

# Field Flows of Dark Energy

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Scalar field dark energy evolving from a long radiation- or matter-dominated epoch has characteristic dynamics. While slow-roll approximations are invalid, a well defined field expansion captures the key aspects of the dark energy evolution during much of the matter-dominated epoch. Since this behavior is determined, it is not faithfully represented if priors for dynamical quantities are chosen at random. We demonstrate these features for both thawing and freezing fields, and for some modified gravity models, and unify several special cases in the literature.

## I. INTRODUCTION

The acceleration of the cosmic expansion can be explained by a multitude of models suggested in the literature but few or none derived from first principles. We look, therefore, for common characteristics among classes of models and for physically motivated generic behaviors. One example of effective characterization is the designation of the thawing and freezing patterns of dark energy evolution [1]. These involve scalar fields leaving or approaching the behavior of a cosmological constant in the dark energy equation of state  $w$  and its time variation  $w' = dw/d \ln a$ . Here we seek further unifying features in the evolution of the dark energy field.

For inflation a slow-roll approach, neglecting higher time derivatives in the field evolution, can be used. This is not valid for dark energy, even at early times when  $w$  may be near  $-1$  and the dark energy is a small contribution to the total energy density, and we must develop a different formalism. Several suggestions for specific approximations exist in the literature and we will see that these can be unified into a single approach. This can also be extended in part to the freezing class of fields, traditionally difficult to characterize generically.

One key conclusion is that one must take into account that our universe is old: the scalar field has been evolving for many Hubble times in a background that was initially radiation-dominated then matter-dominated. This defines particular initial conditions and determines the dynamical behavior. Employing random, Monte Carlo initial conditions may lead to underrepresentation of thawing or freezing behavior, due to neglecting the physics of a long evolution.

In §II we show why slow roll conditions are invalid for dark energy and discuss new methods for evaluating the dynamics. We provide a specific example of a complete solution in §III. Identifying a particular characteristic combination of parameters in §IV, we show how the physics constrains the dark energy evolution within the classes of dark energy. We investigate extending this relation to some modified gravity scenarios in §V.

## II. FIELD DYNAMICS

Cosmic acceleration is given by the condition  $\ddot{a} > 0$  on the scale factor  $a$ , where dots represent time derivatives. For a state of near exponential expansion, the Hubble parameter  $H = \dot{a}/a$  follows

$$|\dot{H}/H^2| \ll 1, \quad (1)$$

so the Hubble parameter is nearly constant. If the scalar field  $\phi$  provides the dominant contribution to the expansion, as it does for inflation, then the potential must be nearly constant, leading to an equivalent condition  $|V_\phi/V| \ll 1$  where  $V_\phi = dV/d\phi$  and we work in units with  $8\pi G = 1$ . This is often referred to as the (first) slow-roll parameter. Another implication is that the field acceleration  $\ddot{\phi} \ll 3H\dot{\phi}$  and so can be neglected in the equation of motion, or Klein-Gordon equation.

However, these slow-roll conditions on  $V$  or  $\ddot{\phi}$  rely on scalar field domination, and this is not valid for the dark energy field during its evolution, even today (since the matter density  $\Omega_m$  is not negligible). In [2], ratios of terms in the Klein-Gordon equation were defined as

$$X \equiv \frac{\ddot{\phi}}{H\dot{\phi}}, \quad (2)$$

$$Y \equiv \frac{\ddot{\phi}}{V_\phi} = \frac{-X}{X+3}, \quad (3)$$

and shown today (when  $\Omega_m \approx 0.28$ ) to be far from a slow-roll regime. In Fig. 1 we show explicitly that even for a thawing model that has equation of state today  $w_0 = -0.9$ , near the de Sitter  $w = -1$  value, no term in the equation of motion is negligible compared to other terms, and also that  $|V_\phi/V| \ll 1$  does not hold, at any point in the evolution.

