\right\} = \mathbf{0} \end{aligned}$$

and with a more general vector of observable variables y_t defined in (16) we have

$$\begin{aligned} x_t &= T x_{t-1} + e_t \\ m_t &= \bar{\Phi} m_{t-1} + \bar{\Gamma} e_t \\ y_t &= \begin{pmatrix} N \\ T_{(i\in B)} \end{pmatrix} x_t + \begin{pmatrix} \mathbf{0} \\ I \end{pmatrix} m_t + \begin{pmatrix} \mu_t \\ v_t \end{pmatrix} \\ E\left\{e_t e_t'\right\} &= Q \qquad E\left\{e_t\right\} = \mathbf{0} \\ E\left\{v_t v_t'\right\} &= \bar{\mathbf{H}} \qquad E\left\{v_t\right\} = \mathbf{0} \\ E\left\{\mu_t \mu_t'\right\} &= H \qquad E\left\{\mu_t\right\} = \mathbf{0} \end{aligned} \tag{22}$$

Obtaining the likelihood function of (22) is analogous to the steps shown for (17). We have the result again that the Likelihood function depends on the reduced form parameters contained in $\bar{\Phi}$ and $\bar{\Gamma}$. So basically, to incorporate an outsider forecasts as observables for signal extraction, we need to specify a measurement error that is the sum of a standard measurement error term v_t and an autocorrelated and correlated with the structural shocks term m_t.

5 Conclusions

In "Rational Expectations Econometrics" the forecasts of professional forecasters can be used as sources of information for model estimation and to characterize the professional forecaster underlying signal extraction mechanism. For both objectives a fairly general specification that links the PF forecasts with a DSGE model can be derived. The main feature of the specification is the presence of reduced form parameters that capture the relationship between the deviations of the PF relative to an ideal complete information forecaster and the structural shocks of the model.

The reduced form parameters found allow to obtain the Log-Likelihood function of the DSGE model incorporating the professional forecaster forecasts as observables and also the reduced form parameters characterize the shocks of the economy that the professional forecasters miss (don't learn about them). The explicit dependence shown of the reduced form parameters of the gain matrix of the PF and the structure of the economy is relevant information to construct priors for this parameters.

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