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Abstract. When a society's overall deprivation or living standard is assessed in a multi-attribute framework, the following procedure is often used. First, for each attribute, a summary index is constructed to reflect a society's performance in relation to this attribute. Then, an indicator of the overall performance of the society in terms of all the attributes together is constructed. This paper discusses a difficulty associated with this procedure. We show that the difficulty lies in its inability to reconcile two highly attractive ethical principles - the first reflecting a requirement of treating individuals symmetrically and the second reflecting a requirement for equity-sensitivity. This problem implies that this widely-used procedure must lead to possibly untenable conclusions, and that it is necessary to adopt alternative procedures. The alternative procedure must permit describing a society's overall deprivation or living standard as an aggregate of the comprehensive deprivations or living standards experienced by the individuals in the society.

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1 Introduction

Over the last decade or so, economists have increasingly measured the deprivation and living standards of societies in a multi-attribute framework. There has been a growing realization that, while income-based measures of deprivation and living standards are useful, they have important conceptual limitations (see Sen 1985, 1987 for foundational statements of this criticism), and, to overcome these limitations, one needs to think directly in terms of other valuable attributes such as outcomes in (or the specific means to achieve) health, education, housing, personal security, etc. Such thinking underlies, for instance, the measurement exercises undertaken in the United Nations Development Programme's Human Development Reports, in which a country's achievement is assessed on the basis of measures of three attributes: health, education, and real income per capita. For each of these three dimensions of 'human development', a summary index is constructed to reflect a country's performance in that dimension of human development. Based on these three summary indices, an indicator of the overall achievement of the country in terms of all the three dimensions ('the Human Development Index') is constructed and countries of the world are then ranked on the basis of this indicator. The UNDP's Human Poverty Index, presented in the same reports, is constructed in a similar way (although its focus is on aggregate deprivations rather than achievements) and is also used to rank countries. Many other analysts and institutions have produced rankings of overall achievement (or deprivation) of populations (e.g. Morris (1979)) based on such a method, although the UNDP's exercises are likely the best known and most influential.

Analytically, these rankings are examples of a broad approach of the following type. Given a set of attributes, the achievement or deprivation of each individual in a group is measured for each of these attributes, so that, for the group, we have a table or matrix of indices with each index in the table indicating how well an individual is doing in terms of a specific attribute. An overall indicator for the group is then constructed from the given table and is used for comparing the living standards or deprivation levels of the group in different situations. Owing to the nature of data available, the construction of the overall indicator often uses the following strategy. First, for each attribute, one establishes an index which summarizes how the individuals are deemed to be doing in terms of this attribute. Next, one aggregates the summary indices constructed in the first stage for different attributes so as to derive an overall indicator of how the society is deemed to be doing in terms of all the attributes taken together. An approach of this general kind may seem to have the merit of being practical, especially if data concerning the levels of living standards or deprivations experienced by members of the group are only available from independent sources for each attribute.

In this paper, we address the following issue. How good is this seemingly attractive way of arriving at an overall indicator of a society's deprivation or living standards? In particular, can the strategy outlined above yield satisfactory overall indicators of the deprivation or living standards of groups? We show that the answer to this question is essentially negative: in general, the two-stage procedure that we have described above must fail to satisfy certain highly plausible requirements.[1] In much of our discussion, we shall focus on the measurement of deprivation, but we shall indicate how the analysis can

[1]We are most grateful to Thomas Pogge for informing us that he had previously described examples of the difficulty faced by a two-stage procedure in taking account of equity concerns [see Pogge (1989, 2002)]. We demonstrate here that such examples are instances of broader necessities.

be readily extended to the measurement of living standards (see Section 6).

2 Notation and definitions

Let $N = \{1, \ldots, n\}$ be a given finite set of individuals ($n \geqslant 2$) and let $F = \{f_1, \ldots, f_m\}$ be a given finite set of attributes ($m \geqslant 2$). Let $M = \{1, ..., m\}$. For every $j \in M$, let R^j be a non-empty set of real numbers; we assume that $\#R^j \geqslant 2$. Given our focus on the measurement of deprivation, we shall assume that, for every $j \in M$, $0 \in R^j \subseteq [0, 1]$ and we shall interpret R^j as the different levels of deprivation in terms of attribute f_j that an individual may possibly have, with 0 indicating the absence of any deprivation in terms of the attribute under consideration (while our main concern is with the measurement of deprivation, in Section 6 we shall also consider other intuitive interpretations of our framework with corresponding interpretations of the numbers in R^j). For our purpose, it is enough to assume that the numbers in R^j have an ordinal significance so that, if $\alpha, \beta \in R^j$ and $\alpha > \beta$, then α represents a higher level of deprivation in terms of attribute f_j than β, but we do not rule out the possibility of these numbers having cardinal significance. The economy achieved by assuming no more than ordinal significance for the different possible levels of deprivation for any attribute is important insofar as deprivation may not be cardinally measurable in the case of many attributes such as health.

