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DISCRETE TRACY–WIDOM OPERATORS

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Abstract Integrable operators arise in random matrix theory, where they describe the asymptotic eigenvalue distribution of large self-adjoint random matrices from the generalized unitary ensembles. We consider discrete Tracy–Widom operators and give sufficient conditions for a discrete integrable operator to be the square of a Hankel matrix. Examples include the discrete Bessel kernel and kernels arising from the almost Mathieu equation and the Fourier transform of Mathieu's equation.

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1. Introduction

We consider Tracy–Widom operators arising from first-order recurrence relations

$$a(j+1) = T(j)a(j), \quad j = 1, 2, \dots, \tag{1.1}$$

where $a(j)$ is a real 2×1 vector and $T(x)$ is a 2×2 real matrix with entries that are rational functions of x, and such that $\det T(j) = 1$. Then with the skew-symmetric matrix

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

the Tracy–Widom operator $K : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ has matrix

$$K(m,n) = \frac{\langle Ja(m), a(n)\rangle}{m-n}, \quad m \neq n, \tag{1.2}$$

with respect to the usual orthonormal basis. In specific examples, there are natural ways of defining the diagonal $K(n,n)$, as we discuss below. We recall that a real sequence $(\phi_j) \in \ell^2(\mathbb{N})$ gives a Hankel matrix $\Gamma_\phi = [\phi_{j+k-1}]_{j,k=1}^{\infty}$; clearly Γ_ϕ is symmetric, and Γ_ϕ is Hilbert–Schmidt if and only if $\sum_{n=1}^{\infty} n|\phi_n|^2$ converges [**14**].

We consider whether such a K may be expressed as $K = \Gamma_\phi^2$, where Γ_ϕ is a Hankel matrix that is self-adjoint and Hilbert–Schmidt.

A significant example from random matrix theory is the discrete Bessel kernel

$$B(m,n) = \sqrt{\theta}\frac{J_m(2\sqrt{\theta})J_{n-1}(2\sqrt{\theta}) - J_n(2\sqrt{\theta})J_{m-1}(2\sqrt{\theta})}{m-n}, \tag{1.3}$$

as considered by Borodin *et al*. [**5**] and Johansson [**11**]. They showed that B is the square of the Hilbert–Schmidt Hankel matrix $[J_{m+k-1}(2\sqrt{\theta})]$, and thus obtained information about the spectrum of B itself.

Tracy and Widom observed that many of the fundamentally important kernels in random matrix theory have the form

$$W(x,y) = \frac{f(x)g(y) - f(y)g(x)}{x-y}, \quad x \neq y, \tag{1.4}$$

where f, g are bounded real functions in $L^2(0,\infty)$ such that

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{bmatrix} f(x) \\ g(x) \end{bmatrix} = \begin{bmatrix} \alpha(x) & \beta(x) \\ -\gamma(x) & -\alpha(x) \end{bmatrix}\begin{bmatrix} f(x) \\ g(x) \end{bmatrix}, \tag{1.5}$$

and $\alpha(x), \beta(x)$ and $\gamma(x)$ are real rational functions [**16**]. Then there is a bounded linear operator $W : L^2(0,\infty) \to L^2(0,\infty)$ given by

$$Wh(x) = \int_0^\infty W(x,y)h(y)\,\mathrm{d}y.$$

Of particular importance is the case in which W is a trace-class operator such that $0 \leqslant W \leqslant I$, since such a W is associated with a determinantal point process as in [**15**]. In order to verify this property in special cases, Tracy and Widom showed that W is the square of a Hankel operator Γ which is self-adjoint and Hilbert–Schmidt. The spectral theory and realization of Hankel operators is well understood [**8, 13**], so such a factorization is valuable. In [**3, 4**], we developed this method further and considered which differential equations lead to kernels that can be factored as squares of Hankel operators. The factorization theorems involve the notion of operator monotone functions.

Here we consider various discrete Tracy–Widom kernels and their factorization as Hankel products, using the formal analogy between differential equations and difference equations which suggests likely factorization theorems. We can write the matrix in (1.5) as $J\Omega(x)$, where $\Omega(x)$ is real and symmetric, and then consider the analogous one-step transition matrix to be $T(x) = \exp(J\Omega(x))$. The functions $x \mapsto x$ and $x \mapsto -1/x$ are operator monotone increasing on $(0,\infty)$, and they appear in the transition matrices for the discrete Bessel kernel in § 3 and the discrete analogue of the Laguerre kernel in § 4.

The functions $x \mapsto x^2$ and $x \mapsto -1/x^2$ are not operator monotone increasing by [**9**], and so we cannot hope to have simple factorization theorems when they appear in the transition matrix T. In the special case of the parabolic cylinder equation $-\phi_n''(x) + (\frac{1}{4}x^2 - \frac{1}{2})\phi_n(x) = n\phi_n(x)$, Aubrun [**1**] recovered a factorization of the corresponding kernel

$$K(x,y) = \frac{\phi_n(x)\phi_{n-1}(y) - \phi_{n-1}(x)\phi_n(y)}{x-y}$$

in the form

$$K = \Gamma_\phi \Gamma_\psi + \Gamma_\psi \Gamma_\phi, \tag{1.6}$$

where Γ_ϕ and Γ_ψ are bounded and self-adjoint Hankel operators. The parabolic cylinder function is a confluent form of Mathieu's functions associated with the elliptic cylinder, since the parabola is the limiting case of an ellipse as the eccentricity increases to 1 [**17**, p. 427]. Hence, it is natural to factorize kernels associated with Mathieu's equation

$$\frac{\mathrm{d}^2 u}{\mathrm{d}z^2} + (\alpha + \beta \cos z) u(z) = 0 \tag{1.7}$$

in the form (1.6). In § 5 we consider the first-order difference equation associated with the Fourier transform of Mathieu's equation. In § 6, we consider almost Mathieu operators.

