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On Lie Algebras All of Whose Minimal Subalgebras Are Lower Modular[#]

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ABSTRACT

The main purpose of this paper is to study Lie algebras L such that if a subalgebra U of L has a maximal subalgebra of dimension one then every maximal subalgebra of U has dimension one. Such an L is called lm(0)-algebra. This class of Lie algebras emerges when it is imposed on the lattice of subalgebras of a Lie algebra the condition that every atom is lower modular. We see that the effect of that condition is highly sensitive to the ground field F. If F is algebraically closed, then every Lie algebra is lm(0). By contrast, for every algebraically non-closed field there exist simple Lie algebras which are not lm(0). For the real field, the semisimple lm(0)-algebras are just the Lie algebras whose Killing form is negative-definite. Also, we study when the simple Lie algebras having a maximal subalgebra of codimension one are lm(0), provided that $\mathrm{char}(F) \neq 2$. Moreover, lm(0)-algebras lead us to consider certain other classes of Lie algebras and the largest ideal of an arbitrary Lie algebra L on which the action of every element of L is split, which might have some interest by themselves.

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1. INTRODUCTION

Throughout L will denote a finite-dimensional Lie algebra over a field F. The relationship between the structure of L and that of the lattice $\mathscr{L}(L)$ of all subalgebras of L has been studied by many authors. Much is known about modular subalgebras (modular elements in $\mathscr{L}(L)$) through a number of investigations including Amayo and Schwarz (1980), Gein (1987a,b), Varea (1989, 1990, 1993). Modular subalgebras of dimension greater than one which are not quasi-ideals were exhibited in Varea (1993). Other lattice conditions, together with their duals, have also been studied. These include semimodular, upper semimodular, lower semimodular, upper modular, lower modular and their respective duals (see Bowman and Towers, 1989, for definitions). For a selection of results on these conditions see Gein (1976), Varea (1983, 1999), Gein and Varea (1992), Lashi (1986), Towers (1986, 1997), Bowman and Varea (1997). Moreover, it has been proved that none non-solvable locally finite-dimensional Lie algebra admits a lattice isomorphism on a solvable Lie algebra, except the three-dimensional non-split simple, provided that the ground field is perfect of characteristic not 2 or 3 (see Gein and Varea, 1992).

Many of the lattice conditions imposed so far have proved to be very strong, forcing the algebra to be abelian, almost abelian, supersolvable, a μ-algebra (this means that every proper subalgebra has dimension one) or an algebra direct sum of the above. Typically, see Gein (1987a), Varea (1993, 1999). In this paper we shall introduce a condition that is less restrictive.

Recall that an element U of a lattice $\mathscr{L}$ is called *lower modular* in $\mathscr{L}$ if, given any element B of $\mathscr{L}$ with $U \vee B$ covering U, then B covers $U \wedge B$. A subalgebra U of a Lie algebra L is called *lower modular* in L (lm in L) if it is a lower modular element in the lattice of subalgebras of L.

In this paper, we impose the condition that every minimal subalgebra of L is lm in L. We prove that this condition is equivalent to the condition that if a subalgebra U of L has a maximal subalgebra of dimension one then every maximal subalgebra of U has dimension one. We shall call such an algebra lm(0). The situation depends essentially on the ground field. For example, we will obtain that if the field is algebraically closed then all Lie algebras are lm(0), and over other any field there are even simple Lie algebras which are not lm(0). On the other hand, for each element a of any Lie algebra L, denote by $S_L(a)$ the largest subalgebra of L containing a on which $\operatorname{ad} a$ is split. This subalgebra was introduced in Barnes and Newell (1970). In our study on lm(0)-algebras, we obtain some properties of the intersection $S(L)$ of all $S_L(a)$ which might have some interest by themselves.

In Sec. 2 we obtain several properties of the subalgebra $S(L)$ which will be used in the sequel. We prove that if L' is nilpotent then $L/C_L(S(L)_L)$ is supersolvable and every chief factor of L below $S(L)_L$ is one-dimensional. If $\sqrt{F} \not\leq F$ and $\operatorname{char}(F) = 0$, then $S(L)$ is supersolvable. Also, we prove that if $\operatorname{char}(F) = 0$ and if T is a Levi subalgebra of a Lie algebra L, then $S(L) \trianglelefteq L$ and $S(L) + T$ decomposes into a direct sum of ideals A and B such that $S(A) = A$ and $S(B) = 0$.

In Sec. 3 we assemble some general results on lm(0)-algebras. We prove that every homomorphic image of $S(L)$ is lm(0). Over an algebraically closed field *every* Lie algebra is lm(0), whereas over any algebraically non-closed field there are simple Lie algebras that are not lm(0). We prove that either $S(L) = L$, $L/S(L)_L$ is semi-simple or else $L/S(L)_L$ is not lm(0). Also, in this section we introduce some other classes of Lie algebras which might have some interest by themselves.

Section 4 is concerned with solvable lm(0)-algebras over arbitrary fields. It is shown that every strongly solvable lm(0)-algebra with trivial Frattini ideal is supersolvable, and that every strongly solvable, non-supersolvable, Lie algebra is an extension of a Lie algebra that is not lm(0) by an lm(0)-algebra.

In the next two sections many of the results require the underlying field to have characteristic zero. Non-solvable lm(0)-algebras are considered in Sec. 5. A major result classifies such algebras having an abelian radical. In Sec. 6 we determine the Lie algebras all of whose proper homomorphic images are lm(0).

Section 7 concerns lm(0)-algebras over a field F of characteristic $p > 0$. First, we prove that the derived subalgebra of a centerless ad-semisimple Lie algebra has no non-singular derivations, provided that F is perfect and $p > 3$. Then, we obtain that every ad-semisimple Lie algebra over such a field F is lm(0). Finally we investigate when the simple Lie algebras having a maximal subalgebra of codimension one are lm(0). In particular we consider the Zassenhaus algebras.

Throughout L will denote a finite-dimensional Lie algebra over a field F. An element A of a lattice $\mathscr{L}$ is said to be an atom (resp. co-atom) if it is minimal (resp. maximal) in $\mathscr{L}$. Let A, B be elements of a lattice $\mathscr{L}$. We say that B covers A if $A < B$ and A is maximal in B. If L is a Lie algebra, we denote by $\mathscr{L}(L)$ the lattice of all subalgebras of L. A Lie algebra L is said to be strongly solvable if its derived subalgebra, L', is nilpotent. We shall denote the nilradical of L by $\mathrm{Nil}(L)$. If U is a subalgebra of L, we denote by U_L the largest ideal of L contained in U and by $C_L(U)$ the centralizer of U in L. We shall denote the center of L by $Z(L)$.

2. THE SUBALGEBRA $S(L)$

Following Barnes and Newell (1970), for each element $a \in L$ we denote by $S_L(a)$ the largest subalgebra of L containing a on which $\mathrm{ad}\, a$ is split. We denote by $S(L)$ the intersection of all $S_L(a)$. In this section we obtain several properties of the subalgebra $S(L)$ which will be used in the sequel. Note that $S(L) = L$ means that $\mathrm{ad}\, x$ is split on F for every $x \in L$. In this case, we will say that the Lie algebra L is *completely split*; while if $S(L) = 0$, we will say that L is *completely non-split*. We start with the following lemma which is easily checked.

Lemma 2.1. *Let L be any Lie algebra. Let $U \leq S(L)$ and $N \trianglelefteq L$ such that $N \leq S(L)$. Then $S(U) = U$ and $S(L/N) = S(L)/N$.*

We say that an ideal I of a Lie algebra L is *supersolvably immersed* in L if every chief factor of L below I is one dimensional. Clearly, every one dimensional ideal of

L is contained in $S(L)$. Now we obtain the following result which is an extension of Lemma 2.4 of Barnes and Newell (1970).