Let $(a_{ij})_{n\times m}$ be an $n \times m$ matrix of real numbers such that, for all $i \in N$ and all $j \in M$, $a_{ij} \in R^j$. For all $i \in N$ and all $j \in M$, a_{ij} will be interpreted as the level of individual i's deprivation in terms of attribute f_j. We shall refer to the matrix $(a_{ij})_{n\times m}$ as a *deprivation matrix*. The $n \times m$ matrices $(a_{ij})_{n\times m}$, $(a'_{ij})_{n\times m}, \ldots$, will be denoted by A, A', etc. The class of all such $n \times m$ matrices will be denoted by V. For every $A = (a_{ij})_{n\times m} \in V$ and every individual p, let $a_{p\bullet}$ denote the row vector $(a_{p1}, \ldots, a_{pm})$, indicating individual p's deprivation levels in terms of the m attributes. Likewise, for every $A = (a_{ij})_{n\times m} \in V$ and every attribute f_j, let $a_{\bullet j}$ denote the column vector $(a_{1j}, \ldots, a_{nj})$, indicating each individual's deprivation level in terms of the attribute f_j. Let $A, A' \in V$, $i \in N$, and $j \in M$. We say that $a_{i\bullet}$ and $a'_{i\bullet}$ are $j-$variants if and only if $a_{ij} \neq a'_{ij}$ and $a_{ik} = a'_{ik}$ for all $k \in M - \{j\}$, that is, if and only if $a_{i\bullet}$ and $a'_{i\bullet}$ are identical except that $a_{ij} \neq a'_{ij}$.

Let $\succeq$ be a reflexive (but not necessarily transitive or connected) binary relation over V. We shall call such a binary relation an *overall group depri-*

vation ranking (OGDR). For all $A, A' \in V, A \succeq A'$ denotes that the overall deprivation of the group N in the social situation given by A is deemed at least as high as the overall group deprivation in the social situation described by A'. For all $A, A' \in V$, [$A \succ A'$ iff ($A \succeq A'$ and not($A' \succeq A$)] and [$A \sim A'$ iff ($A \succeq A'$ and $A' \succeq A$)]. $A \succ A'$ indicates that the overall group deprivation is deemed strictly greater in the social situation A than in the social situation A', and $A \sim A'$ indicates that the overall group deprivation is deemed identical in the two social situations.

It may be noted that, in much of the literature on multi-dimensional deprivation, the group deprivation measure specifies exactly one real number (representing the level of overall group deprivation) for each deprivation matrix. Of course, any such group deprivation measure induces an ordering over V. Since our formal results are negative, in the interests of greater generality we have chosen a framework in which the binary relation $\succeq$ over V rather than the real numbers attached to different deprivation matrices is the primitive concept. Our negative results proved for the binary relation $\succeq$ will encompass, *a fortiori*, corresponding results for the case where, for every $A \in V$, we have a real number representing the level of overall group deprivation corresponding to A.

3 Anonymity and non-invariance

In this section, we shall consider some appealing properties which may be imposed on an OGDR, $\succeq$, defined in the last section, and which are related to the invariance of the OGDR to permutations of different kinds.

3.1 Anonymity concepts

Consider two deprivation matrices A and B such that A and B are identical except that, for some two individuals s and t, s's deprivation levels under A are identical to t's deprivation levels under B, and t's deprivation levels under A are identical to s's deprivation levels under B; that is, $a_{s\bullet} = b_{t\bullet}$, $a_{t\bullet} = b_{s\bullet}$, and $a_{k\bullet} = b_{k\bullet}$ for all $k \neq s, t$. If one believes that all individuals should be treated symmetrically, then one would find it appealing to require that A and B, as specified above, should be associated with the same overall group deprivation level. Formally, this property is captured by the following:

Anonymity. $\succeq$ satisfies anonymity if and only if, for all $A, B \in V$ and all $s, t \in N$, if ($a_{i\bullet} = b_{i\bullet}$ for all $i \in N - \{s, t\}$, $a_{s\bullet} = b_{t\bullet}$, and $a_{t\bullet} = b_{s\bullet}$), then $A \sim B$.

Anonymity essentially requires that names of individuals should not play any role in constructing an OGDR and an OGDR should be neutral with respect to individuals' names. Having said this, we note that anonymity would not be a reasonable property to postulate for $\succeq$ if we want to build into our measure any extra concern (beyond that already taken into account in the assessment of their individual deprivation levels, registered in the deprivation matrix) about the deprivation of specific subgroups in N, such as women, ethnic minorities, people belonging to a lower caste, etc. In the absence of any such special concern about any subgroup, however, the property seems most reasonable and has been used extensively in the literature on poverty and deprivation.