For a compact and self-adjoint operator W, the spectrum consists of real eigenvalues λ_j, which may be ordered so that the sequence of singular numbers $s_j = |\lambda_j|$ satisfies $s_1 \geqslant s_2 \geqslant \cdots$. While the factorization $W = \Gamma^2$ immediately determines the spectrum of W from the spectrum of Γ, a factorization (1.6) imposes bounds upon the singular numbers of K in terms of the eigenvalues of Γ_ϕ and Γ_ψ.

2. Factoring discrete Tracy–Widom operators as squares of Hankel matrices

Theorem 2.1. *Let $T(j)$ and $B(j)$ be 2×2 real matrices, and let $(a(j))$ be a sequence of real 2×1 vectors such that*

$$a(j+1) = T(j) a(j), \quad j \in \mathbb{N}, \tag{2.1}$$

$$a(j) \to 0, \quad j \to \infty, \tag{2.2}$$

$$\sum_{j=1}^{\infty} \|B(j) a(j)\|^2 < \infty. \tag{2.3}$$

Suppose further that there exists a real symmetric matrix C with eigenvalues 0 and λ, where $\lambda < 0$, such that

$$\frac{T(n)^{\mathrm{T}} J T(m) - J}{m-n} = B(n)^{\mathrm{T}} C B(m), \quad m \neq n,\ m, n \in \mathbb{N}. \tag{2.4}$$

Let $\phi(j) = |\lambda|^{1/2} \langle v_\lambda, B(j) a(j) \rangle$, where v_λ is a real unit eigenvector corresponding to λ. Then Γ_ϕ is compact and $K = \Gamma_\phi^2$ has entries

$$K(m,n) = \frac{\langle J a(m), a(n) \rangle}{m-n}, \quad m \neq n,\ m, n \in \mathbb{N}. \tag{2.5}$$

Proof. Let $K(m,n)$ be as in (2.5) and let

$$G(m,n) = K(m,n) - \sum_{k=1}^{\infty} \phi(m+k-1)\phi(n+k-1), \quad m \neq n,\ m, n \in \mathbb{N}, \tag{2.6}$$

where the infinite sum converges because of the condition (2.3). Observe that $a(j) \to 0$ implies that the first term in $G(m,n)$ tends to 0 as m or $n \to \infty$, and that the same is true of the Hankel sum:

$$\left|\sum_{k=1}^{\infty} \phi(m+k-1)\phi(n+k-1)\right| \leqslant \left(\sum_{k=1}^{\infty} |\phi(m+k-1)|^2\right)^{1/2} \left(\sum_{k=1}^{\infty} |\phi(n+k-1)|^2\right)^{1/2} \to 0 \quad (2.7)$$

as m or $n \to \infty$. Hence, $G(m,n) \to 0$ as m or $n \to \infty$. Now let U be the real orthogonal matrix with v_λ in the first column, and the eigenvector of C corresponding to the eigenvalue 0 in the second column. Then $U^{\mathrm{T}}CU = \operatorname{diag}(\lambda, 0)$, and we have

$$\begin{aligned} K(m+1,n+1) - K(m,n) &= \frac{1}{m-n}(\langle JT(m)a(m), T(n)a(n)\rangle - \langle Ja(m), a(n)\rangle) \\ &= \frac{1}{m-n}\langle (T(n)^{\mathrm{T}}JT(m) - J)a(m), a(n)\rangle \\ &= \langle B(n)^{\mathrm{T}}CB(m)a(m), a(n)\rangle \\ &= \langle U \operatorname{diag}(\lambda,0)U^{\mathrm{T}}B(m)a(m), B(n)a(n)\rangle \\ &= \lambda\langle \operatorname{diag}(1,0)U^{\mathrm{T}}B(m)a(m), \operatorname{diag}(1,0)U^{\mathrm{T}}B(n)a(n)\rangle \\ &= -\phi(m)\phi(n). \end{aligned} \quad (2.8)$$

The above calculation and the equality

$$\sum_{k=1}^{\infty} \phi(m+k)\phi(n+k) - \phi(m+k-1)\phi(n+k-1) = -\phi(m)\phi(n) \quad (2.9)$$

together imply that $G(m+1,n+1) = G(m,n)$, and so in fact $G(m,n) = 0$ for all $m,n \in \mathbb{N}$, which gives

$$K(m,n) = \sum_{k=1}^{\infty} \phi(m+k-1)\phi(n+k-1), \quad m \neq n,\ m,n \in \mathbb{N}, \quad (2.10)$$

where the right-hand side is the (m,n)th entry of the square of a Hankel matrix Γ_ϕ. Furthermore, we observe that K is the composition of the discrete Hilbert transform with the compact operators $\ell^2(\mathbb{N};\mathbb{C}) \to \ell^2(\mathbb{N};\mathbb{C}^2)$ given by $(x_n) \mapsto (Ja(n)x_n)$ and the adjoint of $(x_n) \mapsto (a(n)x_n)$; so K is compact. We deduce that Γ_ϕ is also compact. □