Proposition 2.2. *Let F be an arbitrary field.*

(1) *Let L' be nilpotent. Then the following hold*:

  (a) *Every minimal ideal of L contained in $S(L)$ is one-dimensional.*
  (b) *$S(L)_L$ is the largest ideal of L which is supersolvably immersed in L, and $L/C_L(S(L)_L)$ is supersolvable.*

(2) (*Lemma* 2.4 *of Barnes and Newell*, 1970). *If $S(L)'$ is nilpotent, then $S(L)$ is supersolvable.*

Proof. (1) Let A be a minimal ideal of L contained in $S(L)$. As L' is nilpotent, $A \leq Z(\mathrm{Nil}(L))$. Then we can define a representation $\rho : L/\mathrm{Nil}(L) \longrightarrow A$ by means of $\rho(x + \mathrm{Nil}(L))(a) = [x, a]$ for every $x \in L$. Since $L' \leq \mathrm{Nil}(L)$, we have that $\rho(L/\mathrm{Nil}(L))$ is a commuting family of split linear mappings. Hence these linear maps have a common eigenvector. Minimality of A implies that $\dim A = 1$. To prove (b), let H/K be a chief factor of L below $S(L)_L$. By using Lemma 2.1 and (a) we obtain that $\dim H/K = 1$. The last assertion in (b) follows from Varea (1989).

(2) is a direct consequence of (1) and Lemma 2.1.

Lemma 2.3. *Let* $\mathrm{char}(F) = 0$. *Then, $S(L)$ is a characteristic ideal of L.*

Proof. Note that $S(L)$ is invariant under every automorphism of L. So, the result follows from Theorem 3.1 of Towers (1973) and Chevalley (1968).

Let P be a simple Lie algebra of characteristic zero. As $S(P)$ is an ideal of P, we have that either $S(P) = 0$ or $S(P) = P$. When $\sqrt{F} \not\leq F$, we see that $S(P) = 0$ (since P contains a subalgebra isomorphic to sl(2) which is not completely split). Now, let T be a semisimple Lie algebra. As $S(T)$ is an ideal of T, there exists an ideal $K(T)$ of T such that $T = S(T) \oplus K(T)$. We see that $K(T)$ is the sum of the minimal ideals of T which are completely non-split and $S(T)$ is the sum of those which are completely split. When $\sqrt{F} \not\leq F$, $S(T) = 0$.

Theorem 2.4. *Let* $\mathrm{char}(F) = 0$. *Let T be any Levi subalgebra of a Lie algebra L. Let $T = S(T) \oplus K(T)$ be the decomposition of T into its completely split and completely non-split components. Then the following hold*:

(i) $[S(L), K(T)] = 0$;
(ii) $S(S(L) + S(T)) = S(L) + S(T)$: *that is $S(L) + S(T)$ is completely split*;
(iii) *$S(L) + T$ is a direct sum of a completely split Lie algebra and a completely non-split semisimple Lie algebra*; *and*
(iv) *If $\sqrt{F} \not\leq F$, then $S(L)$ is supersolvable.*

Proof. (i) We may suppose without loss of generality that $K(T)$ is simple. For short, put $K = K(T)$. As $S(K) = 0$, there must exist an element $x \in K$ such that $\mathrm{ad}_K(x)$ is not split on F. Let $x = s + n$ be the decomposition of x into its semisimple and nilpotent components, $s, n \in K$, respectively. We see that $\mathrm{ad}_K(s)$ is not split on F either. It is well-known that there exists a Cartan subalgebra H of K containing s. As $S(L)$ is an ideal of L (see Lemma 2.3), we have that $S(L)$ is a K-module. This yields that $\mathrm{ad}(s)|_{S(L)}$ is semisimple too (see Jacobson, 1979). As $\mathrm{ad}(s)|_{S(L)}$ splits on F, we get that $\mathrm{ad}(s)|_{S(L)}$ is diagonalizable on F. On the other hand, let Ω be an algebraic closure of F and consider the Lie algebra $L_\Omega = L \otimes_F \Omega$ over Ω. We see that H_Ω is a Cartan subalgebra of K_Ω and that K_Ω is semisimple. Let

$$K_\Omega = H_\Omega \oplus \Sigma (K_\Omega)_\alpha$$

be the decomposition of K_Ω into its root spaces relative to H_Ω. As $\mathrm{ad}_K s$ is not split on F, it follows that $\alpha(s) \notin F$ for some root α. Let α be such a root. Put $(K_\Omega)_\alpha = \Omega e_\alpha$. Let $a \in S(L)$ be an eigenvector of $\mathrm{ad}(s)|_{S(L)}$ and let $t \in F$ be its corresponding eigenvalue. Then we see that $[a, e_\alpha] = 0$. Otherwise $t + \alpha(s)$ would be an eigenvalue of $\mathrm{ad}(s)|_{S(L)}$ and then $t + \alpha(s) \in F$, which is a contradiction. This yields that $K_\Omega \cap C_{L_\Omega}(S(L))_\Omega \neq 0$ and hence $K \cap C_L(S(L)) \neq 0$. As K is simple, it follows that $K \leq C_L(S(L))$, as required.

(ii) Clearly, $S(L) \cap T \trianglelefteq S(T)$. Since $S(T)$ is semisimple, there exists an ideal N of $S(T)$ such that $S(T) = (S(L) \cap T) \oplus N$. As $N \leq S(T)$, we see that N is completely split. Write $U = S(L) + S(T)$. We have $U = S(L) + N$ and $S(L) \cap N = 0$. Let $0 \neq x \in U$. We want to prove that $\mathrm{ad}_u(x)$ is split. Decompose $x = a + b$ where $a \in S(L)$ and $b \in N$. Let Ω be an algebraic closure of F and let $U_\Omega = U \otimes_F \Omega$. Let $\alpha \in \Omega$ be an eigenvalue of $\mathrm{ad}_{U_\Omega}(x)$. We need to prove that $\alpha \in F$. We have that there exists $0 \neq y \in U_\Omega$ such that $[y, x] = \alpha y$. Decompose $y = a' + b'$ where $a' \in S(L)_\Omega$ and $b' \in N_\Omega$. We have

$$[y, x] = [a', a] + [a', b] + [b', a] + [b', b] = \alpha(a' + b').$$

As $[a', a] + [a', b] + [b', a] \in S(L)_\Omega$ and $[b', b] \in N_\Omega$ and since $S(L)_\Omega \cap N_\Omega = 0$, it follows that $[b', b] = \alpha b'$ and $[a', a] + [a', b] + [b', a] = \alpha a'$. If $b' \neq 0$, we see that α is an eigenvalue of $\mathrm{ad}_N(b)$. So, $\alpha \in F$ since $S(N) = N$. Now assume $b' = 0$. Then we have $a' \neq 0$ and $[a', a + b] = \alpha a'$. This yields that α is an eigenvalue of $\mathrm{ad}|_{S(L)}(a + b)$ and hence $\alpha \in F$, since $S(L) \leq S_L(a + b)$. We deduce that $\mathrm{ad}_u x$ is split on F, for every $x \in U$, so that $S(U) = U$, as required.

(iii) Since $S(L) \cap T \leq S(T)$ and $[S(L), K(T)] = 0$, we have that $S(L) + T = (S(L) + S(T)) \oplus K(T)$. So, (iii) follows from (ii).

(iv) From $\sqrt{F} \not\leq F$, it follows that $S(T) = 0$. Since $S(L) \cap T \leq S(T)$ and $S(L) \trianglelefteq L$, it follows that $S(L)$ is solvable. So, $S(L)'$ is nilpotent. By Proposition 2.2(2), we have that $S(L)$ is supersolvable. The proof is complete.

