ON UPPER-MODULAR SUBALGEBRAS OF A LIE ALGEBRA

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Abstract   This paper is a further contribution to the extensive study by a number of authors of the
subalgebra lattice of a Lie algebra. We give some necessary and some sufficient conditions for a subalgebra
to be upper modular. For algebraically closed fields of any characteristic these enable us to determine the
structure of Lie algebras having abelian upper-modular subalgebras which are not ideals. We then study
the structure of solvable Lie algebras having an abelian upper-modular subalgebra which is not an ideal
and which has trivial intersection with the derived algebra; in particular, the structure is determined
for algebras over the real field. Next we classify non-solvable Lie algebras over fields of characteristic
zero having an upper-modular atom which is not an ideal. Finally, it is shown that every Lie algebra
over a field of characteristic different from two and three in which every atom is upper modular is either
quasi-abelian or a μ-algebra.

1. Introduction

This paper is a further contribution to the extensive study by a number of authors of
the subalgebra lattice of a Lie algebra. A subalgebra U of a Lie algebra L is called *upper
modular* in L if, whenever B is a subalgebra of L which covers $U \cap B$ (that is, such
that $U \cap B$ is a maximal subalgebra of B), then $\langle U, B \rangle$ covers U. (Here, $\langle U, B \rangle$ denotes
the subalgebra of L generated by U and B.) This concept was introduced in [**10**] and
studied further in [**2**]. Our objective here is to gain deeper insight into the structure of
Lie algebras L containing an upper-modular subalgebra with certain extra properties,
and to the position of such a subalgebra within L.

In §2 we assemble some results relating upper-modular subalgebras to ideals in the
algebra. We also give a straightforward characterization of upper-modular subalgebras of
supersolvable Lie algebras. In §3 we give some sufficient conditions and some necessary
conditions for a subalgebra to be upper modular. For algebraically closed fields of any

characteristic, we determine the structure of Lie algebras having abelian upper-modular subalgebras which are not ideals. Section 4 is concerned with studying the structure of solvable Lie algebras L having an abelian upper-modular subalgebra U which is not an ideal of L and such that $U \cap L^2 = 0$. We determine this structure when the ground field is the real numbers; for more general fields, we need to impose the condition that L^2 is abelian.

In § 5 we classify non-solvable Lie algebras over fields of characteristic zero having an upper-modular minimal subalgebra which is not an ideal. Such algebras exist if and only if $\sqrt{F} \not\subseteq F$, where F is the underlying field. The final section is devoted to Lie algebras L in which every minimal subalgebra is upper modular in L. We obtain that, over a field of characteristic different from 2, 3, such an algebra is either quasi-abelian or a μ-algebra.

Throughout, L will denote a finite-dimensional Lie algebra over a field F. There will be no assumptions on F other than those specified in individual results. The symbol '$\oplus$' will denote a vector space direct sum. We will use the notation $A \subseteq B$ to mean that A is a subset of B, whereas $A \subset B$ indicates that A is a proper subset of B. Standard results in Lie algebras are taken from [**6**].

2. Preliminaries

If U is a subalgebra of L, then the *normalizer* of U in L is the set

$$N_L(U) = \{x \in L : [x, U] \subseteq U\}.$$

First we give two results that have been proved elsewhere, but which will be used extensively later.

Lemma 2.1. *Let U be upper modular in L. Then $N_L(U) = U$ or L.*

Proof. This is Lemma 1.5 of [**10**]. $\qquad\square$

Theorem 2.2. *Let U be upper modular in L and let I be an ideal of L such that $U \subseteq I \neq L$. Then U is an ideal of L.*

Proof. This is Lemma 2.2 of [**9**]. $\qquad\square$

The *centre* of L, $Z(L)$, is defined by $Z(L) = \{x \in L : [x, L] = 0\}$. If U is a subalgebra of L, the *core* of U, U_L, is the largest ideal of L contained in U. We now have the following corollary to the above result.

Corollary 2.3. *Let U be an upper-modular subalgebra of L which is not an ideal of L. Then $Z(L) \subseteq U_L$.*

Proof. Clearly, $Z(L) \subseteq N_L(U) = U$. $\qquad\square$

For $x \in L$, the *Engel subalgebra* of x, $E_L(x)$, is the Fitting null-component of L relative to $\operatorname{ad} x$. With a mild restriction on the field, we can improve the above result when U is an abelian upper-modular subalgebra of L.

Theorem 2.4. *Let L be a Lie algebra of dimension n over a field F which has at least n elements and let U be an abelian upper-modular subalgebra of L which is not an ideal of L. Then $U_L = Z(L)$.*

Proof. We have that U is nilpotent and self-normalizing, and so is a Cartan subalgebra of L. It follows from Theorem 4.6 of [**1**] that U is minimal Engel in L, and so $U = E_L(x)$ for some $x \in U$. Write $L = E_L(x) \oplus \bar{E}(x)$, where $\bar{E}(x)$ is the Fitting one-component of L relative to $\operatorname{ad} x$. Then $L = U \oplus \bar{E}(x)$ and

$$[L, U_L] = [\bar{E}(x), U_L] \subseteq U_L \cap [\bar{E}(x), E_L(x)] \subseteq U_L \cap \bar{E}(x) = 0.$$

Thus $U_L \subseteq Z(L)$. Now combine this with the Corollary 2.3. $\square$

A subalgebra Q of L is called a *quasi-ideal* of L if $[Q, V] \subseteq Q + V$ for every subspace V of L. It is easy to see that quasi-ideals of L are always upper-modular subalgebras of L. When L is supersolvable, the upper-modular subalgebras of L are precisely the quasi-ideals of L, as is shown next.