### A. General

With all terms retained, the Klein-Gordon equation for the scalar field  $\phi$  is given by

$$\ddot{\phi} + 3H\dot{\phi} = -V_\phi. \quad (4)$$

The ratios  $X$ ,  $Y$  of the terms are of interest for several reasons: 1) they indicate whether the potential driving

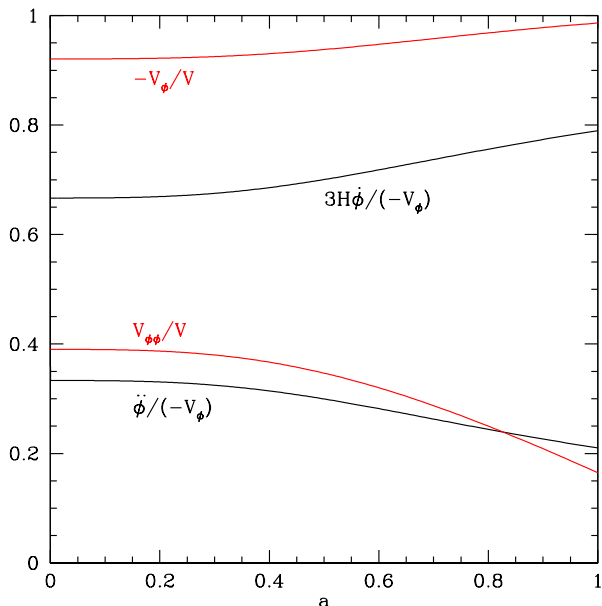


FIG. 1: Even a model evolving only from  $w(z \gg 1) = -1$  to  $w_0 = -0.9$  (here a PNGB model with  $f = 1$ , see §II C) cannot be described by a slow-roll formalism. Generically, none of the ratios of terms in the scalar field equation of motion, or the potential derivatives, is small enough to permit neglect of a term or to allow use of the slow-roll approximation.

term or Hubble friction term is more influential (as used in [1] to motivate thawing and freezing behaviors), 2) the ratios can be rephrased in terms of the tracks in the equation-of-state phase space  $w'$ - $w$ , and 3) under certain conditions, such as tracking (constant  $w$  determined by the background expansion [3, 4]),  $X$  and  $Y$  will be constant.

To begin, we obtain an implicit solution by multiplying Eq. (4) through by an integrating factor  $\exp\{3 \int_{\tau}^t dt' H(t')\}$ , where the lower limit is arbitrary. Integrating from some early time  $t_1$  to a later time  $t_2$  yields

$$\begin{aligned} \dot{\phi}(t_2) &= e^{3 \int_{t_2}^{t_1} dt' H(t')} \dot{\phi}(t_1) \\ &= - \int_{t_1}^{t_2} dt e^{3 \int_{t_2}^t dt' H(t')} V_{\phi}(t). \end{aligned} \quad (5)$$

Note that

$$e^{3 \int_{t_2}^{t_1} dt' H(t')} = (a_1/a_2)^3, \quad (6)$$

so for  $t_1$  early enough relative to  $t_2$  this factor is very small and the term involving  $\dot{\phi}(t_1)$  is irrelevant. That is, the initial speed does not matter due to the high Hubble drag in the early universe. This leaves (taking the scale

factor  $a_1 \rightarrow 0$ )

$$\begin{aligned} \dot{\phi}(t) &= - \int_{-\infty}^t dt' e^{3 \int_t^{t'} dt'' H(t'')} V_{\phi}(t') \\ &= - \int_0^a \frac{da'}{a' H(a')} \left(\frac{a'}{a}\right)^3 V_{\phi}(a'). \end{aligned} \quad (7)$$

Note that the ratio of the friction term and the driving term in the Klein-Gordon equation can be evaluated as

$$\frac{3H\dot{\phi}}{V_{\phi}} = -3 \int_0^a \frac{da'}{a'} \left(\frac{a'}{a}\right)^3 \frac{H(a) V_{\phi}(a')}{H(a') V_{\phi}(a)}. \quad (8)$$

It is also convenient to consider the ratio of the field acceleration to the driving term,

$$\begin{aligned} Y \equiv \frac{\ddot{\phi}}{V_{\phi}} &= \frac{-V_{\phi} - 3H\dot{\phi}}{V_{\phi}} \\ &= 3 \int_0^a \frac{da'}{a'} \left(\frac{a'}{a}\right)^3 \left[ \frac{H(a) V_{\phi}(a')}{H(a') V_{\phi}(a)} - 1 \right]. \end{aligned} \quad (9)$$

This is a formal solution because both  $H$  and  $V_{\phi}$  depend on  $\phi$  itself.