Corollary 2.5. *Let* $\mathrm{char}(F) = 0$. *Assume that* $R(L) \leq S(L)$. *Then* L *is a direct sum of a completely split Lie algebra (supersolvable in the case where* $\sqrt{F} \not\leq F$*) and a completely non-split semisimple Lie algebra.*

Note that $R(L) \leq S(L)$ whenever $R(L') \leq S(L)$.

3. GENERAL RESULTS ON lm(0)-ALGEBRAS

First we give the following result:

Proposition 3.1. *Let F be any field. For a Lie algebra L the following are equivalent*:

(i) *Every minimal subalgebra of L is lower modular.*
(ii) *If a subalgebra U of L has a maximal subalgebra of dimension one, then every maximal subalgebra of U has dimension one.*

Proof. (i) $\Longrightarrow$ (ii) Let $x \in U \leq L$ such that Fx is maximal in U. Let M be a maximal subalgebra of U distinct from Fx. We see that $Fx \vee M = U$. As Fx is lm in L, it follows that $M \cap Fx$ is maximal in M. Since $M \cap Fx = 0$, $\dim M = 1$.

(ii) $\Longrightarrow$ (i) Let $0 \neq x \in L$. Assume that Fx is maximal in $Fx \vee B$ for some subalgebra B of L. If $Fx \leq B$, then $Fx \vee B = B$. So, $Fx \cap B$ is maximal in B. Then suppose $Fx \not\leq B$. We have that B is a proper subalgebra of $Fx \vee B$. By (ii), $\dim B = 1$. This yields that $Fx \cap B$ is maximal in B and hence Fx is lm in L.

A Lie algebra satisfying the two equivalent conditions in Proposition 3.1 is called lm(0)-*algebra*. A lattice $\mathscr{L}$ is called lm(0) if every atom is lower modular. As a first consequence we obtain the following characterization of lattices of subalgebras of lm(0)-algebras.

Corollary 3.2. *Let* $\mathrm{char}(F) \neq 2, 3$. *Let $\mathscr{L}$ be the lattice of subalgebras of a Lie algebra. Then $\mathscr{L}$ is* lm(0) *if and only if the interval* $[0:B]$ *of $\mathscr{L}$ is a modular lattice for every element B of $\mathscr{L}$ covering an atom.*

Proof. Let L be a Lie algebra over F such that $\mathscr{L} \cong \mathscr{L}(L)$. Let us first suppose $\mathscr{L}$ is lm(0). Let B be an element of $\mathscr{L}$ covering an atom A of $\mathscr{L}$. Let U denote the subalgebra of L corresponding to B. Then U has a one-dimensional maximal subalgebra. By Proposition 3.1 it follows that every proper subalgebra of U has dimension one. So, the subalgebra lattice $\mathscr{L}(U)$ of U is modular. As the interval $[0:B]$ of the lattice $\mathscr{L}$ is isomorphic to the lattice $\mathscr{L}(U)$, it follows that $[0:B]$ is a modular lattice. In order to prove the converse, let U be a subalgebra of L having a maximal subalgebra A of dimension one. We have that U covers the atom A in the lattice of subalgebras of L. Then, by hypothesis, the lattice of subalgebras of U is modular. By Corollary 5 of Varea (1995), it follows that every proper subalgebra of U has dimension one. By using Proposition 3.1, we obtain that $\mathscr{L}$ is lm(0). The proof is now complete.

An easy consequence of Proposition 3.1 is the following.

Corollary 3.3.

(i) *Every supersolvable Lie algebra is* lm(0).
(ii) *For every Lie algebra L, each homomorphic image of $S(L)$ is* lm(0).
(iii) *Over algebraically closed fields, EVERY Lie algebra is* lm(0).

Proof. (i) follows from the well-known result that every maximal subalgebra of a supersolvable Lie algebra has codimension one.

(ii) By Lemma 2.1 it suffices to prove that $S(L)$ is lm(0) for every Lie algebra L. To do that, let U be a subalgebra of $S(L)$ having a maximal subalgebra M of dimension one. Pick $0 \neq x \in M$ and consider the action of x on the vector space U/M. Since $\mathrm{ad}_{S(L)}x$ is split, there exist $u \in U$, $u \notin M$, and $\alpha \in F$ such that $[x, u] \equiv \alpha u (\mathrm{mod}\, M)$. It follows that $M + Fu$ is a subalgebra of U. By the maximality of M, we have $M + Fu = U$. So, $\dim U = 2$. Therefore $S(L)$ is lm(0).

(iii) follows from (ii) and the fact that $S(L) = L$ for every Lie algebra L over an algebraically closed field.

For algebraically non-closed fields, the situation is quite different. Here we will prove that, for any such fields, there are simple Lie algebras which are not lm(0). In the next section, we will prove that every strongly solvable Lie algebra can be obtained as an extension of a Lie algebra which is not lm(0) by an lm(0)-algebra. Also, we note that the three-dimensional split simple Lie algebra is lm(0) whenever $\sqrt{F} \leq F$ or $\mathrm{char}(F) = 2$, but it is not lm(0) in the case where $\sqrt{F} \nleq F$ and $\mathrm{char}(F) \neq 2$.

Proposition 3.4. *Let L be a simple, but not central-simple, Lie algebra having an element x such that* $\mathrm{ad}\, x$ *has a nonzero eigenvalue in F. Then L is not* lm(0).

Proof. By our hypothesis, there exists an element $x \in L$ such that $\mathrm{ad}\, x$ has a nonzero eigenvalue t in F. So, there exists $e \in L$ such that $[e, x] = te$. Put $x' = t^{-1}x$. Then, we have $[e, x'] = e$. Let Γ be the centroid of L. As L is not central-simple, $\Gamma \neq F$. Then, we can take $\gamma \in \Gamma$, $\gamma \notin F$. Let n be the degree of the minimum polynomial of γ over F. So $n > 1$. Consider the vector subspace A of L spanned by $e, \gamma(e), \ldots, \gamma^{n-1}(e)$. We see that $e, \gamma(e), \ldots, \gamma^{n-1}(e)$ is a basis for A and that A is an abelian subalgebra of L. Also, we see $[e, \gamma(x')] = \gamma([e, x']) = \gamma(e)$ and $[\gamma^i(e), \gamma(x')] = \gamma([\gamma^i(e), x']) = \gamma(\gamma([\gamma^{i-1}(e), x'])) = \gamma^{i+1}([e, x']) = \gamma^{i+1}(e)$, for every $1 \leq i \leq r$. As γ^n can be decomposed into a linear combination of $1, \gamma, \ldots, \gamma^{n-1}$ with coefficients in F, it follows that $[A, \gamma(x')] \subseteq A$. We see that the corresponding matrix to the transformation $\mathrm{ad}(\gamma(x'))|_A$ is the companion matrix to the minimum polynomial of γ over F. So $\mathrm{ad}(\gamma(x'))|_A$ has no eigenvalues in F. This yields that L is not lm(0).

Corollary 3.5. *For every algebraically non-closed field F, there exist simple Lie algebras which are not* lm(0).

Proof. Pick an element ω in an algebraic closure Ω of F such that $\omega \notin F$. By Proposition 3.4, the Lie algebra over F obtained from the three-dimensional split simple Lie algebra over $F(\omega)$ by restricting the field of scalars, is not lm(0).

Next we study the class of lm(0)-algebras and relations between it and certain other classes of Lie algebras. These classes might have some interest by themselves.

If $\mathscr{X}$ is a class of Lie algebras, we will denote by $s\mathscr{X}$ the class of all subalgebras of $\mathscr{X}$-algebras.