Theorem 2.5. *Let L be supersolvable and let U be upper modular in L. Then U is a quasi-ideal of L.*

Proof. Suppose that $x \in L \setminus U$. Then $U \cap Fx = 0$, which is covered by Fx. Hence $\langle U, x \rangle$ covers U and so, since $\langle U, x \rangle$ is supersolvable, $\langle U, x \rangle = U + Fx$. $\square$

3. General results

Lemma 3.1. *Let L be a Lie algebra and let U be a subalgebra of L. Let A be any subspace of L such that $L = U \oplus A$. Then U is upper modular in L if and only if U is maximal in $\langle U, a \rangle$, for every $0 \neq a \in A$.*

Proof. The 'only if' part is trivial. To prove the converse, take a subalgebra S of L such that $S \cap U$ is maximal in S. Pick $0 \neq s \in S$, $s \notin U$. We have $S = \langle U \cap S, s \rangle$. We have $\langle U, s \rangle \supseteq \langle U \cap S, s \rangle = S$. This yields $\langle U, S \rangle \supseteq \langle U, s \rangle \supseteq \langle U, S \rangle$, from which $\langle U, S \rangle = \langle U, s \rangle$. Now decompose $s = a + u$, where $a \in A$ and $u \in U$. As $s \notin U$, we see that $a \neq 0$. Also, we have $\langle U, s \rangle = \langle U, a \rangle$. By our hypothesis, U is maximal in $\langle U, a \rangle$. Consequently, U is maximal in $\langle U, S \rangle$. Hence U is upper modular in L. $\square$

Now, for algebraically closed fields of any characteristic, we are able to determine the structure of Lie algebras having abelian upper-modular subalgebras which are not ideals.

Theorem 3.2. *Let F be an algebraically closed field of any characteristic. Let L be a Lie algebra and let U be an abelian subalgebra of L which is not an ideal of L. Then U is upper modular in L if and only if $L = A \oplus U$, where either $[A, U] = 0$, or A is an abelian ideal of L and every element of U acts as a scalar on A.*

Proof. Assume that U is upper modular in L. By Lemma 2.1, we have that U is a Cartan subalgebra of L. Let $L = L_1(U) \oplus U$ be the Fitting decomposition of L relative to U. Let $0 \neq a \in L_1(U)$. Then U is maximal in $\langle U, a \rangle$ because of its upper modularity. As $L_1(U)$ is a U-submodule of L, so is $L_1(U) \cap \langle U, a \rangle$. Since U is abelian, we have that $\{(\operatorname{ad} u)|_{L_1(U) \cap \langle U, a \rangle} : u \in U\}$ is a set of simultaneously triangulable linear mappings. So there must exist $0 \neq b \in L_1(U) \cap \langle U, a \rangle$ such that $[u, b] = \alpha(u)b$ for every $u \in U$, where $\alpha(u) \in F$. Then we have that $U + Fb$ is a subalgebra of $\langle U, a \rangle$ strictly containing U. Therefore, $U + Fb = \langle U, a \rangle$. This yields $L_1(U) \cap \langle U, a \rangle = Fb$, and hence $Fa = Fb$. Consequently, every non-zero element of $L_1(U)$ is a common eigenvector of $\operatorname{ad} u$ for each $u \in U$. From this it follows that $\operatorname{ad} u$ acts as a scalar on $L_1(U)$, so that $(\operatorname{ad} u)|_{L_1(U)} = \alpha(u)1$, where $\alpha(u) \in F$. Then, by using the Jacobi identity, we see that $[L_1(U), L_1(U)] = 0$ if $\alpha \neq 0$. This yields that either U acts trivially on $L_1(U)$, or $L_1(U)$ is an abelian ideal of L.

To prove the converse, let $0 \neq a \in A$. By our hypothesis, we have $[u, a] = \alpha(u)a$, where $\alpha(u) \in F$, for every $u \in U$. So $U + Fa$ is a subalgebra of L. Therefore, $\langle U, a \rangle = U + Fa$ and U is maximal in $\langle U, a \rangle$. By Lemma 3.1, it follows that U is upper modular in L. $\square$

Note that the condition that F is algebraically closed in Theorem 3.2 can be removed whenever the subalgebra U is splitting.

Next, we give sufficient conditions to guarantee that a subalgebra is upper modular.

Lemma 3.3. *Let $L = U \oplus A$ be a Lie algebra, where U is a subalgebra of L and A is a U-submodule of L.*

(i) *Let B be any irreducible U-submodule of A. Then $\langle U, B \rangle = \langle U, b \rangle$ for every $0 \neq b \in B$.*

(ii) *If $\langle U, a \rangle \cap A$ is irreducible as a U-submodule, for every $0 \neq a \in A$, then U is upper modular in L.*

Proof. (i) We have

$$[U, \langle U, b \rangle \cap A] \subseteq [U, A] \cap [\langle U, b \rangle, \langle U, b \rangle] \subseteq A \cap \langle U, b \rangle,$$

so that $\langle U, b \rangle \cap A$ is a U-submodule of A. Since $0 \neq b \in B \cap \langle U, b \rangle$ and B is irreducible, we have $B = B \cap \langle U, b \rangle$. This yields $\langle U, B \rangle \subseteq \langle U, b \rangle \subseteq \langle U, B \rangle$, whence $\langle U, b \rangle = \langle U, B \rangle$, as required.