To conclude this subsection, we discuss a property related to Anonymity. Consider two pairs of deprivation matrices (A, B) and (A', B') such that A and A' are identical and B and B' are identical except that, for some two individuals s and t, s's deprivation levels under A are identical to t's deprivation levels under A', and t's deprivation levels under A are identical to s's deprivation levels under A', and s's deprivation levels under B are identical to t's deprivation levels under B', and t's deprivation levels under B are identical to s's deprivation levels under B'; that is, $a_{s\bullet} = a'_{t\bullet}$, $a_{t\bullet} = a'_{s\bullet}$, $b_s\bullet = b'_{t\bullet}$, $b_{t\bullet} = b'_{s\bullet}$, and $a_{k\bullet} = a'_{k\bullet}$ and $b_{k\bullet} = b'_{k\bullet}$ for all $k \neq s, t$. If it is again believed that all individuals should be treated symmetrically, then it would be reasonable to require that the ranking between A and B must be identical to the ranking between A' and B'. Formally, this property is captured by the following:

Invariance of Matrix Ranking to Permutations of Individuals. $\succeq$ satisfies anonymity if and only if, for all $A, B, A', B' \in V$ and all $s, t \in N$, if ($a_{i\bullet} = a'_{i\bullet}, b_{i\bullet} = b'_{i\bullet}$ for all $i \in N - \{s, t\}$, $a_{s\bullet} = a'_{t\bullet}$, $b_{s\bullet} = b'_{t\bullet}$, $a_{t\bullet} = a'_{s\bullet}$ and $b_{t\bullet} = b'_{s\bullet}$), then $A \succeq B \Leftrightarrow A' \succeq B'$.

It may be noted that, though conceptually Invariance of Matrix Ranking to Permutations of Individuals is a stronger requirement on $\succeq$ than Anonymity on $\succeq$, the two properties are logically independent if $\succeq$ is not connected and transitive.

3.2 Non-invariance and related properties

To motivate the next property of an OGDR, $\succeq$, we consider the following simple situations. Let there be only two individuals, 1 and 2, and two attributes, education(f_1) and health (f_2). Suppose, to start with, we have a deprivation matrix A given below:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

so that individual 1 is deemed deprived in education but not in health, and individual 2 is deemed deprived in health but not in education. Consider now another situation in which individual 1 is deprived in both education and health, and individual 2 is deprived in neither. The deprivation matrix for this situation is given by the matrix B below:

$$B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

The fact that, in B, 1 has all the different types of deprivation in the society while such deprivations are split between the two individuals in A could be significant in comparing overall group deprivations represented by A and B: it may be reasonable to say that A and B do not contain the same degree of overall group deprivation. This is the basic idea of the next property, non-invariance, that we introduce below:

Non-invariance. $\succeq$ satisfies *non-invariance* if and only if there exist $A, B \in V$, $s, t \in N$, and $k \in M$, such that ($a_{\bullet j} = b_{\bullet j}$ for all $j \in M - \{k\}$), ($a_{ik} = b_{ik}$ for all $i \in N - \{s, t\}$), ($a_{sk} = b_{tk}$), ($a_{tk} = b_{sk}$), and (not($A \sim B$)).

Formally, non-invariance requires the existence of some deprivation matrix A , some individuals s and t and some attribute f_k such that, if, starting with A, we interchange the deprivation levels of s and t with respect to attribute f_k in A, but keep everything else in A unchanged, then the new deprivation matrix will have a different level of overall group deprivation as compared to A.

To further our understanding of the intuition behind non-invariance, we consider the following two other properties which are formally stronger than non-invariance but have a more immediate intuitive appeal. First, consider a situation represented by the following deprivation matrix C with two individuals and three attributes:

$$C = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Note that, under C, individual 1 may be said to be unambiguously more deprived than individual 2: individual 1 is more deprived than individual 2 in terms of each of attributes f_1 and f_2, and both individuals have the same level of deprivation of attribute f_3. Now, suppose that individual 1 becomes even more deprived with an increase in 1's deprivation in terms of attribute f_3 from 0 to 0.5 (1's deprivation remains the same for attributes f_1 and f_2), while individual 2's deprivation levels for the three attributes remain the same as under C. As a consequence, we have the situation represented by a deprivation matrix D below:

$$D = \begin{pmatrix} 1 & 1 & 0.5 \\ 0 & 0 & 0 \end{pmatrix}$$

Next, suppose that, starting with C, individual 2, who is less deprived than individual 1 under C, becomes more deprived in that 2's deprivation level for attribute f_3 increases from 0 to 0.5, the same as 1's deprivation level for attribute f_3 in D. This situation is illustrated by the deprivation matrix E given below:

$$E = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0.5 \end{pmatrix}$$

Comparing D with E, there seem to be good reasons to conclude that the overall group deprivation level under D is greater than the overall group deprivation level under E: D is obtained from C by making a "poorer" individual even more so while E is obtained from C by making a "richer" individual poorer. Formally, we have the following:

Equity Principle I. $\succeq$ satisfies *equity principle I* if and only if, for all $A, B, C \in V$, for all $s, t \in N$, and for all $k \in M$, if $[(a_{i\bullet} = b_{i\bullet} = c_{i\bullet}$ for all $i \in N - \{s, t\})$, $(a_{s\bullet} > a_{t\bullet},\ a_{sk} = a_{tk},\ a_{t\bullet} = b_{t\bullet}$, and $a_{s\bullet} = c_{s\bullet})$, and ($a_{s\bullet}$ and $b_{s\bullet}$ are $k-$variants, $a_{t\bullet}$ and $c_{t\bullet}$ are $k-$variants, and $b_{sk} = c_{tk} > a_{sk} = a_{tk})]$, then $B \succ C$.