The first class we introduce is defined in terms of the lattice theory: let $\mathscr{Y}$ denote the class of Lie algebras L such that if an atom of $\mathscr{L}(L)$ is a co-atom so is every atom. The class $\mathscr{Y}$ is a very large class. Indeed

Lemma 3.6. (i) *For any field, the only Lie algebras which are not in $\mathscr{Y}$ are those Lie algebras L such that $L = L' + Fx$ with x acting irreducibly on L' and $\dim L' > 1$, and the simple Lie algebras of rank one having a one-dimensional maximal subalgebra and subalgebras of dimension greater than one.*

(ii) *If $\mathrm{char}(F) = 0$, then a Lie algebra L is not in $\mathscr{Y}$ if and only if either $L = A + Fx$ where A is a proper minimal abelian ideal of L and $\dim A > 1$, or $L \cong \mathrm{sl}(2)$ and $\sqrt{F} \not\leq F$.*

Proof. This is straightforward.

Corollary 3.7. (i) $\mathrm{lm}(0) = s\mathscr{Y}$.

(ii) *If $\mathrm{char}(F) = 0$, then L is minimal non-$\mathrm{lm}(0)$ (this means that every proper subalgebra of L is $\mathrm{lm}(0)$ but L is not) if and only if $L \notin \mathscr{Y}$.*

Next, we introduce the class $\mathscr{P}_1$ of Lie algebras L such that every minimal ideal of L is one dimensional or $L = 0$. This class of Lie algebras is contained in the class $\mathscr{P}_2$ of Lie algebras L in which every minimal ideal lies in $S(L)$. Let $\mathscr{P}_3$ be the class of Lie algebras L such that every abelian ideal of L is contained in $S(L)$. So that $L \in \mathscr{P}_3$ if and only if the transformation $\mathrm{ad}(x)|_A$ is split for every abelian ideal A of L and every $x \in L$. Let $\mathscr{P}_4$ be the class of Lie algebras L such that either $S(L) \neq 0$ or $L = 0$.

Some relationships between these classes are given in the following result.

Theorem 3.8. (i) *For any field, $\mathscr{P}_1 \subseteq \mathscr{P}_2 \subseteq \mathscr{P}_4 \cap \mathscr{Y}$, and $\mathrm{lm}(0) \subseteq s\mathscr{P}_3$.*

(ii) *If $\mathrm{char}(F) = 0$, then $s\mathscr{P}_4 \subseteq \mathrm{lm}(0)$ and every Lie algebra in $s\mathscr{P}_1$ is solvable.*

Proof. (i) Clearly, $\mathscr{P}_1 \subseteq \mathscr{P}_2$ and $\mathscr{P}_2 \subseteq \mathscr{P}_4$. Now let $L \in \mathscr{P}_2$. To prove that $L \in \mathscr{Y}$, assume that L has a maximal subalgebra M of dimension one. Put $M = Fx$. Take a minimal ideal N of L. We have $N \leq S(L)$ and so $\mathrm{ad}(x)|_N$ is split. Thus there exists $0 \neq y \in N$ such that $[x, y] = ty$ for some $t \in F$. This yields $\dim L = 2$ and hence $L \in \mathscr{Y}$. Now let L be $\mathrm{lm}(0)$. To prove that $L \in s\mathscr{P}_3$, it suffices to show that $L \in \mathscr{P}_3$. Let A be an abelian ideal of L. Suppose $A \not\leq S(L)$. Then there exists $x \in L$, $x \notin A$ such that $A \not\leq S_L(x)$. Let $K_L(x)$ be the $\mathrm{ad}(x)$-invariant subspace of L such that $L = S_L(x) + K_L(x)$ and $S_L(x) \cap K_L(x) = 0$ (see Barnes and Newell, 1970). We see that $K_L(x) \cap A \neq 0$ and $(K_L(x) \cap A) + Fx$ is a subalgebra of L. Take a subalgebra M of $(K_L(x) \cap A) + Fx$ containing Fx and such that Fx is maximal in M. We have that $(M \cap K_L(x)) \cap A$ is an ideal of M and a maximal subalgebra of M. Since L is $\mathrm{lm}(0)$, it follows that $\dim(M \cap K_L(x) \cap A) = 1$. This yields that

$\mathrm{ad}(x)|_{K_L(x)\cap A}$ has an eigenvalue in F, which is a contradiction. The proof of (i) is now complete.

(ii) As $\mathrm{char}(F)=0$, every nonsolvable Lie algebra has a semisimple subalgebra. Since, clearly, a Lie algebra in $s\mathscr{P}_1$ contains no semisimple subalgebras, it follows that every Lie algebra in $s\mathscr{P}_1$ is solvable. It remains only to show that $s\mathscr{P}_4 \subseteq \mathrm{lm}(0)$. Let $L \in s\mathscr{P}_4$. We need only prove that $L \in \mathscr{Y}$. By Lemma 2.3 we have $S(L) \trianglelefteq L$. This yields that for each $0 \neq x \in L$ there exists a nonzero element $y \in S(L)$ such that $[x, y] = ty$, where $t \in F$. So, either $\dim L \leq 2$ or L has no maximal subalgebras of dimension one. From this it follows that $L \in \mathscr{Y}$.

Later in this paper, we show examples of Lie algebras L which are lm(0) and such that $S(L) = 0$ (so that, in general, lm(0) is not contained in $\mathscr{P}_4$).

Corollary 3.9. *Let L be any Lie algebra. Then either L is completely split,* $L/S(L)_L$ *is semisimple or else* $L/S(L)_L$ *is not* lm(0).

Next, we give some properties of the classes above introduced. Let L be a Lie algebra which is isomorphic to the direct sum of the Lie algebras L_1 and L_2. A subalgebra U of L is said to be a *sub-direct summand* of L if the canonical projections $\pi_1 : U \longrightarrow L_1$ and $\pi_2 : U \longrightarrow L_2$ are both surjective. A class $\mathscr{X}$ of Lie algebras is called R_0-closed if every sub-direct summand of $L_1 \oplus L_2$ is in $\mathscr{X}$ whenever L_1 and L_2 both lie in $\mathscr{X}$ (or equivalently if, whenever $L/A \in \mathscr{X}$ and $L/B \in \mathscr{X}$, where A and B are ideals of the Lie algebra L, it always follows that $L/A \cap B \in \mathscr{X}$).

Lemma 3.10. *Let* $\mathscr{X}$ *be a class of Lie algebras which is* R_0*-closed. Then the class* $s\mathscr{X}$ *is* R_0*-closed too.*

Proof. Let $L_1,\ L_2 \in s\mathscr{X}$. Write $L = L_1 \oplus L_2$. Let $U \leq L$. We see that U is a sub-direct summand of $\pi_1(U) \oplus \pi_2(U)$. Since $\pi_i(U) \leq L_i \in s\mathscr{X}$, for $i = 1, 2$, it follows that $\pi_i(U) \in \mathscr{X}$. Then, by our hypothesis, $U \in \mathscr{X}$ too.

Proposition 3.11. *The classes* $\mathscr{P}_i$ *for* $1 \leq i \leq 4$ *and the class* $\mathscr{Y}$ *are all* R_0*-closed and hence so are the classes* $s\mathscr{P}_i$ *for* $1 \leq i \leq 4$ *and the class of* lm(0)*-algebras.*

Proof. This is straightforward.

We will denote by $\mathrm{Asoc}(L)$ the sum of all abelian minimal ideals of the Lie algebra L and call it the *abelian socle* of L. The *Frattini subalgebra*, $\mathrm{Fr}(L)$, of a Lie algebra L is defined to be the intersection of all maximal subalgebras of L. It is well-known that $\mathrm{Fr}(L)$ is an ideal of L whenever either L is solvable or else $\mathrm{char}(F) = 0$, (see Towers, 1973). However, for any algebraically closed field of characteristic greater than 7, there exist simple Lie algebras having non-trivial Frattini subalgebra (see Varea, 1993). We will denote by $\phi(L)$ the largest ideal of L contained in $\mathrm{Fr}(L)$. A Lie algebra L is said to be *ϕ-free* if $\phi(L) = 0$.