## B. Asymptotic

Some instructive cases can be evaluated directly. Consider a model where  $V_{\phi}$  is nearly constant in an epoch dominated by a component with equation of state  $w_b$ , so that  $H \sim a^{-3(1+w_b)/2}$ . This is generic for the thawing class. Evaluating Eq. (9) gives

$$Y_{\infty} = -\frac{1+w_b}{3+w_b}, \quad (10)$$

or  $Y_{\infty} = -1/3$  for the matter dominated era,  $Y_{\infty} = -2/5$  for the radiation dominated era.

## C. Thawing Evolution

The evolution of the thawing field can be determined iteratively from Eq.(7), taking into account that initially the matter contribution dominates in  $H$  and that the field moves relatively little from its initial value. Taylor expanding the potential around that initial field value (which without loss of generality can be set to zero),

$$\begin{aligned} V(\phi) &= V_i + V_{\phi i} \phi + \frac{1}{2} V_{\phi \phi i} \phi^2 + \dots \\ H^2 &= \frac{1}{3} [\rho_m(a) + \rho_{\phi}(a)] = \frac{1}{3} [\rho_m(a) + V_i + \dots] \end{aligned} \quad (11)$$

$$H^2 = \frac{1}{3} [\rho_m(a) + \rho_{\phi}(a)] = \frac{1}{3} [\rho_m(a) + V_i + \dots] \quad (12)$$

where the subscript  $i$  denotes the value at  $\phi = \phi_i = 0$ ,  $\rho_m$  is the matter density, and  $\rho_\phi$  is the dark energy density<sup>1</sup>.

At lowest order we find

$$\dot{\phi}(a) = -\frac{2}{9} \frac{V_{\phi i}}{H_0 \Omega_m^{1/2}} a^{3/2} \quad (13)$$

$$\phi(a) = -\frac{2}{27} \frac{V_{\phi i}}{H_0^2 \Omega_m} a^3 = -\frac{2}{9} \frac{V_{\phi i}}{\rho_m(a)}. \quad (14)$$

In second order

$$\dot{\phi}(a) = -\frac{2}{9} \frac{V_{\phi i} a^{3/2}}{H_0 \Omega_m^{1/2}} \left[ 1 - \frac{3}{5} \left( \frac{V_i}{2\rho_m(a)} + \frac{2V_{\phi\phi i}}{9\rho_m(a)} \right) \right] \quad (15)$$

$$\phi(a) = -\frac{2}{9} \frac{V_{\phi i}}{\rho_m(a)} \left[ 1 - \frac{2}{5} \left( 1 + \frac{V_{\phi\phi i}}{6V_i} \right) \frac{V_i}{\rho_m(a)} \right]. \quad (16)$$

Thus the criterion for the validity of the first order solution for  $\phi$  is mainly that  $V_i/\rho_m(a) \ll 1$ , i.e. dark energy does not dominate, as expected. There are no slow roll  $V_\phi/V \ll 1$  or  $V_{\phi\phi}/V \ll 1$  conditions or other stringent condition on the potential derivatives. Indeed one could argue that having  $V$  and its derivatives be of the same order in Planck units is natural, as in the technically natural PNCB model [5] with  $V = V_0 [1 + \cos(\phi/f)]$  where  $f$  is a symmetry energy scale of order unity. If  $V_{\phi\phi}/V \gg 1$ , we see from Eqs. (15)–(16) we might have to reevaluate the field expansion; see for example the next subsection.

From these expressions we can calculate  $Y$  directly, substituting in the first order corrections to  $H$ ,  $\phi$ , and  $V$ :

$$Y(a) = -\frac{1}{3} + \left( \frac{2}{15} + \frac{8}{135} \frac{V_{\phi\phi i}}{V_i} \right) \frac{V_i}{\rho_m(a)}. \quad (17)$$

The parameter  $Y$  starts off constant, at  $Y_\infty$ , only changing as the scalar field rolls sufficiently far or its energy density starts to dominate the Hubble expansion. This behavior is evident in Fig. 1. Also note that  $Y$  is not particularly small, and hence no one term dominates in the Klein-Gordon equation and ignoring  $\ddot{\phi}$  is invalid.