(ii) By Lemma 3.1, we need only prove that U is maximal in $\langle U, a \rangle$ for every $0 \neq a \in A$. To see this, let M be a subalgebra of $\langle U, a \rangle$ such that $U \subseteq M \subseteq \langle U, a \rangle$. We have $M = (M \cap A) \oplus U$. On the other hand, we see that $M \cap A$ is a U-submodule of A and $M \cap A \subseteq \langle U, a \rangle \cap A$. So either $M \cap A = 0$, and then $M = U$, or $M \cap A = \langle U, a \rangle \cap A$, and then $M = \langle U, a \rangle$. $\square$

Theorem 3.4. *Let $L = U \oplus A$ be a Lie algebra over any field F, where U is a subalgebra of L and A is a U-submodule of L. Assume, in addition, that the following hold.*

(i) *Each element of A lies in an irreducible U-submodule of L.*

(ii) *$B + U$ is a subalgebra of L, for every irreducible U-submodule B of A.*

Then U is upper modular in L.

Proof. Let $0 \neq a \in A$. By (i), there exists a U-submodule B of A containing a. By Lemma 3.3 (i), we have $\langle U, a \rangle = \langle U, B \rangle$. Since $U + B$ is a subalgebra, we have $\langle U, B \rangle = U + B$, whence $\langle U, a \rangle \cap A = (U + B) \cap A = B$. It follows from Lemma 3.3 (ii) that U is upper modular in L. $\qquad \square$

Condition (i) of Theorem 3.4 is clearly satisfied whenever $U = Fu$ and the minimum polynomial of $(\operatorname{ad} u)|_A$ is irreducible. Also, it is satisfied in the case where the ground field is infinite and perfect, U is abelian and the minimum polynomial of $(\operatorname{ad} u)|_A$ is irreducible for every $u \in U$, as is shown by the following.

Proposition 3.5. *Let $L = U \oplus A$ be a Lie algebra over an infinite perfect field, where U is an abelian subalgebra of L, A is a U-submodule of L and the minimum polynomial of $(\operatorname{ad} u)|_A$ is irreducible for every $u \in U$. Then each element $a \in A$ lies in an irreducible U-submodule of A.*

Proof. Let Ω be the algebraic closure of F, and put $A_\Omega = A \otimes_F \Omega$. Let

$$\sigma : U \to gl(A) : u \mapsto (a \mapsto [a, u])$$

be the representation corresponding to the U-module A. Pick $u \in U$ and decompose A into its generalized eigenspaces relative to $\sigma(u)$. Since $[\sigma(u), \sigma(v)] = 0$ for every $v \in U$, each generalized eigenspace is a submodule of A. Also, since the minimum polynomial of $\sigma(u)$ is irreducible for every $u \in U$, the transformations form a set of commuting semisimple operators. It follows that A_Ω can be decomposed into

$$(A_\Omega)_{\alpha_i} = \{a \in A_\Omega : \tilde{\sigma}(a) = \alpha_i(u)a \ \forall u \in U\},$$

where $\tilde{\sigma}$ is the representation of A_Ω induced by σ and $1 \leqslant i \leqslant r$.

If $r = 1$, then $A = \sum_{i=1}^n Fa_i$, where $[u, a]_i = \alpha_u a_i$ for all $u \in U$, and the result is clear.

So suppose that $r > 1$. Then, as in Lemma 3.2 of [**5**], there exists $u_0 \in U$ such that $\alpha_i(u_0) \neq \alpha_j(u_0)$ for every $1 \leqslant i \neq j \leqslant r$. Pick $0 \neq a \in A$, and put $C = \sum_{k=0}^\infty a(\operatorname{ad} u_0)^k$, so that C is a cyclic subspace of A relative to $\sigma(u_0)$. Now $(A_\Omega)_{\alpha_i}$ is just the eigenspace of A_Ω corresponding to the eigenvalue $\alpha_i(u_0)$ relative to $\tilde{\sigma}(u_0)$ for every $1 \leqslant i \leqslant r$. Thus

$$C_\Omega = \bigoplus_{i=1}^r C_\Omega \cap (A_\Omega)_{\alpha_i}.$$

Let $c \in C$ and write $c = c_1 + \cdots + c_r$, where $c_i \in C_\Omega \cap (A_\Omega)_{\alpha_i}$. Then, for every $u \in U$,

$$\sigma(u)(c) = \sum_{i=1}^r \tilde{\sigma}(u)(c_i) = \sum_{i=1}^r \alpha_i(u)(c_i) \in C_\Omega \cap A = C,$$

so C is a U-submodule of A. It is irreducible as such, as $\sigma(u_0)$ has an irreducible minimum polynomial, and contains a. $\qquad \square$

Now we give necessary conditions for a subalgebra to be upper modular.