We finish this section giving, for a ϕ-free and $\mathscr{P}_1$-algebra L, a relationship between the dimensions of L, the center of L and the abelian socle of L.

Proposition 3.12. *Let F be any field. Let $L \in \mathscr{P}_1$ be ϕ-free. Then*

$$\dim L + \dim Z(L) \leq 2(\dim \mathrm{Asoc}(L)).$$

Proof. If L is abelian, there is nothing to prove. Then assume L is non-abelian. Since $\phi(L) = 0$, by Theorem 7.3 of Towers (1973) there exists $B \leq L$ such that $L = \mathrm{Asoc}(L) + B$ and $B \cap \mathrm{Asoc}(L) = 0$. We see that B contains no nonzero ideals of L; since otherwise, B would contain a minimal ideal of L which is of dimension one (because $L \in \mathscr{P}_1$), a contradiction. On the other hand, we have $\mathrm{Asoc}(L) = Z(L) \oplus A_1 \oplus \cdots \oplus A_r$ where each A_i is an abelian minimal ideal of L and $r \geq 0$. We have $r > 0$, since otherwise we would have $B \triangleleft L$, which is a contradiction. Also, we have $\dim A_i = 1$ for every i. Write $A_i = Fa_i$. Define $\rho_i : L \longrightarrow Fa_i$ by means of $\rho_i(x) = [a_i, x]$ for every $x \in L$. Since $a_i \notin Z(L)$, we see $\dim L/C_L(a_i) = 1$ for every i. Write $C = C_L(a_1) \cap \cdots \cap C_L(a_r)$. We see that $[C \cap B, L] = [C \cap B, \mathrm{Asoc}(L) + B] \leq [C \cap B, B] \leq C \cap B$. This yields, $C \cap B \triangleleft L$ and hence $C \cap B = 0$. So, $C = \mathrm{Asoc}(L)$, giving $\dim(L/\mathrm{Asoc}(L)) \leq r$. We have $\dim L \leq r + \dim \mathrm{Asoc}(L) = 2r + \dim Z(L)$. Therefore, $\dim L + \dim Z(L) \leq 2(\dim \mathrm{Asoc}(L))$.

4. ON SOLVABLE lm(0)-ALGEBRAS OVER ARBITRARY FIELDS

A Lie algebra L is said to be *strongly solvable* if its derived subalgebra L' is nilpotent. It is well-known that for fields of characteristic zero, every solvable Lie algebra is strongly solvable (see Jacobson, 1979). For arbitrary fields, every super solvable Lie algebra is strongly solvable. For algebraically closed fields, every strongly solvable Lie algebra is supersolvable (Proposition 2.2(2)).

Theorem 4.1.

(i) *For solvable Lie algebras,* $s\mathscr{P}_1 \subseteq s\mathscr{P}_2 = \mathrm{lm}(0) = s\mathscr{P}_3 = s\mathscr{P}_4$.
(ii) *For strongly solvable Lie algebras,* $s\mathscr{P}_1 = \mathrm{lm}(0)$.

Proof. (i) From Theorem 3.8 and Corollary 3.7 it follows that $s\mathscr{P}_1 \subseteq s\mathscr{P}_2 \subseteq \mathrm{lm}(0) \subseteq s\mathscr{P}_3$. For solvable Lie algebras it is trivial that $\mathscr{P}_3 \subseteq \mathscr{P}_2 \cap \mathscr{P}_4$. Let $0 \neq L \in s\mathscr{P}_4$ and let L be solvable. We need only to prove that L is lm(0). Assume that L is not lm(0). We may suppose, without loss of generality, that every proper subalgebra of L is lm(0). By Corollary 3.7, we have $L \notin \mathscr{Y}$. By Lemma 3.6, L has a unique abelian minimal ideal A of dimension greater than one and codimension one in L. Let $x \in L$, $x \notin A$. We see that Fx is maximal in L and $\mathrm{ad}(x)|_A$ is not split. This yields that $S_L(x) = Fx$ and therefore $S(L) = 0$. This contradicts the fact that $L \in \mathscr{P}_4$. The proof of (i) is complete.

(ii) Let L be an lm(0)-algebra which is strongly solvable. We need only to prove that $L \in \mathscr{P}_1$. To do that, let A be a minimal ideal of L. By Theorem 3.8, it follows that $A \leq S(L)$. Then by Proposition 2.2, $\dim A = 1$. This completes the proof.

Proposition 4.2. *If L is strongly solvable and $\phi(L) = 0$, then either L is supersolvable or L is not* lm(0).

Proof. Let L be strongly solvable and let $\phi(L) = 0$. Assume that L is lm(0). Then, by Theorem 4.1, we have that every minimal ideal of L is one dimensional. This yields that every maximal subalgebra of L which does not contain Asoc(L) has codimension one in L. On the other hand, since $\phi(L) = 0$ we have Nil(L) = Asoc(L) (see Theorem 7.4 of Towers, 1973). It follows that $L/\text{Asoc}(L)$ is abelian, since L' is nilpotent. This yields that every maximal subalgebra of L has codimension one in L. Hence, by using Theorem 7 of Barnes (1967), we conclude that L is supersolvable.

Next, we prove that every strongly solvable, non-supersolvable Lie algebra has homomorphic images which are NOT lm(0)-algebras.

Corollary 4.3. *Let F be any field. Let L be strongly solvable but not supersolvable. Then, none of the Lie algebras $L/S(L)_L$, $L/\phi(L)$, $L/(S(L)_L \cap \phi(L))$ and $L/(S(L)_L + \phi(L))$ is* lm(0).

Proof. By Proposition 2.2(2), we have that $S(L) \neq L$. Thus $L/S(L)_L$ is not lm(0) by Corollary 3.9. By Theorem 6 of Barnes (1967), we have that $L/\phi(L)$ is not supersolvable. So, $L/\phi(L)$ is not lm(0) by Proposition 4.2. To prove that $L/(S(L)_L \cap \phi(L))$ is not lm(0), we may suppose without loss of generality that $S(L)_L \cap \phi(L) = 0$ and $\phi(L) \neq 0$. Then, we can take an abelian minimal ideal A of L contained in $\phi(L)$. Since $A \not\leq S(L)$, by Theorem 3.8, it follows that L is not lm(0). What remains to prove is that the Lie algebra $L/(S(L)_L + \phi(L))$ is not lm(0). By Proposition 2.2, we have that $(S(L)_L + \phi(L))/\phi(L)$ is a supersolvably immersed ideal of $L/\phi(L)$. This yields that $L/(S(L)_L + \phi(L))$ is not supersolvable, since otherwise we would have that $L/\phi(L)$ is supersolvable and then so is L, which is a contradiction. On the other hand, we see that $\phi(L/S(L)_L) = S(L)_L + \phi(L)$. So, the algebra $L/(S(L)_L + \phi(L))$ is ϕ-free. Then, the result follows from Proposition 4.2. The proof is now complete.

Corollary 4.4. *Every strongly solvable, non-supersolvable Lie algebra is an extension of a Lie algebra which is not* lm(0) *by an* lm(0)*-algebra.*

Proof. Let L be strongly solvable but not supersolvable. By Corollary 3.3(ii), we have that $S(L)_L$ is lm(0). By Corollary 4.3, we have that $L/S(L)_L$ is not lm(0).