It is similarly straightforward to determine the leading correction to the dark energy equation of state  $w$  by substituting the first order expressions into

$$w = \frac{\frac{1}{2}\dot{\phi}^2 - V}{\frac{1}{2}\dot{\phi}^2 + V} \quad (18)$$

to find

$$1 + w = \frac{4}{27} \frac{V_{\phi i}^2}{V_i \rho_m(a)}. \quad (19)$$

<sup>1</sup> One can avoid specifying  $H$  explicitly by rewriting the Klein-Gordon equation as

$$\phi'' + \frac{3}{2} (1 + \Omega_\phi - (1 + w)\Omega_\phi) \phi' + 3(1 - \Omega_\phi) \frac{V_\phi(\phi)}{\rho_m(a)} = 0,$$

where a prime denotes  $d/d \ln a$ . Since  $\Omega_\phi$ ,  $1 + w$ , and  $V_\phi/\rho_m(a)$  are all small at early times, this form is convenient for identifying the order of each term.

The time variation of the equation of state is

$$w' = 3(1 + w), \quad (20)$$

the result obtained by [1]. That is, the evolution of the field in a matter-dominated universe fixes the asymptotic behavior of the dark energy for such thawing fields.

#### D. Non-analytic Potentials

As alluded to above, when the derivatives of the potential become large at a point, the Taylor expansion approach can break down. Consider a class of potentials with a singularity at some  $\phi_*$ , e.g.

$$V = V_* - V_n (\phi - \phi_*)^n, \quad (21)$$

with  $n$  non-integral and positive (we discuss negative  $n$  below) and  $\phi > \phi_*$ . This represents an inverted (concave) potential with the field rolling away from a maximum at  $\phi = \phi_*$  (eventually to negative infinity but that will not concern us regarding early time behavior).

We can find a thawing solution by trying

$$\phi = \phi_* + A a^\nu \quad (22)$$

in the Klein-Gordon equation during the matter-dominated era (easily generalized to other background evolution), or equivalently into Eq. (7). The result is

$$\nu = \frac{3}{2 - n} \quad (23)$$

$$A = \left[ \frac{V_n}{H_0^2 \Omega_m} \frac{2n(2 - n)^2}{9(4 - n)} \right]^{1/(2 - n)}. \quad (24)$$

The ratio of the field acceleration to potential slope terms, the equation of state, and derivative of the equation of state become

$$Y = \frac{-n}{4 - n} \quad (25)$$

$$1 + w = \frac{H_0^2 \Omega_m}{V_*} A^2 \nu^2 a^{3n/(2 - n)} \quad (26)$$

$$w' = \frac{3n}{2 - n} (1 + w) \quad (27)$$

as the field starts to roll.

Note that for  $n > 2$ , in this Ansatz the field starts with a large kinetic energy, or equivalently  $w$  is positive and large, so we restrict to  $0 \leq n < 2$  (for  $n = 2$  the Ansatz fails). As  $n \rightarrow 2$ , the dark energy shoots away from  $w = -1$ , acting more like sublimation than thawing. For the two integer values of  $n$  within this range,  $n = 0, 1$ , the potential has no singularities and these results agree with the Taylor expansion of the previous subsection. As well, if the field starts frozen away from the singularity then Taylor expanding the field works and the  $w' = 3(1 + w)$  trajectory is the early-time solution.

If instead we consider negative  $n$ , we can ignore the  $V_*$  term at early times and (making the potential convex) we end up with a tracking field [6]. The equations for  $v$ ,  $A$ , and  $Y$  above still hold but now

$$w_{n<0} = -\frac{2}{2-n}. \quad (28)$$

and thus  $w' = 0$ .