Theorem 3.6. *Let $L = U \oplus A$ be a Lie algebra, where U is a subalgebra of L and $A = N \oplus V$ for some non-zero solvable ideal N of L and an irreducible U-submodule V of L (possibly zero). Let $0 \neq B$ be an irreducible U-submodule of A. Assume that U is upper modular in L. Then the following hold.*

(i) *$\langle U, a \rangle \cap A$ is irreducible as a U-submodule, for every $0 \neq a \in A$.*

(ii) *If $B \subseteq N$, then B is an abelian subalgebra of L.*

(iii) *$\langle U, b \rangle = \langle U, B \rangle = U + B$ for every $0 \neq b \in B$.*

(iv) *A is completely reducible as a U-module.*

(v) *If U is abelian, then the minimum polynomial of $(\mathrm{ad}\, u)|_A$ is irreducible, for every $u \in U$.*

Proof. (i) First, take $0 \neq n \in N$. Since N is an ideal of L, the subspace $N + U$ is a subalgebra of L. So $\langle U, n \rangle \subseteq N + U$. This gives $\langle U, n \rangle \cap N = \langle U, n \rangle \cap A$. Also, we have that $\langle U, n \rangle \cap N$ is an ideal of $\langle U, n \rangle$. On the other hand, since $Fn \cap U = 0$, we have that U is maximal in $\langle U, n \rangle$ because of its upper modularity. It follows that $\langle U, n \rangle \cap N$ is a minimal ideal of $\langle U, n \rangle$. As N is solvable, we find that $\langle U, n \rangle \cap N$ is a solvable minimal ideal of $\langle U, n \rangle$ and so is abelian. This yields that $\langle U, n \rangle \cap A$ is irreducible as a U-module. Now, take $a \in A$, $a \notin N$. Write $A_a = \langle U, a \rangle \cap A$. Suppose $A_a \cap N \neq 0$. Then there exists $0 \neq n \in A_a \cap N$. We have $\langle U, n \rangle \subseteq \langle U, a \rangle$. Since U is maximal in $\langle U, a \rangle$, it follows that $\langle U, a \rangle = \langle U, n \rangle$. Then, by the above, we have that $\langle U, n \rangle \cap A$ is irreducible and $\langle U, n \rangle \cap A = \langle U, n \rangle \cap N$. But then we have $a \in A_a \subseteq N$, which is a contradiction. Therefore, $A_a \cap N = 0$. Since A_a is a U-submodule of A, we have that $0 \neq A_a \cong (A_a + N)/N$ is a U-submodule of $A/N \cong V$. Irreducibility of V implies that $A_a \cong V$ is irreducible as a U-submodule, as required.

(ii) Assume that $B \subseteq N$. Pick $0 \neq b \in B$. By Lemma 3.3 (i), we have $\langle U, B \rangle = \langle U, b \rangle$. So $B \subseteq \langle U, b \rangle \cap A$. Moreover, in the proof of (i), we have obtained that $\langle U, b \rangle \cap A$ is an abelian subalgebra of L.

(iii) If $B \subseteq N$, then $B + U$ is a subalgebra of L because B is abelian (by (ii) above). Assume then that $B \not\subseteq N$. Then we have $N \cap B = 0$ by the irreducibility of B. Thus $A = N \oplus B$, since A/N is irreducible. Now pick $0 \neq b \in B$. By Lemma 3.3 (i), $\langle U, B \rangle = \langle U, b \rangle$. By (i), $\langle U, b \rangle \cap A$ is irreducible as a U-module. This gives $B = \langle U, B \rangle \cap A$. On the other hand, since $L = U \oplus A$, we have $\langle U, B \rangle = U + \langle U, B \rangle \cap A$. Therefore, $\langle U, B \rangle = U + B$.

(iv) Since $A = \sum_a (A \cap \langle U, a \rangle)$, it follows from (i) that A is completely reducible as a U-module.

(v) Assume that U is abelian. Then we know that the minimum polynomial of $(\mathrm{ad}\, u)|_B$ is irreducible, for every irreducible U-module B and for every $u \in U$. By (i), we have that every non-zero element of A is contained in an irreducible U-submodule of A. From this follows (v). $\qquad\square$

4. Solvable Lie algebras

First we consider the case where the ground field is the real field. In this case, we are able to determine the structure of solvable Lie algebras L having an abelian upper-modular subalgebra U which is not an ideal of L and such that $U \cap L^2 = 0$.

The *abelian socle* of L, $\mathrm{Asoc}(L)$, is the sum of the minimal abelian ideals of L. The *Frattini subalgebra* of L, $F(L)$, is the intersection of all the maximal subalgebras of L, and the *Frattini ideal* of L, $\phi(L)$, is the largest ideal of L contained in $F(L)$. If $\phi(L) = 0$, we say that L is ϕ-*free*.