Notice that Corollary 4.4 fails for solvable Lie algebras over fields of prime characteristic. Indeed, we will see that there are solvable non-supersolvable Lie algebras L such that $S(L) = L$. So, by Corollary 3.3(ii), every homomorphic image of such an L is lm(0).

Next we want to consider solvable lm(0)-algebras L which are not supersolvable. In the case where L is strongly solvable, it follows from Proposition 4.2 that such an L must have non-trivial Frattini subalgebra. We will determine the lm(0)-algebras in the class of solvable, non-supersolvable Lie algebras all of whose proper subalgebras are supersolvable (called minimal non-supersolvable for short). The algebras L in this class are determined in Elduque and Varea (1986). In the case where L' is not nilpotent, there it is proved that $\operatorname{ad} x$ is split for every $x \in L$, so that $S(L) = L$. Then, by Corollary 3.3(ii), L/N is lm(0) for every ideal N of L. In the case where L' is nilpotent, we see that $S(L) = \phi(L)$. It follows that $L/\phi(L)$ is not lm(0), by Corollary 4.3. However, we see that L is lm(0) whenever $\phi(L) \neq 0$. Therefore we have the following:

Proposition 4.5. *Let F be an arbitrary field. For a solvable, minimal non-supersolvable Lie algebra L, the following are equivalent*:

(i) *L is NOT* lm(0).
(ii) *L is strongly solvable and $\phi(L) = 0$.*
(iii) *L has a basis $e_1, \ldots, e_r$, x with non-trivial product given by $[e_i, x] = e_{i+1}$, for every $1 \le i < r$ and $[e_r, x] = c_0e_1 + \cdots + c_{r-1}e_r$, where the polynomial $\lambda_r - c_{r-1}\lambda_{r-1} - \cdots - c_0$ is irreducible in $F[\lambda]$ and $r > 1$.*

Note that the minimal non-supersolvable Lie algebras with non-nilpotent derived subalgebra and with trivial Frattini subalgebra are examples of solvable lm(0)-algebras having no minimal ideals of dimension one.

5. NON-SOLVABLE lm(0)-ALGEBRAS OF CHARACTERISTIC ZERO

In this section F is assumed to be of characteristic zero. First we see that the problem of the classification of lm(0)-algebras is reduced in some sense to the classification of solvable lm(0)-algebras.

Theorem 5.1. *Let L be a nonsolvable Lie algebra. Then the following hold*:

(i) *If $\sqrt{F} \le F$, then L is* lm(0) *if and only if every solvable subalgebra of L is* lm(0).
(ii) *If $\sqrt{F} \not\le F$, then L is* lm(0) *if and only if every solvable subalgebra of L is* lm(0) *and L has no subalgebras isomorphic to* sl(2).

Proof. We prove (i) and (ii) together. As the class of lm(0)-algebras is closed by subalgebras, it suffices to prove the "only if" part. Let L be a counterexample of minimal dimension. Then we see that L is minimal non-lm(0). By Corollary 3.7(ii), it follows that $L \notin \mathcal{Y}$. By Lemma 3.6(ii), we have that either L is solvable or $L \cong \mathrm{sl}(2)$ and $\sqrt{F} \not\le F$, which is a contradiction.

A Lie algebra L is said to be *anisotropic* if it has no nonzero ad-nilpotent elements. A Lie algebra L is called *ad-semisimple* if $\mathrm{ad}\, x$ is semisimple for every $x \in L$. Note that if L is ad-semisimple, then $L/Z(L)$ is semisimple. It is known that for perfect fields a Lie algebra L is anisotropic if and only if it is ad-semisimple. It is easy to see and well-known that if L is ad-semisimple, then zero is the only eigenvalue of $\mathrm{ad}\, x$ for every $x \in L$ and that every solvable subalgebra of L is abelian (see Farnsteiner 1983). From Theorem 5.1, it follows that every ad-semisimple Lie algebra is lm(0).

Next, we study semisimple Lie algebras which are lm(0). In the case where $\sqrt{F} \not\leq F$, we have the following.

Corollary 5.2. *Let $\sqrt{F} \not\leq F$. Then, a semisimple Lie algebra L is* lm(0) *if and only if it is anisotropic.*

Proof. Assume that L is lm(0). As $\sqrt{F} \not\leq F$, the Lie algebra sl(2) is not lm(0). So, L cannot contain any subalgebra isomorphic to sl(2). This yields that L has no nonzero ad-nilpotent element; since otherwise, such an element would be immersed in a subalgebra of L isomorphic to sl(2), according to Theorem 17, p. 100 of Jacobson (1979), which is a contradiction. This gives that L is anisotropic. The converse follows from Theorem 5.1.

Corollary 5.2 covers the case where the ground field F is the real number field. We recall that a real semisimple Lie algebra is anisotropic if and only if it is compact (that is, its Killing bilinear form is negative definite). So that, the only real semisimple Lie algebras which are lm(0) are the compact ones.

Our next task is to study lm(0) algebras which are neither solvable nor semisimple. We see that lm(0)-algebras with abelian solvable radical, as well as Lie algebras all of whose solvable subalgebras are supersolvable (for short, $\mathscr{M}$-algebras), satisfy the condition assumed in Corollary 2.5. We will see that the classes lm(0) and $\mathscr{M}$ are closely related. Now we obtain the following:

Theorem 5.3. *Let L be a Lie algebra. Then the following hold*:

(i) *If $L \in \mathscr{M}$, then L is a direct sum of a completely split Lie algebra and a completely non-split semisimple Lie algebra.*

Now assume in addition that $R(L)$ is abelian. Then

(ii) *If $\sqrt{F} \not\leq F$, L is* lm(0) *if and only if L is a direct sum of an abelian Lie algebra and an anisotropic semisimple Lie algebra*; *and*

(iii) *If $\sqrt{F} \leq F$, L is* lm(0) *if and only if L is a direct sum of a completely split Lie algebra and a completely non-split, semisimple* lm(0)*-algebra.*

Proof. (i) Assume $L \in \mathscr{M}$. We claim that $R(L) \leq S(L)$. To see this, let $x \in L$. Since the subalgebra $R(L) + Fx$ is solvable, by our hypothesis it is supersolvable. This yields that $\mathrm{ad}(x)|_{R(L)}$ is split and therefore $R(L) \leq S_L(x)$. Hence, $R(L) \leq S(L)$, as claimed. Then the result follows from Corollary 2.5. Assertions (ii) and (iii) follow from Theorem 3.8 and Corollaries 5.2 and 2.5.

Corollary 5.4. *Let $\sqrt{F} \not\leq F$. Let L be a Lie algebra such that $R(L)$ is supersolvable. Then, the following hold*:

(i) *$L' \in \mathscr{M} \Longleftrightarrow L \in \mathscr{M}$.*

(ii) *if L' is* lm(0) *and if $R(L')$ is abelian, then L is* lm(0).

Proof. (i) Assume $L' \in \mathscr{M}$. Then, $R(L') \leq S(L')$ (see the proof of Theorem 5.3). Let $L = R(L) + T$ be a Levi decomposition of L. We see $L' = R(L') + T$. As $\sqrt{F} \not\leq F$, we have $S(T) = 0$. Then, by Theorem 2.4, it follows that $[R(L'), T] = 0$, whence T is an ideal of L'. This gives that T is the only Levi subalgebra of L'. Since every Levi subalgebra of L must be contained in L', we see that T is the only Levi subalgebra of L. Hence $T \triangleleft L$ and $L = R(L) \oplus T$. Now, let B be a maximal solvable subalgebra of L. We see that $B = R(L) \oplus (B \cap T)$. Since $B \cap T$ is a solvable subalgebra of L', it is supersolvable. This yields that B is supersolvable too and hence $L \in \mathscr{M}$. The converse is clear.