Equity principle I is highly attractive when comparing overall group deprivations. What it says is this. Suppose that we start with a deprivation matrix A where individual s is at least as deprived as individual t in terms of every attribute and strictly more deprived in terms of some attribute, and, further, in terms of some attribute k, s's deprivation is the same as t's deprivation. In this case, s can be considered to be unambiguously more deprived than t. First, suppose, other things remaining the same in the matrix A, s's deprivation increases from a_{sk} to some higher level, say, α, and, as a result, A changes to B. Next, suppose that, other things remaining the same in A, a_{tk} increases to α and, as a result, A changes to C. Equity principle I requires that the overall deprivation of the society be deemed higher in B than in C. Equity principle I is an extension of a very familiar idea in the context of measuring income poverty (where income is the only relevant dimension of deprivation): if, to start with, there are two individuals who are both poor but one of them is poorer than the other, then a reduction in the income of the poorer person, other things remaining the same, will be considered to give rise to a worse situation in terms of overall (group) poverty as compared to an equal reduction in the income of the less poor person, other things remaining the same.

To illustrate the intuition behind our second equity principle, we consider a society with two individuals, 1 and 2, and three attributes, f_1, f_2 and f_3. Consider first a situation represented by the following deprivation matrix X:

$$X = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Note that, in X, individual 1 is deprived in terms of each attribute while individual 2 is not deprived in terms of any attributes, so that individual 1 may be said to be unambiguously more deprived than individual 2. Now assume that, as far as attributes f_1 and f_3 are concerned, there are no changes in the deprivation of either individual, while, for attribute f_2, there is a switch of the two individual's deprivation levels. The new situation is depicted by the matrix Y below:

$$Y = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

It seems reasonable to claim that the situation under X should be deemed to involve a greater overall group deprivation level than the situation under Y. The normative rationale for such a claim may be thought to be related to that underpinning the Pigou-Dalton transfer principle in the literature of income inequality or (more immediately) the "prioritarian" principle described in philosophical literature [see e.g. Parfit (1997); see also Anand

(1983), Appendix E, for a demonstration that in the single attribute context this principle is entailed by an ”egalitarian’ social welfare function]: the “transfer” of an amount of deprivation of an attribute from a more deprived individual to a less deprived individual should be deemed to reduce the overall group deprivation level. Formally, this property is captured by the following statement:

Equity Principle II. $\succeq$ satisfies *equity principle II* if and only if, for all $A, B \in V$, for all $s, t \in N$, and for some $k \in M$, if $[(a_{i\bullet} = b_{i\bullet}$ for all $i \in N - \{s,t\})$ and (for all $j \in M - \{k\}$, $a_{sj} = b_{sj} > a_{tj} = b_{tj}$ and $a_{sk} = b_{tk} > a_{tk} = b_{sk})]$, then $A \succ B$.

To conclude this section, we note that an OGDR satisfying either equity principle I or equity principle II must satisfy non-invariance (though the converse is not true). Since both equity principle I and equity principle II have much intuitive appeal, it is difficult to resist non-invariance (as that would entail denying both equity principles simultaneously). .

4 Two-stage procedures for aggregating deprivation matrices

We now consider two different procedures for deriving a binary relation $\succeq$ over V, which we shall call the row-first two stage procedure and the column-first two stage procedure respectively. Before introducing the formal definitions of these two procedures, it may be helpful to consider a simple example that illustrates the difference between the two procedures.

Example 1 *Consider the case where* $N = \{1, 2\}$ *and* $F = \{f_1, f_2\}$.