Theorem 4.1. *Let F be the real field. Let L be a solvable Lie algebra and let U be an abelian subalgebra which is not an ideal of L and such that $U \cap L^2 = 0$. Then the following are equivalent.*

(i) *U is upper modular in L.*

(ii) *$L^2 \subseteq \mathrm{Asoc}(L)$, $L = L^2 \oplus U$ and the minimum polynomial of $(\mathrm{ad}\, u)|_{L^2}$ is irreducible for every $u \in U$.*

(iii) *L is ϕ-free, L^2 is abelian, U is a Cartan subalgebra of L and the minimum polynomial of $(\mathrm{ad}\, u)|_{L^2}$ is irreducible for every $u \in U$.*

Proof. (i) $\Rightarrow$ (ii). By Lemma 2.1, we have that U is a Cartan subalgebra of L. So $L = L^2 \oplus U$ and L^2 is just the Fitting one-component of L relative to U. By Theorem 3.6 (v), we have that the minimum polynomial, $m_u(x)$, of $(\mathrm{ad}\, u)|_{L^2}$ is irreducible for every $u \in U$. It is well known that there exists $u_0 \in U$ acting non-singularly on L^2. If $m_{u_0}(x) = (x - t)$ for some $t \in F$, then we see that L^2 is precisely the eigenspace of L corresponding to the eigenvalue t relative to $\mathrm{ad}\, u_0$. So, by using the Jacobi identity, we obtain $[L^2, L^2] = 0$. Now suppose $\deg(m_{u_0}(x)) = 2$. Let Ω be the complex field. Decompose $m_{u_0}(x) = (x - \alpha)(x - \beta)$, where $\alpha, \beta \in \Omega$. Let $L_\Omega = L \otimes_F \Omega$, and decompose $(L^2)_\Omega = V_\alpha \oplus V_\beta$ into its eigenspaces relative to $\mathrm{ad}\, u_0$. We have $[V_\alpha, V_\beta] \subseteq V_{\alpha+\beta} = 0$, since u acts non-singularly on L^2. Since, clearly, the other products are zero, we have $[L^2, L^2] = 0$. We conclude that L^2 is abelian. Then we see that the irreducible U-submodules of L^2 are precisely the minimal ideals of L contained in L^2. Since, by Theorem 3.6 (iv), L^2 is completely reducible as a U-submodule, it follows that $L^2 \leqslant \mathrm{Asoc}(L)$.

(ii) $\Rightarrow$ (i). Since $L^2 \subseteq \mathrm{Asoc}(L)$, it follows that L^2 is completely reducible as a U-submodule of L and that the irreducible U-submodules of L^2 are precisely the minimal ideals of L contained in L^2. Since L^2 is abelian and $L = L^2 \oplus U$, it follows from Proposition 3.5 and Theorem 3.4 that U is upper modular.

(ii) $\Leftrightarrow$ (iii). See Proposition 1 of [**7**]. $\qquad\square$

Clearly, subalgebras U of dimension one which are not ideals satisfy the conditions in Theorem 4.1. In this case, we are able to give the multiplication table of such Lie algebras.

Corollary 4.2. *Let F be the real field. Then a solvable Lie algebra L has a non-ideal minimal subalgebra of L which is upper modular if and only if one of the following holds.*

(i) L has a basis $a_1, \ldots, a_r, u$ with product given by $[u, a_i] = a_i$ for every i and the other products are zero.

(ii) L has a basis $a_1, \ldots, a_r, b_1, \ldots, b_r, u$ with product given by $[u, a_i] = b_i$, $[u, b_i] = \beta a_i + \alpha b_i$ for every i, where $\alpha, \beta \in F$ and the polynomial $x^2 - \alpha x - \beta$ is irreducible over F.

For any infinite perfect field, we need to impose the condition that L^2 is abelian.

Theorem 4.3. *Let F be any infinite perfect field. Let L be a metabelian Lie algebra. Let U be an abelian subalgebra which is not an ideal of L and such that $L^2 \cap U = 0$. Then the following are equivalent.*

(i) *U is upper modular.*

(ii) *$L^2 \leqslant \mathrm{Asoc}(L)$, $L = L^2 \oplus U$ and the minimum polynomial of $(\mathrm{ad}\, u)|_{L^2}$ is irreducible for every $u \in U$.*

(iii) *L is ϕ-free, U is a Cartan subalgebra of L and the minimum polynomial of $(\mathrm{ad}\, u)|_{L^2}$ is irreducible for every $u \in U$.*

Proof. (i) $\Rightarrow$ (ii). Since $U + L^2$ is an ideal of L properly containing U, Theorem 2.2 implies that $L = L^2 \oplus U$. From Theorem 3.6 (iv), it follows that L^2 is completely reducible as a U-submodule of L. On the other hand, since L^2 is abelian, we see that the irreducible U-submodules of L^2 are precisely the minimal ideals of L contained in L^2. This yields $L^2 \subseteq \mathrm{Asoc}(L)$. Moreover, by Theorem 3.6 (v), it follows that the minimum polynomial of $(\mathrm{ad}\, u)|_{L^2}$ is irreducible for every $u \in U$. The proof of the remaining parts are as in Theorem 4.1. $\qquad\square$

For arbitrary fields, we have the following result.

Theorem 4.4. *Let L be a solvable Lie algebra over any field F, and let U be an abelian subalgebra of L which is not an ideal of L and such that $U \cap L^2 = 0$. Let U be upper modular. Then, $L = L^2 \oplus U$, L^2 is completely reducible as a U-submodule of L, every irreducible U-submodule of L^2 is an abelian subalgebra of L and the minimum polynomial of $(\mathrm{ad}\, u)|_{L^2}$ is irreducible for every $u \in U$.*

Proof. This follows from Theorem 3.6. $\qquad\square$

For minimal subalgebras, the converse of Theorem 4.4 holds.