Proposition 2.2. *Let $a(j)$, $T(j)$ and $B(j)$ satisfy conditions (2.1) and (2.2), and suppose further that*

$$\sum_{j=1}^{\infty} j\|B(j)a(j)\|^2 < \infty. \quad (2.11)$$

Now let $K = \Gamma_\phi^2$ be as in Theorem 2.1. Then

(i) *K is a positive semidefinite and trace-class operator,*

(ii) *for each $n \in \mathbb{N}$, there exist self-adjoint Hankel operators Γ_n, where Γ_n has rank at most n, such that*

$$s_n(K) = \|\Gamma_\phi - \Gamma_n\|^2, \tag{2.12}$$

so $\Gamma_n^2 \to K$ as $n \to \infty$.

Proof. (i) The Hilbert–Schmidt norm of Γ_ϕ satisfies

$$\|\Gamma_\phi\|_{HS}^2 = \sum_{k=1}^{\infty} k\phi(k)^2 < \infty.$$

Hence, $K = \Gamma_\phi^2$ is of trace class.

(ii) Since Γ_ϕ is self-adjoint, the singular numbers satisfy $s_n(K) = s_n(\Gamma_\phi^2) = s_n(\Gamma_\phi)^2$. By the Adamyan–Arov–Krein theorem [**14**], there exists a unique Hankel operator Γ_n with rank at most n such that $s_n(\Gamma_\phi) = \|\Gamma_\phi - \Gamma_n\|$. Evidently, Γ_n^* is also a Hankel operator of rank at most n such that $s_n(\Gamma_\phi) = \|\Gamma_\phi - \Gamma_n^*\|$, so, by uniqueness, $\Gamma_n = \Gamma_n^*$. We have $\|\Gamma_n - \Gamma_\phi\| \to 0$ as $n \to \infty$, so $\Gamma_n^2 \to \Gamma_\phi^2$ as $n \to \infty$. □

Definition 2.3. For a compact and self-adjoint operator W on a Hilbert space H, the spectral multiplicity function $\nu_W : \mathbb{R} \to \{0, 1, \dots\} \cup \{\infty\}$ is given by

$$\nu_W(\lambda) = \dim\{x \in H : Wx = \lambda x\}, \quad \lambda \in \mathbb{R}. \tag{2.13}$$

Proposition 2.4. *Let K be as in Proposition 2.2. Then the following hold:*

(i) *$\nu_K(0) = 0$ or $\nu_K(0) = \infty$;*

(ii) *$\nu_K(\lambda) < \infty$ and $\nu_K(\lambda) = \nu_{\Gamma_\phi}(\sqrt{\lambda}) + \nu_{\Gamma_\phi}(-\sqrt{\lambda})$ for all $\lambda > 0$;*

(iii) *if $\nu_K(\lambda)$ is even, then $\nu_{\Gamma_\phi}(\sqrt{\lambda}) = \nu_{\Gamma_\phi}(-\sqrt{\lambda})$;*

(iv) *if $\nu_K(\lambda)$ is odd, then $|\nu_{\Gamma_\phi}(\sqrt{\lambda}) - \nu_{\Gamma_\phi}(-\sqrt{\lambda})| = 1$.*

Proof. Part (i) follows from Beurling's theorem (see [**14**, p. 15]), while (ii) is elementary. Peller *et al*. show in [**13**] that, for any compact and self-adjoint Hankel operator Γ_ϕ, the spectral multiplicity function satisfies $|\nu_{\Gamma_\phi}(\lambda) - \nu_{\Gamma_\phi}(-\lambda)| \leqslant 1$. Using this, and (ii), statements (iii) and (iv) follow immediately. □

3. The discrete Bessel kernel

We show how Theorem 2.1 can be applied to the discrete Bessel kernel to recover a result from [**5**, **11**].

Proposition 3.1. *Let $J_n(z)$ be the Bessel function of the first kind of order n, let $J_n = J_n(2\sqrt{\theta})$, where $\theta > 0$; let $\phi(n) = J_{n+1}$ and $a(n) = [\sqrt{\theta}J_n, J_{n+1}]^{\mathrm{T}}$. Then the Hankel operator Γ_ϕ is Hilbert–Schmidt, and $B = \Gamma_\phi^2$ has entries*

$$B(m,n) = \frac{\langle Ja(m), a(n)\rangle}{m-n}, \quad m \neq n,\ m,n \in \mathbb{N}. \tag{3.1}$$

Proof. It is clear that (2.1) holds, since we have the recurrence relation

$$J_{n+2}(2z) = \frac{n+1}{z} J_{n+1}(2z) - J_n(2z), \tag{3.2}$$

giving $a(n+1) = T(n)a(n)$, where

$$T(n) = \begin{bmatrix} 0 & \sqrt{\theta} \\ \dfrac{-1}{\sqrt{\theta}} & \dfrac{n+1}{\sqrt{\theta}} \end{bmatrix}. \tag{3.3}$$