### E. Unifying Relations

Not only the dynamical trajectories but the relations between the dark energy density and the equation of state have characteristic behavior for each class of models. Combining Eq. (19) with the first order expression for the dark energy density,

$$\Omega_\phi = \frac{V}{\rho_m(a)}, \quad (29)$$

shows that

$$\Omega_\phi = \frac{27}{4} \frac{1+w}{\lambda^2}, \quad (30)$$

where  $\lambda = -V_\phi/V$ , here considered to lowest order in an expansion in  $\phi$ . (Note that generally  $\lambda^2 \equiv 2\epsilon$ , where  $\epsilon$  is the conventional first slow-roll parameter.) We return to this relation between the parameters in §IV; now we simply explore some implications of the existence of such a relation.

Note that to first order the relation between  $\Omega_\phi$  and  $w$  does not depend on higher derivatives of  $V$  than the first derivative, and the relation between  $w'$  and  $w$  does not depend on  $V$  or its derivatives at all to this order. This is part of the unifying power of such an analysis, that *any* scalar field dominated by the background Hubble friction must behave in a simple, determined manner.

We can now compare our result to other approaches in the literature that assumed specific potentials or approximations. The simplest case is the limit  $V = \text{constant}$  or  $\lambda \rightarrow 0$ . This is of course just the cosmological constant and the equation of state never leaves  $w = -1$ . (Skating models [7, 8] have a large initial motion  $\dot{\phi}_i$  but this quickly redshifts away, as  $a^{-3}$ , and the field comes to rest at  $w = -1$ .) Next is the linear potential [9, 10], where  $V_\phi = \text{constant}$ , discussed in the next section. In general, though, potentials will have higher order derivatives that are not zero or depend nontrivially on the first derivative.

Thawing models have been studied with approximations, such as taking  $\lambda = \text{constant}$  (turning the exponential potential's tracking behavior [11, 12] into thawing by starting it from a frozen state) but approximating  $\Omega_\phi$  or  $w$  [13]. Indeed [13] noted a version of the relation (30) then holds asymptotically. Another parameter investigated in a first order expansion about a constant value is  $\kappa \equiv -\lambda/(1+X/3)$  [14]. Explicitly incorporating Eq. (20), [15] expanded in the energy density  $\Omega_\phi$  about

the asymptotic solution to form an ‘‘algebraic thawing’’ model, which is actually valid to second order. Interestingly, current data show the algebraic thawing model is statistically a better fit than  $\Lambda\text{CDM}$  [16].

All of these cases follow the unifying first order solution (30) but diverge at higher order. Each one basically chooses different ways to truncate or close the hierarchy of higher order equations. To understand how quickly deviation from the unified solution, or of the validity of the field expansion approach, occurs, we note it requires particular combinations of  $\Omega_\phi$  and  $1+w$  to be much smaller than one. So we do not expect these analytic field solutions to be valid up to the present. Nevertheless, in many cases they are good approximations until surprisingly recent times; e.g. for the model of Fig. 1 the relationship (30) holds to 3% (8%) until  $z = 2$  (1). We discuss this further in §IV.

### III. DARK ENERGY DENSITY AND THE LINEAR POTENTIAL

In addition to understanding the dark energy equation of state, we are often interested in the observables directly, such as the Hubble parameter or the distance-redshift relation. These involve the dark energy density, given by the sum of the potential  $V(\phi)$  and kinetic energy  $\dot{\phi}^2/2$  discussed in the previous section.

The dark energy density at some epoch relative to its current value is given by the ratio

$$\xi \equiv \frac{\rho_\phi(a)}{\rho_{\phi,0}} = \frac{\frac{1}{2}\dot{\phi}^2 + V}{\frac{1}{2}\dot{\phi}_0^2 + V_0}, \quad (31)$$

where the subscript 0 indicates current values. The current value for the equation of state is

$$w_0 = \frac{\frac{1}{2}\dot{\phi}_0^2 - V_0}{\frac{1}{2}\dot{\phi}_0^2 + V_0}. \quad (32)$$