Proposition 4.5. *Let L be a solvable Lie algebra over any field F and let $u \in L$. Then Fu is upper modular in L if and only if either of the following hold.*

(i) *Fu is an ideal of L.*

(ii) *$L = L^2 \oplus Fu$, every subspace of L^2 that is invariant and irreducible under $\mathrm{ad}\, u$ is an abelian subalgebra of L and the minimum polynomial of $(\mathrm{ad}\, u)|_{L^2}$ is irreducible.*

Proof. Suppose that (i) or (ii) holds; ideals are clearly upper modular, so suppose (ii) holds. Then $(\operatorname{ad} u)|_{L^2}$ is semisimple, so L^2 is completely reducible as an Fu-module. Also, from the hypothesis that the minimum polynomial of $(\operatorname{ad} u)|_{L^2}$ is irreducible, it follows that each element in L^2 lies in an irreducible Fu-submodule of L^2. The fact that Fu is upper modular in L is now immediate from Theorem 3.4. The converse follows from Theorem 4.4. $\qquad\square$

5. Non-solvable Lie algebras

For any field of characteristic zero, we are able to classify the non-solvable Lie algebras having an upper-modular minimal subalgebra which is not an ideal. There exist such Lie algebras if and only if $\sqrt{F} \not\subseteq F$.

If $u \in L$, we shall write $C_L(u)$ for $\{x \in L : [x, u] = 0\}$.

Theorem 5.1. *Let L be a non-solvable Lie algebra over a field F of characteristic zero and let $u \in L$. Then Fu is an upper-modular subalgebra of L if and only if one of the following holds.*

(i) *Fu is an ideal of L.*

(ii) *The solvable radical $R(L)$ of L is abelian, u lies in a Levi subalgebra S of L which is three-dimensional simple with basis u, v, w such that $[u, v] = w$, $[v, w] = 4\alpha u$, $[w, u] = 4\beta v$, where $\alpha, \beta \in F$ and $\sqrt{-\beta} \notin F$, $L = R(L) \oplus S$ and the minimum polynomial of $(\operatorname{ad} u)|_{R(L) + Fv + Fw}$ is $(x^2 + 4\beta)$.*

Proof. Let Fu be an upper-modular subalgebra of L which is not an ideal of L. By Levi's theorem, we can write $L = R \oplus S$, where R is a maximal ideal of L containing the radical of L and S is a simple subalgebra of L. Suppose that $u \notin S$. Then $u = r + s$, where $0 \neq r \in R$, $s \in S$. Now there is an element $t \in S$ such that $\langle s, t \rangle = S$ (see, for instance, Theorem 5 of [**8**]). Put $T = \langle u, t \rangle$. Then

$$R + T = R + \langle u, t \rangle = R + \langle s, t \rangle = L,$$

so $T/(T \cap R) \cong L/R \cong S$ is simple.

Suppose that $u \in T \cap R$. Since $T \cap R$ is an ideal of T, it follows from Theorem 2.2 that Fu is an ideal of T. But then $N_L(Fu) \neq Fu$, and so Fu is an ideal of L, by Lemma 2.1, contrary to our assumption. Hence $u \notin T \cap R$.

Suppose now that $T \cap R \neq 0$. Then $T = (T \cap R) \oplus Fu$, since Fu is a maximal subalgebra of T. But this implies that $T/(T \cap R)$ is one dimensional, contradicting the fact that it is simple. This yields $T \cap R = 0$ and $u \in T \cong S$, which is simple. Replacing S by T if necessary, we may assume that $u \in S$.

But Fu is upper modular in S, and hence a maximal subalgebra of S, as above. It follows that S is three-dimensional non-split simple. Moreover, if Σ is a Levi factor of L containing S, then Fu is a maximal subalgebra of Σ, and so $\Sigma = S$ and $R = R(L)$ is the radical of L. Therefore, $L = R(L) \oplus S$. Also, the element u can be complemented by elements v, w to form a basis for S with products given by $[u, v] = w$, $[v, w] = 4\alpha u$,

$[w, u] = 4\beta v$, where $\alpha, \beta \in F$ and $\sqrt{-\beta} \notin F$. Write $V = Fv + Fw$. We see that V is an irreducible Fu-submodule of L and that the minimum polynomial of $(\operatorname{ad} u)|_V$ is $x^2 + 4\beta$. Write $M = R(L) \oplus V$. If $R(L) = 0$, then case (ii) holds; so suppose that $R(L) \neq 0$. By Theorem 3.6, it follows that $R(L)$ is completely reducible as an Fu-module and the minimum polynomial of $(\operatorname{ad} u)|_M$ is irreducible. This gives that the minimum polynomial of $(\operatorname{ad} u)|_{R(L)}$ is also $x^2 + 4\beta$. Moreover, we see that the adjoint action of u on $R(L)$ is non-singular.

Let Ω be an algebraic closure of F and let $R(L)_\Omega = R(L) \otimes_F \Omega$. We can decompose $x^2 + 4\beta = (x - \omega)(x + \omega)$, where $\omega \in \Omega$. This implies that we can decompose $R(L)_\Omega = (R(L)_\Omega)_\omega \oplus (R(L)_\Omega)_{-\omega}$ into its root spaces relative to $\operatorname{ad} u$. We find $[(R(L)_\Omega)_\omega, (R(L)_\Omega)_{-\omega}] \subseteq (R(L)_\Omega)_0 = 0$, and the other products are also equal to zero. This yields that $R(L)$ is abelian. So, we have case (ii).