Note that

$$\frac{T(n)^{\mathrm{T}} J T(m) - J}{m-n} = C, \tag{3.4}$$

where $C = \operatorname{diag}(0, -1)$, which is clearly of rank 1. The non-zero eigenvalue of C is $\lambda = -1$, and a corresponding unit eigenvector is $v_\lambda = [0, 1]^{\mathrm{T}}$, so

$$|\lambda|^{1/2}\langle v_\lambda, a(n)\rangle = J_{n+1} = \phi(n).$$

We now verify condition (2.11), and thus (2.2). Note that

$$\frac{1}{\theta}\sum_{n=1}^{\infty} nJ_{n+1}^2 < \frac{1}{\theta}\sum_{n=1}^{\infty}(n+1)^2 J_{n+1}^2 = \sum_{n=1}^{\infty}(J_{n+2} + J_n)^2 \leqslant 4\sum_{n=1}^{\infty} J_n^2. \tag{3.5}$$

The standard formula [**17**, p. 379]

$$\mathrm{e}^{\mathrm{i}2\sqrt{\theta}\sin\psi} = J_0(2\sqrt{\theta}) + 2\sum_{m=1}^{\infty} J_{2m}(2\sqrt{\theta})\cos 2m\psi + 2\mathrm{i}\sum_{m=1}^{\infty} J_{2m-1}(2\sqrt{\theta})\sin(2m-1)\psi \tag{3.6}$$

and Parseval's identity can be used to show that

$$J_0(2\sqrt{\theta})^2 + 2\sum_{m=1}^{\infty} J_m(2\sqrt{\theta})^2 = 1 \quad \text{for all } \theta > 0,$$

and hence that the sum on the right-hand side of (3.5) is finite. □

4. A discrete analogue of the Laguerre differential equation

In this section we consider a case in which condition (2.2) is violated and we cannot hope to factor the kernel K as the square of a Hankel operator. Nevertheless, we can identify a Toeplitz operator W such that $K - W$ factors as a product of Hankels.

Proposition 4.1. *For $\theta \in \mathbb{R}$, let $(a(j))$ satisfy the recurrence relation $a(j+1) = T(j)a(j)$ with*

$$T(j) = \begin{bmatrix} \theta/(j+1) & -1 \\ 1 & 0 \end{bmatrix} \tag{4.1}$$

and $a(1) = [\theta, 1]^{\mathrm{T}}$. Then there exist polynomials $p_j(\theta)$ of degree j such that

(i) $a(j) = [p_j(\theta), p_{j-1}(\theta)]^{\mathrm{T}}$.

(ii) *The self-adjoint Hankel matrix $\Gamma_\phi = [\phi(j+k-1)]_{j,k=1}^{\infty}$ with entries*

$$\phi(j) = \frac{p_j(\theta)}{j+1} \tag{4.2}$$

is a bounded linear operator such that $\theta\Gamma_\phi^2 = K + W$ where K has entries

$$K(m,n) = \frac{\langle Ja(m), a(n)\rangle}{m-n}, \quad m \neq n,\ m,n \in \mathbb{N}, \tag{4.3}$$

and W is a bounded Toeplitz operator with matrix

$$W(m-n) = \frac{\langle J^{\bar m - \bar n + 1} T_\infty a(1), T_\infty a(1)\rangle}{m-n}, \quad m \neq n,$$

for some 2×2 matrix T_∞, and $\bar m$ and $\bar n$ are the congruence classes of m and n modulo 4.

Proof. When $\theta = 0$, the recurrence relation reduces to $a(n+1) = Ja(n)$, with solution $a(n) = J^{n-1}a(1)$. This gives rise to a kernel

$$K(m,n) = \frac{\langle Ja(m), a(n)\rangle}{m-n} = \frac{\langle J^{m-n+1}a(1), a(1)\rangle}{m-n}, \quad m \neq n, \tag{4.4}$$

which has the shape of a Toeplitz operator, and is a variant on the discrete Hilbert transform with matrix $[1/(m-n)]_{m\neq n}$.

Now suppose $\theta \neq 0$. The matrix J satisfies $J^4 = I$, and so we consider the partial product of the $T(j)$ in bunches of four, with the jth bunch giving

$$B(j) = T(4j)T(4j-1)T(4j-2)T(4j-3) = I - \frac{\theta}{2j}J + O(1/j^2), \quad j \in \mathbb{N}. \tag{4.5}$$

Now we deduce that

$$\|B(j)\|^2 = \|B(j)^*B(j)\| = 1 + O(1/j^2), \tag{4.6}$$

and likewise with $B(j)^{-1}$ in place of $B(j)$; so there exists $C(\theta)$ such that

$$\left.\begin{aligned} \|T(n)T(n-1)\cdots T(2)T(1)\| &\leqslant C(\theta), \\ \|T(1)^{-1}T(2)^{-1}\cdots T(n-1)^{-1}T(n)^{-1}\| &\leqslant C(\theta), \end{aligned}\right\} \quad n = 1,2,\ldots. \tag{4.7}$$

It follows that there exists $\kappa(\theta) > 0$ such that $\kappa(\theta) < \|a(n)\| < \kappa(\theta)^{-1}$ for all n, so (2.2) is violated. We introduce

$$C_k = \exp\left(\theta J \sum_{j=1}^{k} \frac{1}{2j}\right) B(k)B(k-1)\cdots B(1), \tag{4.8}$$

which satisfies $C_{k+1} - C_k = O(1/k^2)$; so the limit

$$T_\infty = \lim_{k\to\infty} C_k \tag{4.9}$$

exists. One can check that

$$\exp\left(\sum_{j=1}^{k+n} \frac{\theta J}{2j}\right)^* \exp\left(\sum_{j=1}^{k+m} \frac{\theta J}{2j}\right) \to I \tag{4.10}$$

as $k \to \infty$, and hence

$$\begin{aligned} &\langle Ja(m+4k), a(n+4k)\rangle \\ &\quad = \langle JT(m+4k)T(m+4k-1)\cdots T(1)a(1), T(n+4k)T(n+4k-1)\cdots T(1)a(1)\rangle \\ &\quad \to \langle J^{\bar{m}+1}T_\infty a(1), J^{\bar{n}}T_\infty a(1)\rangle, \quad k\to\infty. \end{aligned} \tag{4.11}$$