Differentiating the dark energy density with respect to time and using the Klein-Gordon equation, we find

$$\frac{d\xi}{dt} = -\frac{3H\dot{\phi}^2}{V_0} \frac{1-w_0}{2} \quad (33)$$

or

$$\frac{d\xi}{da} = -\frac{3\dot{\phi}^2}{V_0} \frac{1-w_0}{2a}. \quad (34)$$

To remove the explicit appearance of  $\dot{\phi}$  we use

$$1+Y = -\frac{3H\dot{\phi}}{V_\phi}, \quad (35)$$

and to eliminate  $V_0$  we employ

$$\frac{\dot{\phi}_0^2}{V_0} = \frac{V_{\phi,0}^2(1+Y_0)^2}{9H_0^2V_0} = \frac{2(1+w_0)}{1-w_0}, \quad (36)$$

with the result

$$\frac{d\xi}{da} = -3 \frac{1+w_0}{a} \frac{H_0^2}{H^2} \frac{V_\phi^2}{V_{\phi,0}^2} \left( \frac{1+Y(a)}{1+Y_0} \right)^2. \quad (37)$$

Note that the Hubble parameter is given by

$$H^2/H_0^2 = \Omega_{m0} a^{-3} + (1 - \Omega_{m0}) \xi(a) \quad (38)$$

so the density  $\xi(a)$  is an implicit function only of the scale factor  $a$  – except for the dependence on the potential slope  $V_\phi(\phi(a))$ . This dependence occurs for  $Y(a)$  as well through Eq. (9).

In the special case of a potential linear in  $\phi$ , the quantity  $V_\phi$  is constant. This was treated by [9] as one of the first dark energy models and more recently as a textbook example by [17]. Then the two equations (37) and (9) involve only the independent variable  $a$  and the dependent variable  $\xi$ . (This will be a good approximation for any potential with a slope varying with  $a$  sufficiently slowly.) Despite being coupled integro-differential equations, they are actually simple to solve numerically. One starts with an initial approximation for the Hubble parameter with  $\xi$  set to unity for all values of  $a$ , that is initially  $H(a)$  is appropriate to a cosmological constant. In this case, two parameters fix the solution:  $\Omega_\Lambda = 1 - \Omega_{m0}$  and  $w_0$ .

The value of  $Y(a)$  is not known until the solution is found iteratively, but for the first iteration the cosmological constant Ansatz

$$H(a)/H_0 = \sqrt{(1 - \Omega_\Lambda)a^{-3} + \Omega_\Lambda} \equiv h(a) \quad (39)$$

gives

$$1 + Y(a) = \frac{1 - \Omega_\Lambda}{\Omega_\Lambda} \frac{h(a)}{a^3} \left[ \frac{h(a)}{h(a)^2 - \Omega_\Lambda} + \frac{1}{2\Omega_\Lambda^{1/2}} \ln \frac{h(a) - \Omega_\Lambda^{1/2}}{h(a) + \Omega_\Lambda^{1/2}} \right]. \quad (40)$$

The dark energy density calculated with the first order quantities is quite accurate, even when  $w_0$  is not very close to  $-1$ , and convergence upon iteration is extremely rapid. By taking a further derivative of  $\xi$ , one can also derive the relation for the time variation of the dark energy equation of state followed by the linear potential,  $w'_0 \equiv dw/d \ln a(a=1) \approx -1.3(1+w_0)$ . Results found iteratively for the linear potential are shown for  $Y(a)$  and the density  $\xi(a) = \rho_\phi(a)/\rho_{\phi,0}$  in Figs. 2 and 3.

#### IV. FLOW PARAMETER

As we saw in §II E, evolution away from the frozen state involves a synchronized deviation in both  $\Omega_\phi$  and  $1+w$  away from zero in a particular proportional or scaling manner that persists to quite late times. This imposes a constraint that phenomenological models must obey, and parameter priors or Monte Carlo scans must