It remains to show that, in case (ii), Fu must be upper modular in L. We write $V = Fv + Fw$. In this case, $(\operatorname{ad} u)|_{R(L)+Fv+Fw}$ is semisimple, and so $R(L) + V$ is completely reducible as an Fu-module. Also, we have that, for every non-zero element in $R(L) + V$, the cyclic Fu-module generated by it is irreducible as an Fu-module. On the other hand, we see $V + Fu = S$, and so $V + Fu$ is a subalgebra of L. Now let B be any irreducible Fu-submodule of L contained in $R(L) \oplus V$. If $B \subseteq R(L)$, then we have that $B + Fu$ is a subalgebra of L, since $R(L)$ is abelian. Now assume $B \not\subseteq R(L)$. Then $B \cap R(L) = 0$ and $R(L) \oplus B$ is a Fu-submodule of $R(L) \oplus V$. Since the latter is a completely reducible Fu-submodule, there exists a complement C such that $R(L) \oplus B \oplus C = R(L) \oplus V$. Consequently, $\dim B \leqslant 2$. If $\dim B = 1$, then B is abelian and $B + Fu$ is a subalgebra of L. Hence only the case $\dim B = 2$ remains to be considered. Let Ω be an algebraic closure of F and let $B_\Omega = B \otimes_F \Omega$. Decompose, as above, $x^2 + 4\beta = (x - \omega)(x + \omega)$, where $\omega \in \Omega$. Let $B_\Omega = (B_\Omega)_\omega \oplus (B_\Omega)_{-\omega}$ be the decomposition of B_Ω into its root spaces relative to $\operatorname{ad} u$. Now we have that $[u, [(B_\Omega)_\omega, (B_\Omega)_{-\omega}]] = 0$. This yields $[B, B] \subseteq C_L(u)$. On the other hand, as $x^2 + 4\beta$ is the minimum polynomial of $(\operatorname{ad} u)|_{R(L)+B}$, we see that $C_L(u) \cap (R(L) + B) = 0$. This yields $C_L(u) = Fu$, and therefore $[B, B] \subseteq Fu$. It follows that $B + Fu$ is a subalgebra of L. Then, by Theorem 3.4, it follows that Fu is upper modular in L. Now the proof is complete. $\square$

Corollary 5.2. *Let F be any field of characteristic zero. Let L be a non-solvable Lie algebra. Then L has an upper-modular minimal subalgebra which is not an ideal if and only if L has a basis $a_1, \ldots, a_r, b_1, \ldots, b_r, u, v, w$ with product given by $[u, a_i] = b_i$, $[u, b_i] = -4\beta a_i$, $[u, v] = w$, $[v, w] = 4\alpha u$, $[w, u] = 4\beta v$ and the other products zero, where $\alpha, \beta \in F$ and $\sqrt{-\beta} \notin F$.*

Corollary 5.3. *Over fields F of characteristic zero such that $\sqrt{F} \subseteq F$, a Lie algebra L having an upper-modular minimal subalgebra which is not an ideal of L must be solvable.*

6. um(0)-algebras

We shall call L a um(0)-*algebra* if all of its one-dimensional subalgebras are upper modular in L. The Lie algebra L is *almost abelian* if $L = L^2 \oplus Fx$, where L^2 is abelian

and $[x, y] = y$ for all $y \in L^2$; if L is abelian or almost abelian, we say that it is *quasi-abelian*. In [**2**], it was shown that, over a field of characteristic zero, L is a um(0)-algebra if and only if it is quasi-abelian or three-dimensional non-split simple. Here we want to generalize this result by considering a wider class of fields.

A *μ-algebra* is a non-solvable Lie algebra in which every non-zero proper subalgebra is one dimensional. Every μ-algebra L is *supersimple*; that is, every subalgebra of L of dimension greater than one is simple. Over a perfect field of characteristic different from two and three, the only supersimple Lie algebras are the three-dimensional non-split simple Lie algebras (see Proposition 1 of [**4**]). The generalization we obtain is that, over a field of characteristic different from two and three, L is a um(0)-algebra if and only if it is quasi-abelian or a μ-algebra. First we need the following result, which is related to Lemma 4.1 of [**3**].

Theorem 6.1. *Let L be a Lie algebra over any field F and suppose that every two-generated subalgebra of L is either quasi-abelian or a μ-algebra. Then either of the following hold.*

(i) *L is quasi-abelian.*

(ii) *Every two-generated subalgebra of L is a μ-algebra.*

Proof. Suppose first that every two-generated subalgebra of L is quasi-abelian. Then every two-dimensional subspace of L is a subalgebra of L, from which it is clear that every subspace of L is a subalgebra of L. It follows that L is quasi-abelian.