For temporary convenience we introduce

$$\tilde{K}(m,n) = \begin{cases} \dfrac{\langle Ja(m), a(n)\rangle}{(m-n)} & \text{for } m \neq n, \\ 0 & \text{for } m = n. \end{cases} \tag{4.12}$$

The discrete Hilbert transform is bounded on $\ell^2(\mathbb{Z};\mathbb{C}^2)$ by [**7**], so $\tilde{K}$ defines a bounded linear operator on $\ell^2(\mathbb{N})$, but condition (2.2) is violated.

The $p_j(\theta)$ satisfy the recurrence relation

$$p_{n+1}(\theta) + p_{n-1}(\theta) = \frac{\theta}{n+1} p_n(\theta) \tag{4.13}$$

with $p_0(\theta) = 1$ and $p_1(\theta) = \theta$, so clearly $p_j(\theta)$ is a polynomial of degree j such that $|p_n(\theta)| \leqslant \kappa(\theta)^{-1}$ for all θ and n. Furthermore,

$$\frac{T(n)^{\mathrm{T}} JT(m) - J}{m-n} = \begin{bmatrix} \dfrac{-\theta}{(m+1)(n+1)} & 0 \\ 0 & 0 \end{bmatrix}, \tag{4.14}$$

so that, by the calculation in the proof of Theorem 2.1,

$$\tilde{K}(m+1,n+1) - \tilde{K}(m,n) = -\theta \frac{p_m(\theta)p_n(\theta)}{(m+1)(n+1)}. \tag{4.15}$$

We can write

$$\tilde{K}(m,n) - \tilde{K}(m+N+1, n+N+1) = \theta \sum_{k=0}^{N} \frac{p_{m+k}(\theta) p_{n+k}(\theta)}{(m+k+1)(n+k+1)}, \quad m \neq n, \quad (4.16)$$

where the limit

$$\lim_{N\to\infty} \tilde{K}(m+N+1, n+N+1) = W(m-n), \quad m \neq n, \quad (4.17)$$

exists and is finite since the sequence $\phi(j) = p_j(\theta)/(j+1)$ is square summable, so

$$\tilde{K}(m,n) = \theta \sum_{k=0}^{\infty} \frac{p_{m+k}(\theta) p_{n+k}(\theta)}{(m+k+1)(n+k+1)} + W(m-n), \quad m \neq n. \quad (4.18)$$

We now define $W(0) = 0$, and let

$$K(m,n) = \theta \sum_{k=0}^{\infty} \frac{p_{m+k}(\theta) p_{n+k}(\theta)}{(m+k+1)(n+k+1)} + W(m-n), \quad (4.19)$$

so that the matrix of K equals the matrix of $\tilde{K}$, except on the principal diagonal, and the principal diagonal of K is a bounded sequence; hence, K is a bounded linear operator and also satisfies the preceding identities for $m = n$. Let S be the shift operator on $\ell^2(\mathbb{N})$. Now $\theta \Gamma_\phi^2$ equals the limit in the weak operator topology of the sequence $K - S^{*n} K S^n$ as $n \to \infty$, so Γ_ϕ is bounded and hence $W = K - \theta \Gamma_\phi^2$ is also bounded. We recognize the matrix of W as

$$W(m-n) = \frac{\langle J^{\bar{m}-\bar{n}+1} T_\infty a(1), T_\infty a(1) \rangle}{m-n}, \quad m \neq n.$$

□

Remark 4.2. The generating function $f(z) = \sum_{j=0}^{\infty} p_j(\theta) z^j$ satisfies the differential equation

$$(1+z^2) f'(z) + (2z - \theta) f(z) = 0$$

with initial condition $f(0) = 1$, and hence

$$f(z) = \left(\frac{1 - \mathrm{i}z}{1 + \mathrm{i}z} \right)^{\mathrm{i}\theta/2} \frac{1}{1+z^2}. \quad (4.20)$$

For comparison, Laguerre's equation [**16**] may be expressed as

$$\frac{\mathrm{d}}{\mathrm{d}x} \begin{bmatrix} u(x) \\ u'(x) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{1}{4} - \frac{(n+1)}{x} & 0 \end{bmatrix} \begin{bmatrix} u(x) \\ u'(x) \end{bmatrix}, \quad (4.21)$$

with solution $u(x) = x\mathrm{e}^{-x/2} L_n^{(1)}(x)$, where

$$L_n^{(1)}(x) = \frac{x^{-1}\mathrm{e}^x}{n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n} (x^{n+1} \mathrm{e}^{-x}), \quad x > 0, \quad (4.22)$$

is the Laguerre polynomial of degree n and parameter 1. The Laplace transform of u is the rational function

$$\mathcal{L}(u;\lambda) = (n+1)\frac{(\lambda-\frac{1}{2})^n}{(\lambda+\frac{1}{2})^{n+2}}, \quad \operatorname{Re}\lambda > -\tfrac{1}{2}. \tag{4.23}$$

5. The Fourier transform of Mathieu's equation

Let $\beta \neq 0$ be a real number; then there exists a sequence of real values of α such that Mathieu's equation

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + (\alpha + \beta\cos\theta)u(\theta) = 0 \tag{5.1}$$

has a real periodic solution with period 2π or 4π. The odd or even periodic solutions are known as Mathieu functions, and various determinants describe the dependence of the eigenvalues α on β, as in [**12, 17**]. Here we are concerned with some matrices that arise from the Fourier transform of the differential equation.