What we call a row-first procedure will consist of two stages. In the first stage we would use two (possibly identical) functions g_1 *and* g_2 *(corresponding to the two individuals* 1 *and* 2*). For* $i = 1, 2$*, the functions* g_i *may be applied to any deprivation matrix* $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. $g_i\,(i = 1, 2)$ *will aggregate* $a_{i\bullet} = (a_{i1}, a_{i2})$ *to arrive at a real number* α_i *which will indicate the overall deprivation of the individual* i *in the social situation* A*. Thus, corresponding to deprivation matrices* $A, A', \ldots,$ *we would obtain vectors* $(\alpha_1, \alpha_2), (\alpha'_1, \alpha'_2), \ldots$ *representing, for each social situation, the overall* *deprivations experienced by the individuals. The second stage is to rank those different vectors,* $(\alpha_1, \alpha_2), (\alpha'_1, \alpha'_2), \ldots$ *, obtained from the first stage, so as to produce a ranking,* $\succeq$*, of overall group deprivation experienced in different social situations. The row-first approach embodies the idea (which may be referred to as 'normative individualism') that the deprivation attributed to a society must be derived from (or possible to interpret as if it was derived from) the deprivations attributed to individuals.*

A column-first procedure will also consist of two stages. In the first stage, we would use two functions h_1 *and* h_2 *(now corresponding to the two attributes* f_1 *and* f_2*). For* $j \in \{1, 2\}$*, the functions* h_j *may be applied to the deprivation matrix* A *above.* $h_j (j = 1, 2)$ *will aggregate* $a_{\bullet j} = (a_{1j}, a_{2j})$ *to arrive at a real number* β_j*. Thus corresponding to deprivation matrices* A, A'*,... we would obtain vectors* $(\beta_1, \beta_2), (\beta'_1, \beta'_2), \ldots,$*. The second stage is now to rank those different vectors,* $(\beta_1, \beta_2), (\beta'_1, \beta'_2), \ldots$*,obtained from the first stage, so as to produce a ranking,* $\succeq$*, of overall group deprivation experienced in different social situations, as before.*

More generally, but informally, a row-first two-stage procedure for deriving a ranking $\succeq$ over V proceeds in the following fashion: In the first stage we use n functions $g_i : \times_{j \in M} R^j \rightarrow R$ $(i \in N)$, so as to produce, for each deprivation matrix A, an n-tuple of real numbers $(g_1(a_{1\bullet}), \ldots, g_n(a_{n\bullet}))$. For each $i \in N$, let $G_i = \{\alpha : \alpha = g_i(a_{i\bullet})$ for some $A \in V\}$. Let $G = G_1 \times \cdots \times G_n$. For each $i \in N$, $g_i(a_{i\bullet})$ can be interpreted as the level of $i's$ overall deprivation in the social situation represented by A. Thus, the information in each deprivation matrix is compressed to an n-tuple of real numbers $(g_1(a_{1\bullet}), \ldots, g_n(a_{n\bullet}))$. One can call $(g_1(a_{1\bullet}), \ldots, g_n(a_{n\bullet}))$ the vector of overall individual deprivations corresponding to A. In what constitutes the second stage of the procedure, we rank the overall individual deprivation vectors corresponding to different deprivation matrices. This "intermediate" ranking, call it $\succeq^{rc}$over G, is then used to induce a ranking $\succeq$ over V in the following straightforward fashion:

for all $A, A' \in V, A \succeq A'$ if and only if $(g_1(a_{1\bullet}), \ldots, g_n(a_{n\bullet})) \succeq^{rc}$ $(g_1(a'_{1\bullet}), \ldots, g_n(a_{n\bullet}))$.

A column-first two stage procedure for deriving a ranking $\succeq$ over V is, informally, as follows. In the first stage, we use m functions $h_j : (R^j)^n \rightarrow R$

($j \in M$) so as to arrive, for each deprivation matrix A, at an m-tuple of real numbers: $(h_1(a_{\bullet 1}), \ldots, h_m(a_{\bullet m}))$. For every $j \in M$, $h_j(a_{\bullet j})$ can be thought of as the society's deprivation in terms of attribute j. Thus, the information in the deprivation matrix A is compressed into a vector $(h_1(a_{\bullet 1}), \ldots, h_m(a_{\bullet m}))$ of real numbers. For each $j \in M$, let $H_j = \{\beta : \beta = h_j(a_{\bullet j}) \text{ for some } A \in V\}$. Let $H = H_1 \times \cdots \times H_m$. In the second stage of the procedure we rank the m-vectors $(h_1(a_{\bullet 1}), \ldots, h_m(a_{\bullet m}))$, $(h_1(a'_{\bullet 1}), \ldots, h_m(a'_{\bullet m}))$, etc. corresponding to $A, A', \ldots \in V$. Letting $\succeq^{cr}$ denote this "intermediate" ranking over H, the ranking of $\succeq$ over V is induced by the following rule:

> for all $A, A' \in V, A \succeq A'$ if and only if $(h_1(a_{\bullet 1}), \ldots, h_m(a_{\bullet m})) \succeq^{cr} (h_1(a'_{\bullet 1}), \ldots, h_m(a'_{\bullet m}))$.