(ii) Assume L' is lm(0) and $R(L')$ is abelian. By Theorem 5.3, we have $L' = A \oplus B$, where A is an abelian ideal of L' and B is an anisotropic semisimple ideal of L'. We see that B is the only Levi subalgebra of L and so $L = R(L) \oplus B$. As $R(L)$ and B are both lm(0), it follows from Proposition 3.11 that L is lm(0).

Corollary 5.5. *$\mathscr{M} \subseteq \mathrm{lm}(0)$ whenever $\sqrt{F} \leq F$, while the class of semisimple* lm(0)*-algebras is properly contained in the class of semisimple $\mathscr{M}$-algebras whenever $\sqrt{F} \not\leq F$.*

Proof. This follows from Theorem 5.1, Corollary 5.2, and from the fact that sl(2) is in $\mathscr{M}$, but it is not lm(0) whenever $\sqrt{F} \not\leq F$.

6. LIE ALGEBRAS ALL OF WHOSE PROPER HOMOMORPHIC IMAGES ARE lm(0)

In this section, by using previous results in this paper, we are able to determine the Lie algebras all of whose proper homomorphic images are lm(0), which will be called $Q - \mathrm{lm}(0)$ for short. A Lie algebra all of whose proper homomorphic images are supersolvable will be called *Q-supersolvable*.

We start by considering solvable Lie algebras over arbitrary fields.

Theorem 6.1. *Let F be an arbitrary field. Let L be solvable. Then the following hold*:

(i) *Every homomorphic image of L is* lm(0) *if and only if L is completely split*; *and*
(ii) *If L is strongly solvable, then L is $Q - \mathrm{lm}(0)$ if and only if L is Q-supersolvable.*

Proof. (i) This follows from Corollary 3.9 and Corollary 3.3(ii).

(ii) Assume that L is strongly solvable and Q-lm(0). Let us first suppose that $\phi(L) \neq 0$. Then we have that $L/\phi(L)$ is lm(0) and ϕ-free. So, by Proposition 4.2, it follows that $L/\phi(L)$ is supersolvable. Then, by Theorem 6 of Barnes (1967), we have that L is supersolvable. Now assume that $\phi(L) = 0$. Then we have that L/N is lm(0) and ϕ-free for every proper ideal N of L. It follows from Proposition 4.2 again that every proper homomorphic image of L is supersolvable. Thus L is Q-supersolvable, as required. The converse is clear.

Solvable, Q-supersolvable Lie algebras were studied by Towers (1985). He proved that such a Lie algebra L must have the following form: $L = N(L) + U$,

$N(L) \cap U = 0$, where $N(L)$ is the unique minimal ideal of L, $[u, N(L)] = N(L)$ for some $u \in U$, and U is a supersolvable maximal subalgebra of L.

Next we classify the nonsolvable Lie algebras all of whose homomorphic images are lm(0). To do that we need the assumption of characteristic zero for the ground field.

Theorem 6.2. *Let* $\mathrm{char}(F) = 0$. *For a nonsolvable Lie algebra L, the following are equivalent*:

(i) *Every homomorphic image of L is* lm(0).
(ii) *L and* $L/S(L)$ *are both* lm(0); *and*
(iii) *Either* $\sqrt{F} \not\leq F$ *and* $L = U \oplus T$ *where U is a supersolvable ideal of L and T is an anisotropic semisimple ideal of L, or else* $\sqrt{F} \leq F$ *and L is isomorphic to a direct sum of a completely split Lie algebra and a completely non-split, semisimple* lm(0)-*algebra.*

Proof. (i) $\Longrightarrow$ (ii) is trivial.

(ii) $\Longrightarrow$ (iii) Let $L = R(L) + T$ be a Levi decomposition of L. As L is nonsolvable, $T \neq 0$. As $L/S(L)$ is lm(0), by using Corollary 3.9 we obtain that either L is completely split, or else $L/S(L)$ is semisimple. In the former case, we have $\sqrt{F} \leq F$ because $S(T) = T$, and we are done. So assume $L/S(L)$ is semisimple. This implies that $R(L) \leq S(L)$, since $S(L)$ is an ideal of L. Then the result follows from Corollary 2.5.

(iii) $\Longrightarrow$ (i) Let $N \lhd L$. Then we see that L/N is a direct sum of two lm(0) ideals. By Proposition 3.11, it follows that L/N is lm(0).

Corollary 6.3. *Let* $\mathrm{char}(F) = 0$. *For a non-solvable*, *non-semisimple and non*-lm(0) *Lie algebra L, the following are equivalent*:

(i) *L is* $Q -$ lm(0).
(ii) $S(L) = 0$, *L has a unique minimal ideal A that is abelian and* L/A *has the structure given in Theorem* 6.2(iii).

Proof. (i) $\Longrightarrow$ (ii) By Proposition 3.11, it follows that L has only one minimal ideal A. As $R(L) \neq 0$, we have $A \leq R(L)$ and so A is abelian. Now we prove that $S(L) = 0$. Suppose $S(L) \neq 0$. Then $L/S(L)$ is lm(0). By Corollary 3.9 it follows that $L/S(L)$ is semisimple. This yields $R(L) \leq S(L)$. Then by Corollary 2.5 it follows that $L = S(L) \oplus K$, where K is a semisimple ideal of L. As A is the unique minimal ideal of L, we have $K = 0$. This yields that $S(L) = L$ and hence L is lm(0). This contradiction shows that $S(L) = 0$. The last statement follows from Theorem 6.2, since every homomorphic image of L/A is lm(0).

(ii) $\Longrightarrow$ (i) By Theorem 6.2, we have that every homomorphic image of L/A is lm(0). As A is the unique minimal ideal of L, it follows that L is Q-lm(0). Since $S(L) = 0$ and L has abelian minimal ideals, from Theorem 3.8 it follows that L is not lm(0).

7. ON lm(0)-ALGEBRAS OVER FIELDS OF PRIME CHARACTERISTIC

First, we consider ad-semisimple Lie algebras over perfect fields of characteristic greater than three. Before that we need the following lemma:

Lemma 7.1. *Let F be perfect and* $\mathrm{char}(F) = p > 3$. *Let L be ad-semisimple without center. Then L' has no non-singular derivations.*

Proof. Since L/L'' is ad-semisimple and solvable, we have that it is abelian (see Farnsteiner, 1983). Therefore $L' = L''$. Then, by using Theorem 3 and Corollary 2 of Premet (1987), we obtain that $L' \otimes_\Omega F$ is a direct sum of simple ideals which are of classical type. In particular, we have that $L' \otimes_\Omega F$ is restricted and without center. This yields that every derivation of $L' \otimes_\Omega F$ is restricted, see Seligman (1967). By Jacobson (1955), it follows that $L' \otimes_\Omega F$ has no non-singular derivation, and so neither has L'.

Proposition 7.2. *Let F be perfect and* $\mathrm{char}(F) = p > 3$. *Then, every ad-semisimple Lie algebra is* $\mathrm{lm}(0)$.

Proof. Let L be a minimal counterexample. Then, we have that every proper subalgebra of L is lm(0). By Corollary 2.9 (i) it follows that $L \notin \mathcal{Y}$. So, either $L = L' + Fx$ with x acting irreducibly on L' and $\dim L' > 1$ or L is simple of rank one having subalgebras of dimension greater than one (Lemma 3.6(i)). In the former case we see that $Z(L) = 0$ and $\mathrm{ad}x\,|_{L'}$ is a non-singular derivation of L'. This contradicts Lemma 7.1. Therefore L is simple of rank one. By using Theorem 3 and Corollary 2 of Premet (1987), we obtain that L_Ω is a direct sum of simple ideals of classical type. This yields $L_\Omega \cong \mathrm{sl}(2)$. Therefore $\dim L = 3$. Finally, since L is ad-semisimple, it follows that every proper subalgebra of L has dimension one. This contradiction completes the proof.