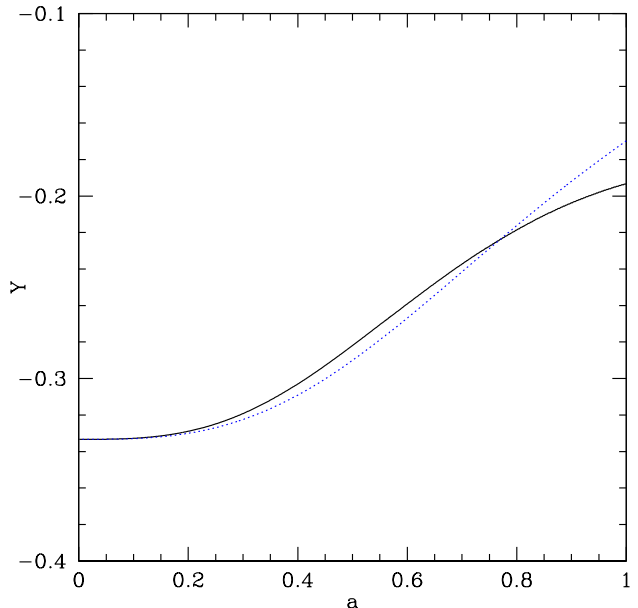


FIG. 2: The acceleration response function  $Y(a)$  for any thawing field has common characteristics. Here we plot the case of a linear potential with  $w_0 = -0.777$ ,  $\Omega_{\phi,0} = 0.76$  (compare to the PNGB case shown in Fig. 1). The solid curve is the exact solution, while the dotted curve shows the analytical approximation for  $w_0 = -1$ ,  $\Omega_{\phi,0} = 0.76$ , which is used as the starting point for the iterative solution.

take into account. Other forms of dark energy, such as modified gravity, might also follow similar scaling relations (see, e.g., [18]) and we investigate this in the next section. Such constraints are important as we lack first principles theories for dark energy, as scalar fields or modified gravity.

Motivated by the relation in Eq. (30), we define a new quantity we call the flow parameter,

$$F \equiv \frac{1+w}{\Omega_\phi \lambda^2}, \quad (41)$$

where  $\lambda = -V_\phi/V$ ,  $\Omega_\phi$ , and  $w$  are all functions of time. This is related to the friction vs. potential slope term ratio  $1+Y$  by

$$F = \frac{1}{12} (1-w)^2 (1+Y)^2, \quad (42)$$

and to the phase space evolution of dark energy as

$$w' = -3(1-w^2)[1 - (3F)^{-1/2}]. \quad (43)$$

These relations follow generally from the Klein-Gordon equation and from the definition of  $Y$ , for all  $a$  and without any requirement for matter domination. We note that  $F$  holds close to its high-redshift, asymptotic value for a substantial part of the evolution, even when  $\Omega_\phi$  is

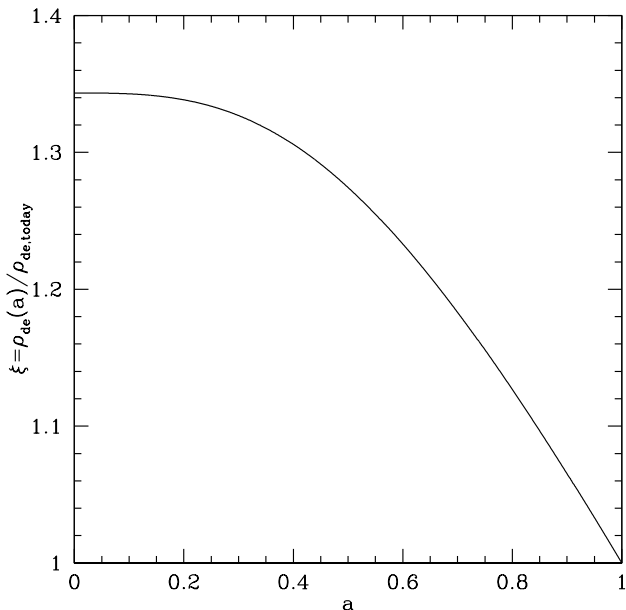


FIG. 3: The dark energy density relative to the present,  $\xi(a)$ , changes little for thawing fields. The curve shows the solution for the linear potential with  $w_0 = -0.777$ ,  $\Omega_{\phi,0} = 0.76$ ; note the density used in the first iteration (dotted curve in Fig. 2) is just the cosmological constant behavior, unity for all scale factors. By contrast, for a typical tracking field  $\xi(a \ll 1) \gg 1$ , e.g.  $\xi(a = 10^{-3}) \approx 10^4$  for  $V \sim \phi^{-2}$ .

not much less than one. Figure 4 shows the evolution of  $F$  for thawing and freezing (in fact modified gravity) examples.