Theorem 5.1. *Suppose that u has Fourier expansion*

$$u(\theta) = \sum_{n=-\infty}^{\infty} b_n \mathrm{e}^{\mathrm{i}n\theta}.$$

Let Γ_u be the Hankel matrix $[b_{j+k-1}]_{j,k=1}^{\infty}$, let Γ_v be the Hankel matrix $[(j+k-1)b_{j+k-1}]_{j,k=1}^{\infty}$ and let $K = (-2/\beta)(\Gamma_u\Gamma_v + \Gamma_v\Gamma_u)$. Then K is a trace class operator on $\ell^2(\mathbb{N})$ such that

$$\operatorname{tr}(K) = \frac{1}{\beta\pi}\int_0^{2\pi}\left|\frac{\mathrm{d}u}{\mathrm{d}\theta}\right|^2 \mathrm{d}\theta$$

and

$$K(j,k) = \frac{b_{j-1}b_k - b_j b_{k-1}}{j-k}, \quad j,k \in \mathbb{N},\ j \neq k. \tag{5.2}$$

Proof. Since u is real, we have $b_m = \bar{b}_{-m}$. The recurrence relation for the Fourier coefficients

$$2(-n^2+\alpha)b_n + \beta b_{n+1} + \beta b_{n-1} = 0 \tag{5.3}$$

may be expressed as the first-order recurrence relation

$$\begin{bmatrix} b_n \\ b_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & \left(\frac{2}{\beta}\right)(n^2-\alpha) \end{bmatrix} \begin{bmatrix} b_{n-1} \\ b_n \end{bmatrix}, \tag{5.4}$$

or in the obvious shorthand $a(n+1) = T(n)a(n)$. Then we have

$$\frac{T(n)^{\mathrm{T}}JT(m) - J}{m-n} = \begin{bmatrix} 0 & 0 \\ 0 & \left(-\frac{2}{\beta}\right)(m+n) \end{bmatrix}. \tag{5.5}$$

We introduce the kernel $\tilde{K}$ by the formula

$$\tilde{K}(m,n) = \frac{\langle Ja(m), a(n)\rangle}{m-n} = \frac{b_n b_{m-1} - b_{n-1} b_m}{m-n},$$

which therefore satisfies

$$\tilde{K}(m+1,n+1) - \tilde{K}(m,n) = \left(-\frac{2}{\beta}\right)(m b_m b_n + n b_n b_m), \tag{5.6}$$

and $\tilde{K}(m,n) \to 0$ as $m,n \to \infty$ in any way such that $m \neq n$. We deduce that

$$\tilde{K}(m,n) = \frac{2}{\beta}\sum_{k=0}^{\infty}(m+k)b_{m+k}b_{n+k} + (n+k)b_{n+k}b_{m+k}, \quad m,n \in \mathbb{N},\ m \neq n.$$

Hence, $\tilde{K}(m,n)$ is the (m,n) entry of the matrix of $K = (2/\beta)(\Gamma_u\Gamma_v + \Gamma_v\Gamma_u)$ for all $m \neq n$.

Since u and u'' are square integrable, the series $\sum_{n=-\infty}^{\infty} n^4 |b_n|^2$ converges; so Γ_u and Γ_v are Hilbert–Schmidt, and K is trace class. Furthermore, we have

$$\begin{aligned} \mathrm{tr}(K) &= \sum_{m=1}^{\infty} K(m,m) \\ &= \frac{4}{\beta}\sum_{m,k=1}^{\infty}(m+k-1)b_{m+k-1}^2 \\ &= \frac{4}{\beta}\sum_{m=1}^{\infty} m^2 b_m^2 \\ &= \frac{2}{\beta}\int_0^{2\pi} |u'(\theta)|^2 \frac{\mathrm{d}\theta}{2\pi}. \end{aligned} \tag{5.7}$$

□

6. Almost Mathieu operators

We introduce the almost Mathieu operator $H : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ by

$$(Hu)_n = u_{n+1} + u_{n-1} + \lambda \cos 2\pi(n\theta + \alpha)u_n \tag{6.1}$$

for $u = (u_n)_{n=-\infty}^{\infty} \in \ell^2(\mathbb{Z})$, where $(Hu)_n$ denotes the nth term in the sequence $Hu \in \ell^2(\mathbb{Z})$. For all real λ, θ, ω and α, the operator H is bounded and self-adjoint, with its spectrum contained in $[-2-|\lambda|, 2+|\lambda|]$. According to the precise values of the parameters, as we discuss below, the spectrum can consist of a mixture of the point spectrum, the continuous spectrum and the singular continuous spectrum.