We can now provide formal definitions of the two types of procedure. Let $\succeq$ be a given reflexive and transitive binary relation defined over V. Then, we define:

> *Row-first two-stage procedure:* For every $i \in N$, let g_i be a function from $\times_{j \in M} R^j$ to R and let $\succeq^{rc}$ be a reflexive and transitive relation defined over G. We say that $\succeq$ can be derived through the row-first two-stage procedure based on $(g_1, ..., g_n; \succeq^{rc})$ if and only if for all $A, A' \in V$, $[A \succeq A'$ if and only if $(g_1(a_{1\bullet}), ..., g_n(a_{n\bullet})) \succeq^{rc} (g_1(a'_{1\bullet}), ..., g_n(a_{n\bullet}))]$.

> *Column-first two-stage procedure:* For every $j \in M$, let h_j be a function from $(R^j)^n$ to R, and let $\succeq^{cr}$ be a reflexive and transitive relation defined over H. We say that $\succeq$ can be derived through the column-first two-stage procedure based on $(h_1, ..., h_m; \succeq^{cr})$ if and only if for all $A, A' \in V$, $[A \succeq A'$ if and only if $(h_1(a'_{\bullet 1}), ..., h_m(a'_{\bullet m})) \succeq^{cr} (h_1(a'_{\bullet 1}), ..., h_m(a'_{\bullet m}))]$.

To conclude this section, we introduce two final concepts. Consider a column-first two-stage procedure based on $(h_1, ..., h_m; \succeq^{cr})$. For all $j \in M$, we say that h_j is *symmetric* iff if and only if, for every permutation σ over $\{1, ..., n\}$ and for all $x, y \in [0,1]^n$ such that $[x_i = y_{\sigma(i)}$ for all $i \in N]$, we have $h_j(x) = h_j(y)$; and we say that $\succeq^{cr}$is *positively* responsive if and only if for all $A, B \in V$, if $(h_1(a_{\bullet 1}), ..., h_m(a_{\bullet m})) > (h_1(b_{\bullet 1}), ..., h_m(b_{\bullet m}))$, then $(h_1(a_{\bullet 1}), ..., h_m(a_{\bullet m})) \succ^{cr} (h_1(b_{\bullet 1}), ..., h_m(b_{\bullet m}))$.

5 Impossibility of deriving overall group deprivation rankings with desirable properties through column-first two-stage procedures

We have earlier argued that anonymity, non-invariance, equity principle I, and equity principle II are independently attractive properties of an OGDR. Suppose that we postulate some of these properties for our OGDR. Can we derive such an OGDR through some 'plausible' two-stage column-first procedure? We explore this issue in this section. As noted earlier, the issue is important since two-stage column-first procedures are very frequently used in applied research on multi-dimensional deprivation.

Proposition 1 *Let $\succeq$ be a transitive overall group deprivation ranking, which satisfies anonymity and non-invariance. Then $\succeq$ cannot be derived through any column-first two-stage procedure based on $(h_1, ..., h_m; \succeq^{cr})$ such that, for every $j \in M$, $\succeq^{cr}$ is positively responsive.*

Proof. Suppose an OGDR $\succeq$ satisfies anonymity and non-invariance, and can be derived through a column-first two stage procedure based on $(h_1, \ldots, h_m; \succeq^{cr})$, where $\succeq^{cr}$ is positively responsive for every $j \in M$. Then we shall show that a contradiction follows. By non-invariance, there exist $A, B \in V$, $s, t \in N$ and $j \in M$ such that $[a_{\bullet k} = b_{\bullet k}$ for all $k \in M - \{j\}]$, $[a_{sj} = b_{tj}, a_{tj} = b_{sj}$, and $a_{ij} = b_{ij}$ for all $i \in N - \{s, t\}]$, and [not$(A \sim B)$]. Without loss of generality, let $s = 1, t = 2$, and $j = 2$. Then

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2m} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3m} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nm} \end{pmatrix}, \quad B = \begin{pmatrix} a_{11} & a_{22} & a_{13} & \cdots & a_{1m} \\ a_{21} & a_{12} & a_{23} & \cdots & a_{2m} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3m} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nm} \end{pmatrix},$$

and not$(A \sim B)$.

Note that, $\succeq$ can be derived through a column-first two stage procedure based on $(h_1, \ldots, h_m; \succeq^{cr})$, where $\succeq^{cr}$ is positively responsive for every $j \in M$. From not$(A \sim B)$, we must have that not$[(h_1(a_{\bullet 1}), h_2(a_{\bullet 2}), h_3(a_{\bullet 3}), \ldots, h_m(a_{\bullet m})) \sim^{cr} (\mathrm{h}_1(b_{\bullet 1}), h_2(b_{\bullet 2}), h_3(b_{\bullet 3}), \ldots, h_m(b_{\bullet m}))]$. Note that $a_{\bullet i} = b_{\bullet i}$ for $i = 1, 3, \ldots, m$. Since $(h_1, \ldots, h_m; \succeq^{cr})$ is such that $\succeq^{cr}$ is positively responsive for every $j \in M$ and not$(A \sim B)$, it must be true that $h_2(a_{\bullet 2}) \neq h_2(b_{\bullet 2})$. Since both $h_2(a_{\bullet 2})$ and $h_2(b_{\bullet 2})$ are real numbers, it must be the case that either $[h_2(a_{\bullet 2}) > h_2(b_{\bullet 2})]$ or $[h_2(a_{\bullet 2}) < h_2(b_{\bullet 2})]$. Without loss of generality, let $[h_2(a_{\bullet 2}) > h_2(b_{\bullet 2})]$. Then, $A \succ B$ by positive responsiveness of $\succeq^{cr}$.