So suppose that L has two-generated subalgebras Q, U, where Q is quasi-abelian and U is a μ-algebra. We claim that we may assume that $Q = \langle x, y \rangle$ and $U = \langle y, z \rangle$ for some $x, y, z \in L$. Clearly, either $Q \cap U = 0$ or $Q \cap U = Fa$ for some $0 \neq a \in L$. Suppose first that $Q \cap U = 0$. Choose any $0 \neq q \in Q$, $0 \neq u \in U$. Then $\langle q, u \rangle$ is either quasi-abelian or a μ-algebra. If it is quasi-abelian, take $x = q$, $y = u$ and $z \in U \setminus \langle q, u \rangle$; if it is a μ-algebra, take $x \in Q \setminus \langle q, u \rangle$, $y = q$ and $z = u$. Now suppose that $Q \cap U = Fa$. Then take $x \in Q \setminus Fa$, $y = a$ and $z \in U \setminus Fa$. This establishes our claim.

Suppose next that $Q \cap U = Fy$ is an ideal of Q, so that $[x, y] = \alpha y$ for some $\alpha \in F$. Then $[y, z] \notin Fy$, whence $Fy \subset \langle y, [y, z] \rangle \subseteq \langle y, z \rangle$, giving $\langle y, [y, z] \rangle = \langle y, z \rangle$. However, $[x + z, y] = \alpha y + [z, y]$, so $\langle y, z \rangle = \langle y, [y, z] \rangle \subseteq \langle x + z, y \rangle$. But now $\langle x + z, y \rangle$ is a two-generated subalgebra of L containing both Q and U, which is impossible.

It follows that $Q \cap U = Fy$ is not an ideal of Q. There is an element $q \in Q$ for which Fq is an ideal of Q, and, without loss of generality, we may assume that $q = x$. Thus $[x, y] = \beta x$ for some $\beta \in F$. Then $[x + z, y] = \beta x + [z, y] \in \langle x + z, y \rangle$, so

$$-\beta z - [y, z] = -\beta z + [z, y] = -\beta (z + x) + \beta x + [z, y] \in \langle x + z, y \rangle.$$

Hence $\beta z + [y, z] \in \langle x + z, y \rangle$. Also, $\beta z + [y, z] \notin Fy$, since $\langle y, z \rangle$ is a μ-algebra, from which it follows that $\langle y, z \rangle = \langle \beta z + [y, z], y \rangle \subseteq \langle x + z, y \rangle$. This means that $\langle x + z, y \rangle$ is a μ-algebra, because otherwise $U = \langle y, z \rangle$ would be solvable. It follows that $\langle y, z \rangle = \langle x + z, y \rangle$ and $x \in \langle x, y \rangle \cap \langle y, z \rangle = Fy$, which is a contradiction. The result now follows. $\qquad\square$

Lemma 6.2. *Let L be a Lie algebra over any field F. Then the following are equivalent.*

(i) *L is supersimple.*

(ii) *Every subalgebra of L is either one dimensional or simple.*

(iii) *L has no two-dimensional subalgebras.*

Proof. The equivalence of (ii) and (iii) is given by Varea in [**11**]. Since two-dimensional subalgebras are never simple, (i) implies (iii). Finally, it is trivial that (ii) implies (i). $\square$

We shall need the following property of supersimple Lie algebras which was given by Varea in [**12**].

Theorem 6.3. *Let L be a supersimple Lie algebra over a field F of characteristic p different from 2, 3. Then, for each $0 \neq x \in L$, there is a $y \in L$ such that $\langle x, y \rangle = L$.*

Proof. This is Theorem 4 of [**12**]. $\square$

The above result has the following consequence for um(0)-algebras.

Lemma 6.4. *Let L be a supersimple um(0)-algebra over a field of characteristic different from 2, 3. Then L is a μ-algebra.*

Proof. Choose any $0 \neq x \in L$. Then, by the above theorem, there is a $y \in L$ such that $\langle x, y \rangle = L$. Now $Fx \cap Fy = 0$, which is a maximal subalgebra of Fy, so Fx is a maximal subalgebra of L. It follows that every one-dimensional subalgebra of L is maximal in L, and hence that L is a μ-algebra. $\square$

We can now prove our main result.

Theorem 6.5. *Let L be any Lie algebra over a field F of characteristic different from 2, 3. Then the following are equivalent.*

(i) *L is a um(0)-algebra.*

(ii) *L is quasi-abelian or a μ-algebra.*

Proof. Suppose first that L is a um(0)-algebra, and let $U = \langle x, y \rangle$ be a two-generated subalgebra of L which is not quasi-abelian. Then U has dimension greater than two, and Fx, Fy are maximal subalgebras of U. Suppose that U has a two-dimensional subalgebra, $S = Fa + Fb$, say. Clearly, we can assume that $x \notin S$, so that $\langle x, s \rangle = U$ for every $0 \neq s \in S$. But now $Fx \cap Fa = 0$, which is a maximal subalgebra of Fx, so Fa is maximal in $\langle x, a \rangle = U$. This contradicts the fact that $Fa \subset S \subset U$. It follows that U has no two-dimensional subalgebras, and so is supersimple, by Lemma 6.2. But now U must be a μ-algebra, by Lemma 6.4. We have shown that every two-generated subalgebra of L is either quasi-abelian or a μ-algebra. We conclude from Theorem 6.1 that either L is quasi-abelian or else every two-generated subalgebra of L is a μ-algebra. In the latter case, L is supersimple, by Lemma 6.2, and hence a μ-algebra, by Lemma 6.4.

The converse is easy to check. $\square$

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