Definition 6.1. Let E be an eigenvalue of H with the corresponding eigenvector (u_n). Say that (u_n) *decays exponentially* if there exist $C, \delta > 0$ and $n_0 \in \mathbb{Z}$ such that

$$|u_n| \leqslant C\mathrm{e}^{-\delta|n-n_0|}, \quad n \in \mathbb{Z}. \tag{6.2}$$

(Typically, δ depends on θ, α and λ in a complicated fashion.) Say that H exhibits *Anderson localization* if its spectrum is pure point and all eigenvectors decay exponentially.

Definition 6.2. Say that $\theta \in \mathbb{R}$ is *Diophantine* if there exist $c(\theta) > 0$ and $r(\theta) > 0$ such that

$$|\sin 2\pi j\theta| \geqslant c(\theta)|j|^{-r(\theta)}, \quad j \in \mathbb{Z} \setminus \{0\}. \tag{6.3}$$

With respect to Lebesgue measure, almost all real numbers are Diophantine [**2**, p. 373]. Clearly, rational numbers are not Diophantine, and Liouville numbers are not Diophantine [**10**].

Following the work of several mathematicians, as summarized in [**6**], Jitomirskaya [**10**] obtained a satisfactory description of the spectrum of the almost Mathieu operator.

Lemma 6.3 (Jitomirskaya [10]). *Suppose that θ is Diophantine. Then there exists a set S_θ such that $\mathbb{R} \setminus S_\theta$ has Lebesgue measure zero, and such that, for all $\lambda > 2$ and all $\alpha \in S_\theta$, the Mathieu operator H has an eigenvalue E such that the corresponding eigenvector (u_n) decays exponentially as $n \to \pm\infty$.*

Moreover, Jitomirskaya proved that the Mathieu operator has pure point spectrum for $\lambda > 2$, and further conjectured that the same conclusion holds for all real α. For comparison, for $\lambda = 2$ there exists a purely singular continuous spectrum, whereas for $0 < \lambda < 2$ there exists a purely absolutely continuous spectrum.

We introduce the Hankel matrices

$$\Gamma_{\mathrm{c}} = [\cos \pi(\alpha + x\theta + k\theta)u_{x+k}]_{x,k=1}^{\infty}, \qquad \Gamma_{\mathrm{s}} = [\sin \pi(\alpha + x\theta + k\theta)u_{x+k}]_{x,k=1}^{\infty}. \tag{6.4}$$

There exists a bounded and measurable function $\Phi : \mathbb{T} \to M_2(\mathbb{C})$ such that

$$\hat{\Phi}(k) = \begin{bmatrix} \cos \pi(\alpha + k\theta)u_k & \sin \pi(\alpha + k\theta)u_k \\ \cos \pi(\alpha + k\theta)u_k & \cos \pi(\alpha + k\theta)u_k \end{bmatrix}; \tag{6.5}$$

the block Hankel operator associated with Φ is $[\hat{\Phi}(j + k)]$, which becomes, after a rearrangement of the block form, the matrix

$$\Gamma_\Phi = \begin{bmatrix} \Gamma_{\mathrm{c}} & \Gamma_{\mathrm{s}} \\ \Gamma_{\mathrm{s}} & \Gamma_{\mathrm{c}} \end{bmatrix}. \tag{6.6}$$

The negative Fourier coefficients of Φ are not uniquely determined by Γ_Φ, but may be chosen advantageously. Note that all of these operators are self-adjoint.

We introduce operators K and L by

$$\begin{bmatrix} L & K \\ K & L \end{bmatrix} = \begin{bmatrix} \Gamma_{\mathrm{c}}^2 + \Gamma_{\mathrm{s}}^2 & \Gamma_{\mathrm{c}}\Gamma_{\mathrm{s}} + \Gamma_{\mathrm{s}}\Gamma_{\mathrm{c}} \\ \Gamma_{\mathrm{c}}\Gamma_{\mathrm{s}} + \Gamma_{\mathrm{s}}\Gamma_{\mathrm{c}} & \Gamma_{\mathrm{c}}^2 + \Gamma_{\mathrm{s}}^2 \end{bmatrix} = \Gamma_\Phi^2. \tag{6.7}$$

Theorem 6.4. *Let u_n be as in Lemma 6.3.*

(i) *The matrices of K and L are given by*

$$\left.\begin{aligned} K(m,n) &= \frac{u_{m-1}u_n - u_{n-1}u_m}{2\lambda\sin\pi\theta(m-n)}, && m\neq n,\ m,n\in\mathbb{N};\\ L(m,n) &= \cos\pi\theta(m-n)\sum_{k=1}^{\infty} u_{m+k}u_{n+k}, && m,n\in\mathbb{N}. \end{aligned}\right\} \tag{6.8}$$

(ii) *Then Γ_Φ^2 is a positive semidefinite trace-class operator.*

(iii) *Further, the eigenvectors of K, Γ_{c} and Γ_{s} decay exponentially as $x\to\infty$.*

(iv) *There exists a bounded and measurable function $\Psi_n : \mathbb{T}\to M_2(\mathbb{C})$ such that the associated Hankel operator $[\hat{\Psi}_n(j+k)]$ has rank less than or equal to n and*

$$s_n(\Gamma_\Phi) = \|\Gamma_\Phi - \Gamma_{\Psi_n}\| = \|\Phi - \Psi_n\|_\infty. \tag{6.9}$$