Consider $A', B' \in V$ such that

$$A' = \begin{pmatrix} a_{21} & a_{22} & a_{23} & \cdots & a_{2m} \\ a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3m} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nm} \end{pmatrix}, \quad B' = \begin{pmatrix} a_{21} & a_{12} & a_{23} & \cdots & a_{2m} \\ a_{11} & a_{22} & a_{13} & \cdots & a_{1m} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3m} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nm} \end{pmatrix},$$

that is, A' is obtained from A by interchanging the first two rows while keeping all other rows unchanged, and B' is obtained from B by interchanging the first two rows while keeping all other rows intact. By anonymity, $A' \sim A$ and $B' \sim B$. $A' \succ B'$ now follows from the transitivity of $\succeq$ and $A \succ B$.

Note that $\succeq$ can be derived through a column-first two stage procedure based on $(h_1, \ldots, h_m; \succeq^{cr})$, where $\succeq^{cr}$ is positively responsive for every $j \in M$. From $A' \succ B'$, we must have $(h_1(a'_{\bullet 1}), h_2(a'_{\bullet 2}), h_3(a'_{\bullet 3}), \ldots, h_m(a'_{\bullet m})) \succ^{cr} (\mathrm{h}_1(b'_{\bullet 1}), h_2(b'_{\bullet 2}), h_3(b'_{\bullet 3}), \ldots, h_m(b'_{\bullet m}))$. Note that $a'_{\bullet i} = b'_{\bullet i}$ for $i = 1, 3, \ldots, m$. It then follows that $h_2(a'_{\bullet 2}) > h_2(b'_{\bullet 2})$. Note, however, that $a'_{\bullet 2} = b_{\bullet 2}$ and $b'_{\bullet 2} = a_{\bullet 2}$. Therefore, $h_2(b_{\bullet 2}) > h_2(a_{\bullet 2})$, a contradiction. ■

Proposition 2 *Let $\succeq$ be an overall group deprivation ranking, which satisfies invariance of matrix ranking to permutations of individuals and non-invariance. Then $\succeq$ cannot be derived through any column-first two-stage procedure based on $(h_1, ..., h_m; \succeq^{cr})$ such that, for every $j \in M$, $\succeq^{cr}$ is positively responsive.*

Proof. The proof of Proposition 2 is similar to Proposition 1 and we omit it. ■

As noted earlier, since equity principle I, as well as equity principle II, implies non-invariance, Propositions 3 and 4 below follow from Propositions 1 and 2.

Proposition 3 *Let $\succeq$ be a transitive overall group deprivation ranking, which satisfies anonymity and either equity principle I or equity principle II. Then $\succeq$ cannot be derived through any column-first two-stage procedure based on $(h_1, ..., h_m; \succeq^{cr})$ such that, for every $j \in M$, $\succeq^{cr}$is positively responsive.*

Proposition 4 *Let $\succeq$ be an overall group deprivation ranking, which satisfies invariance of matrix ranking to permutations of individuals and either equity principle I or equity principle II. Then $\succeq$ cannot be derived through any column-first two-stage procedure based on $(h_1, ..., h_m; \succeq^{cr})$ such that, for every $j \in M$, $\succeq^{cr}$is positively responsive.*

Proposition 5 *Let $\succeq$ be a transitive overall group deprivation ranking which satisfies non-invariance. Then $\succeq$ cannot be derived through any column-first two-stage procedure based on $(h_1, ..., h_m; \succeq^{cr})$ such that, for every $j \in M$, h_j satisfies symmetry.*

Proof. Let $\succeq$ be an OGDR satisfying non-invariance. Then there exist $A, A' \in V$, $s, t \in N$ and $j \in M$ such that $a_{\bullet k} = a'_{\bullet k}$ for all $k \in M - \{j\}$, $a_{ij} = a'_{ij}$ for all $i \in N - \{s, t\}$, $a_{sj} = a'_{tj}$ and $a_{tj} = a'_{sj}$, and not$(A \sim B)$. By following a similar argument as in the proof for Proposition 1, it must be true that (either $A \succ A'$ or $A' \succ A$). Without loss of generality, let $A \succ A'$. Suppose to the contrary that $\succeq$ can be derived through some column-first two-stage procedure based on $(h_1, ..., h_m; \succeq^{cr})$, such that, for every $j \in M$, h_j satisfies symmetry. Since h_j is symmetric, it must be true that $h_j(a_{\bullet j}) = h_j(a'_{\bullet j})$. Note that $a_{\bullet k} = a'_{\bullet k}$ for all $k \in M - \{j\}$. We then obtain $h_k(a_{\bullet k}) = h_k(a'_{\bullet k})$ for all $k \in M$, which implies that $A \sim A'$, which is a contradiction. ∎