Next, we consider Lie algebras L having a maximal subalgebra L_0 of codimension one which does not contain any proper ideal of L. We recall that such a Lie algebra L must be isomorphic to one of the Lie algebras $L_n(\Gamma)$ constructed by Amayo (1976). Assume $\mathrm{char}(F) = p > 2$ and let $\Gamma = \{\gamma_0, \gamma_1, \ldots, \gamma_{n-1}\} \subseteq F$. Then, the Lie algebra $L_n(\Gamma)$ can be defined by having a basis y_{-1}, y_0, $y_1, \ldots, y_{p^{n-2}}$ with product given by $[y_{-1}, y_i] = y_{i-1}$ for $0 \le i \le p^{n-2}$ except when $i = p^{j-1}$ for some j, $[y_{-1}, y_{p^{j-1}}] = y_{p^{j-2}} + \gamma_j y_{p^{n-2}}$ and $[y_i, y_j] = a_{ij} y_{i+j}$ for $0 \le i, j \le p^{n-2}$, where

$$a_{ij} = \binom{i+j+1}{j} - \binom{i+j+1}{i},$$

and where we follow the convention that the binomial coefficient $\binom{r}{s} = 0$ unless $0 \le s \le r$. We mention that $L_n(\{0\})$ is the Zassenhaus algebra $Z_n(F)$ and that $L_1(\{0\})$ is the Witt algebra over F.

Let $\mathrm{char}(F) = p > 2$. We recall that a polynomial $f \in F[X]$ is called a *p-polynomial* if the only powers of X having nonzero coefficients in f are of the form

X^{p^i} for $i \geq 0$. For $n > 0$, we say that the field F has *p- index greater than or equal to n* (written $\mathrm{ind}_p(F) \geq n$) if every p-polynomial f of degree less than or equal to p^n without multiple roots in an algebraic closure of F, has a non-zero root in F. We say that $\mathrm{ind}_p(F) = n$ if $\mathrm{ind}_p(F) \geq n$ but $\mathrm{ind}_p(F) \not\geq n+1$. Finally, $\mathrm{ind}_p(F) = \infty$ if every p-polynomial of degree greater than zero without multiple roots in an algebraic closure of F, has nonzero roots in F.

Theorem 7.3. *Let* $\mathrm{char}(F) = p > 2$. *Let n be a positive integer. Then, the following are equivalent.*

(i) $L_m(\Gamma)$ *is* $\mathrm{lm}(0)$ *for every* Γ *and* $m \leq n$.
(ii) $\mathrm{ind}_p(F) \geq n$ *and* $\sqrt{F} \leq F$.

Proof. (i) $\Longrightarrow$ (ii) Note that the span of y_{-1}, y_0 and y_1 in $Z_n(F)$ is a subalgebra of $Z_n(F)$ which is isomorphic to sl(2). This yields that sl(2) is lm(0) and hence $\sqrt{F} \leq F$. Now, let f be a monic p-polynomial over F of degree $\leq n$ without multiple roots in an algebraic closure of F. So that, f has the form

$$f(X) = X^{p^m} + \beta_{m-1}X^{p^{m-1}} + \cdots + \beta_0 X,$$

where $\beta_0 \neq 0$ and $m \leq n$. Put $\Gamma = \{-\beta_0, \ldots, -\beta_{m-1}\}$. We see that the minimum polynomial of $\mathrm{ad}y_{-1}$ is equal to $f(X)$, where y_{-1} is the first vector in a standard basis for $L_m(\Gamma)$. This algebra is lm(0), by our hypothesis. Take a subalgebra U of $L_m(\Gamma)$ containing Fy_{-1} and such that Fy_{-1} is maximal in U. We see $\dim U/(U \cap L_m(\Gamma)_0) = 1$. By Proposition 3.1 it follows that $\dim U \cap L_m(\Gamma)_0 = 1$. This yields $\dim U = 2$. On the other hand, we see that y_{-1} is self-centralizing in $L_m(\Gamma)$ and that it is not ad-nilpotent. It follows that $\mathrm{ad}(y_{-1})\,|_U$ has a nonzero eigenvalue in F. Therefore $f(X)$ has a nonzero root in F.

(ii) $\Longrightarrow$ (i) Let $m \leq n$. Suppose that $L_m(\Gamma)$ is not lm(0). Then, by Proposition 3.1, there exists a subalgebra U of $L_m(\Gamma)$ which has a maximal subalgebra of dimension one, say Fu, and a maximal subalgebra of dimension greater than one. As U is not lm(0) and since $L_m(\Gamma)_0$ is supersolvable (see Lemma 2.1 of Varea, 1988), we have $U \not\leq L_m(\Gamma)_0$. Then the subalgebra $U \cap L_m(\Gamma)_0$ of U has codimension one in U. Moreover, $U \cap L_m(\Gamma)_0$ contains no nonzero ideals of U (see the proof of Lemma 3.7 of Benkart et al., 1979). This yields that U is simple. Thus U is central simple, since Fu is maximal in U. Therefore, $U \otimes_F \Omega$ is a simple Lie algebra over Ω having a maximal subalgebra of codimension one, where Ω is an algebraic closure of F. By Theorem 3.9 of Benkart et al. (1979), it follows that either $U \otimes_F \Omega \cong \mathrm{sl}(2, \Omega)$ or else $U \otimes_F \Omega \cong Z_r(\Omega)$ for some $r \leq m$. In the former case, we have that U is three-dimensional simple. So, $U \cong \mathrm{sl}(2, F)$ since U has a maximal subalgebra of dimension greater than one. Since $\sqrt{F} \leq F$, we find that the algebra $\mathrm{sl}(2, F)$ has no maximal subalgebras of dimension one, and so neither has U , which is a contradiction. Therefore, $U \otimes_F \Omega$ is a Zassenhaus algebra. By using Theorem 6.1 of Benkart et al. (1979), we get that the characteristic polynomial of $\mathrm{ad}_U(u)$ is equal to the minimum polynomial and has the form

$$\mu(X) = X^{p^r} + \alpha_{r-1}X^{p^{r-1}} + \cdots + \alpha_1 X^p + \alpha_0 X.$$

Since Fu is a Cartan subalgebra of U, we have $\alpha_0 \neq 0$, so that, $\mu(X)$ has no multiple roots in Ω. Then, by our hypothesis, $\mu(X)$ has a nonzero root t in F. Therefore, there exists $0 \neq y \in U$ such that $Fu \neq Fy$ and $[u, y] = ty$. Maximality of Fu implies $\dim U = 2$, which is a contradiction.

Finally, we consider the Zassenhaus algebras $Z_n(F)$.

Corollary 7.4. *Let F be perfect and* $\mathrm{char}(F) > 2$. *Then the following hold*:

(i) *$Z_n(F)$ is* lm(0) *if and only if* $\mathrm{ind}_p(F) \geq n$ *and* $\sqrt{F} \leq F$; *and*
(ii) *Every Zassenhaus algebra is* lm(0) *if and only if* $\mathrm{ind}_p(F) = \infty$ *and* $\sqrt{F} \leq F$.

Proof. Note that, for each n we have $L_n(\Gamma) \cong Z_n(F)$ for every Γ, since F is perfect and $\mathrm{char}(F) > 3$ (see Corollary 2.3 of Varea, 1988). Also, note that if $m < n$, then $Z_m(F)$ is isomorphic to a subalgebra of $Z_n(F)$. So, if $Z_n(F)$ is lm(0), so is $Z_m(F)$ for every $m < n$. The result follows from these notes and Theorem 7.3.

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