(v) *The eigenvalues of K, L, Γ_{s} and Γ_{c} decay exponentially.*

Proof. (i) First we observe that the formula for $K(m,n)$ makes sense for $m\neq n$ since θ is irrational. The discrete Mathieu equation

$$u_{n+1} + u_{n-1} + \lambda\cos 2\pi(n\theta+\alpha)u_n = Eu_n \tag{6.10}$$

gives the system

$$\begin{bmatrix} u_n\\ u_{n+1}\end{bmatrix} = \begin{bmatrix} 0 & 1\\ -1 & E-\lambda\cos 2\pi(n\theta+\alpha)\end{bmatrix}\begin{bmatrix} u_{n-1}\\ u_n\end{bmatrix}. \tag{6.11}$$

Writing $T(n)$ for the one-step transition matrix, we have

$$\frac{T(n)^{\mathrm{T}}JT(m) - J}{2\lambda\sin\pi\theta(m-n)} = -\begin{bmatrix} 0 & 0\\ 0 & 1\end{bmatrix}(\sin\pi(n\theta+\alpha)\cos\pi(m\theta+\alpha)+\cos\pi(n\theta+\alpha)\sin\pi(m\theta+\alpha)); \tag{6.12}$$

so with $a(m) = [u_{m-1}, u_m]^{\mathrm{T}}$ we introduce

$$\tilde{K}(m,n) = \frac{\langle Ja(m), a(n)\rangle}{2\lambda\sin\pi\theta(m-n)}, \tag{6.13}$$

which satisfies

$$\begin{aligned} &\tilde{K}(m+1,n+1) - \tilde{K}(m,n)\\ &\qquad = (\sin\pi(n\theta+\alpha)\cos\pi(m\theta+\alpha) + \cos\pi(n\theta+\alpha)\sin\pi(m\theta+\alpha))u_n u_m \end{aligned}$$

and $\tilde{K}(m+k,n+k)\to 0$ as $k\to\infty$. Hence, by comparing entries of the products, we find that $\tilde{K} = \Gamma_{\mathrm{c}}\Gamma_{\mathrm{s}} + \Gamma_{\mathrm{s}}\Gamma_{\mathrm{c}}$.

(ii) This is clear, since the entries of the matrix of Γ_Φ are real and summable.

(iii) For any unit vector φ, we have

$$\Gamma_{\mathrm{c}}\varphi(x) = \sum_{k=0}^{\infty} \cos \pi(\alpha + \theta x + \theta k) u_{x+k} \varphi_k, \tag{6.14}$$

so by the Cauchy–Schwarz inequality we have the uniform bound

$$|\Gamma_{\mathrm{c}}\varphi(x)| \leqslant \biggl(\sum_{k=x}^{\infty} u_k^2\biggr)^{1/2}, \tag{6.15}$$

where the right-hand side decays exponentially as $x \to \infty$. A similar result applies with Γ_{s}, so in particular the eigenvectors of Γ_{c} and Γ_{s} decay exponentially at infinity.

Now let $(\varphi_j)_{j=1}^{\infty}$ be an orthonormal basis of $\ell^2(\mathbb{Z})$ consisting of eigenvectors of Γ_{s} with corresponding eigenvalues σ_j. Then

$$\Gamma_{\mathrm{c}}\Gamma_{\mathrm{s}}\varphi(x) = \sum_{j=1}^{\infty} \sigma_j \langle \varphi_j, \varphi\rangle \Gamma_{\mathrm{c}}\varphi_j(x), \tag{6.16}$$

where

$$\sum_{j=1}^{\infty} |\sigma_j \langle \varphi_j, \varphi\rangle| \leqslant \biggl(\sum_{j=1}^{\infty} \sigma_j^2\biggr)^{1/2} \biggl(\sum_{j=1}^{\infty} \langle \varphi, \varphi_j\rangle^2\biggr)^{1/2} = \|\Gamma_{\mathrm{s}}\|_{c^2} \|\varphi\|_{\ell^2}. \tag{6.17}$$

This and a similar result for $\Gamma_{\mathrm{s}}\Gamma_{\mathrm{c}}$ imply that $|K\varphi(x)|$ decays exponentially as $x \to \infty$.

(iv) This is immediate from the vectorial form of the matrical AAK theorem [**14**].

(v) Since u_n decays exponentially as $n \to \infty$, we can approximate the Hankel matrices with finite matrices up to exponentially small error terms. For instance, we can approximate Γ_{s} by $\Gamma_{\mathrm{s}}^{(N)} = [\sin \pi(\alpha + x\theta + k\theta) u_{x+k} \boldsymbol{I}_{\{(x,k):x+k \leqslant N\}}]$, which has rank less than $N+1$ and the operator norm satisfies

$$\|\Gamma_{\mathrm{s}} - \Gamma_{\mathrm{s}}^{(N)}\| \leqslant \sum_{k=N+1}^{\infty} k|u_k|. \tag{6.18}$$

The s-numbers satisfy

$$s_n(K) \leqslant s_n\left(\begin{bmatrix} K & L \\ L & K \end{bmatrix}\right) = s_n(\Gamma_\Phi^2) = s_n(\Gamma_\Phi)^2. \tag{6.19}$$

□

A vectorial Hankel matrix Γ_Ψ has finite rank if and only if it has a rational symbol with coefficients of finite rank by [**14**, p. 19]. Peller provides a formula for the rank in terms of the coefficients.

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