Note that equity principle I, as well as equity principle II, are stronger requirement than non-invariance. Proposition 5 therefore implies that an OGDR which satisfies either equity principle I or equity principle II cannot be derived through any column-first two-stage procedure based on $(h_1, ..., h_m; \succeq^{cr})$ such that, for every $j \in M$, h_j satisfies symmetry. It is, however, possible to prove the following result (Proposition 6), which is somewhat stronger than this implication .

Proposition 6 *If a transitive overall group deprivation ranking satisfies either equity principle I or equity principle II, then it cannot be derived through any column-first two-stage procedure based on* $(h_1, ..., h_m; \succeq^{cr})$ *such that, for some* $j \in M$, h_j *satisfies symmetry.*

Since the proof of Proposition 6 is similar to that of Proposition 5, we omit it.

6 An extension

So far we have focussed on the problem of measuring deprivation. Our analysis can, however, be readily extended to the measurement of living standards more generally. To do this, one needs to interpret R^j as the set of different levels of achievements in terms of attribute f_j, a higher number in R^j denoting a higher level of achievement with respect to attribute f_j. The requirement that $R^j \subseteq [0, 1]$ was not drawn upon in our proofs and can be laid aside. As before, we shall have $n \times m$ matrices where each real number figuring in the matrix indicates the level of achievement of some individual $i \in N$ in terms of some attribute f_j . The binary relation $\succeq$ defined over all such matrices will now be interpreted in terms of living standards: for all standard of living matrices, A and A', $A \succeq A'$ will now denote that A represents a higher standard of living for the society than A'. The formal definitions of all the properties, excepting equity principles I and II , will remain intact. Changes in the formal definitions of the two equity principles are, however, necessary to retain the plausibility of the two principles under the changed interpretation of $\succeq$. These changes are of the most straightforward type possible, simply reflecting the fact that lower levels of deprivation must be interpreted as higher levels of the living standard. The two equity principles may now be reformulated as follows.

Equity Principle I (for comparing living standards): $\succeq$ satisfies *equity principle I for comparing living standards* if and only if, for all $A, B, C \in V$, for all $s, t \in N$, and for all $k \in M$, if $[(a_{i\bullet} = b_{i\bullet} = c_{i\bullet}$ for all $i \in N - \{s, t\})$, $(a_{s\bullet} > a_{t\bullet}$, $a_{sk} = a_{tk}$, $a_{t\bullet} = b_{t\bullet}$, and $a_{s\bullet} = c_{s\bullet})$, and ($a_{s\bullet}$ and $b_{s\bullet}$ are $k-$variants, $a_{t\bullet}$ and $c_{t\bullet}$ are $k-$variants, and $b_{sk} = c_{tk} > a_{sk} = a_{tk})]$, then $C \succ B$.

Equity Principle II (for comparing living standards): $\succeq$ satisfies *equity principle II for comparing living standards* if and only if, for all $A, B \in V$, for all $s, t \in N$, and for some $k \in M$, if $[(a_{i\bullet} = b_{i\bullet}$ for all $i \in N - \{s, t\})$ and (for all $j \in M - \{k\}$, $a_{sj} = b_{sj} > a_{tj} = b_{tj}$ and $a_{sk} = b_{tk} > a_{tk} = b_{sk})]$, then $B \succ A$.

It is evident that, given these revised versions of the two equity principles, the exact counterparts of Propositions 1 through 4 hold in the context of standard of living comparisons. Thus, the force of the negative results remains intact when we switch from the measurement of deprivation to the measurement of living standards.

7 Conclusion

In this paper, we have discussed a difficulty associated with a commonly used procedure (the column-first two-stage procedure in our terminology) for measuring a society's deprivation or living standard. The difficulty lies in its inherent inability to reconcile two highly plausible normative properties: anonymity and non-invariance, that an overall indicator of a society's deprivation or living standard may be required to have. Anonymity requires that, in constructing such an overall indicator, individuals in a society should be treated symmetrically. Non-invariance, on the other hand, imposes a requirement for the indicator to be equity-sensitive (in a minimal sense appropriate to a multi-attribute context).

In an earlier paper, Dutta, Pattanaik and Xu (2003) showed that, under some mild conditions, the column-first two-stage procedure and the row-first two-stage procedure would not yield the same overall indicator except when one used the sum, $\sum_{i \in N, j \in M} a_{ij}$, to rank alternative deprivation matrices. In that paper, the authors assumed that the two procedures were undertaken employing aggregation functions of a similar form. No such constraint has been imposed in the current exercise.

8 References

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