As seen in Eq. (30), for any thawing model evolving in a matter dominated universe the field flow begins with

$$F = 4/27 \quad (\text{thawer}). \quad (44)$$

This is equivalent to the condition  $w' = 3(1+w)$  at early times, but is preserved for more of the evolution. The other major class of scalar field models is tracking models. These have constant equation of state at early times in a background dominated universe, and from Eq. (43) this implies

$$F = 1/3 \quad (\text{tracker}). \quad (45)$$

This can be viewed as  $Y(a \ll 1) = (1 + w_\infty)/(1 - w_\infty)$ , where  $w_\infty$  is the high redshift dark energy equation of state. The non-analytic thawing solutions with  $0 < n < 1$  interpolate between the regular thawing case (cf.  $n = 1$ ) and tracking models ( $n < 0$ ):

$$F = \frac{4}{3} \left( \frac{2-n}{4-n} \right)^2. \quad (46)$$

Again, there is a substantial range of validity for  $F$  being constant even as  $\Omega_\phi$  grows to an appreciable fraction.

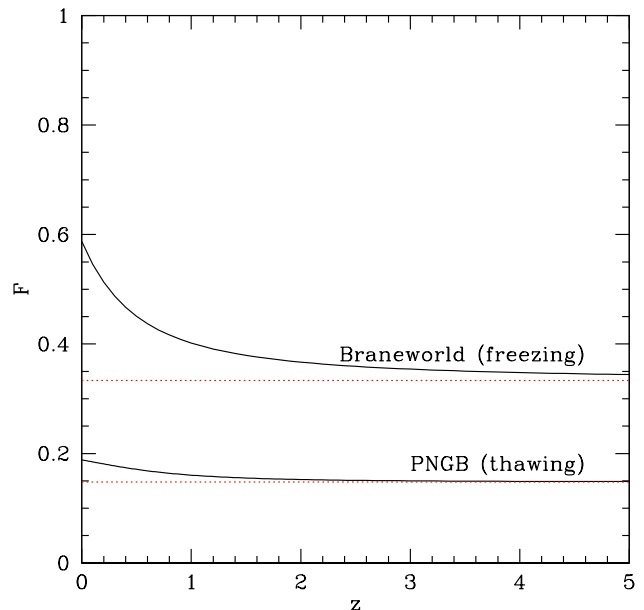


FIG. 4: A particular combination of dark energy parameters,  $F = (1+w)/[\Omega_\phi(dV/d\phi)^2]$ , has fixed high redshift behavior for each class of models,  $4/27$  for thawing and  $1/3$  for freezing (red, dotted lines). Also, it is nearly conserved during the long evolution until dark energy begins to dominate.

This relation imposes a particular high redshift behavior on a whole class of phenomenological models. Effectively, these flow behaviors define physical priors that must be included in any Monte Carlo simulation of scalar field evolution. As shown, they hold until quite recent epochs,  $z \gtrsim 1-2$ , not just for  $a \ll 1$ . Assuming instead a random dynamical state, rather than that determined during the many matter-dominated e-folds of expansion, is equivalent to a fine tuning of the scalar field initial conditions to avoid the natural evolutionary path. See [14] for an interesting analysis of the effect of the difference in priors between those uniform in  $w'-w$  and those uniform in a version of the field parameters.

## V. MODIFIED GRAVITY AND EXPANSION

The accelerating expansion of the universe could indicate a deviation from general relativity rather than the presence of a new scalar field. It is of great interest to explore beyond scalar fields and see whether the evolution of expansion and matter density growth parameters probes modified gravity. For example, [18, 19] motivated a deviation from the general relativistic growth behavior scaling as  $\Omega_w(a)$ , or  $a^3$  in many cases, while [20] assumed a variation as  $a$ , and [21] left it as a fit parameter  $a^s$ . As seen in [21], the observational consequences for the deviation index  $s$  are significant. This is too large and

complex a subject to address here, but we examine how some modified gravity theories respond to the parameters treated above for scalar field explanations of dark energy.

For gravity extended beyond Einstein relativity, the parameters  $\Omega_\phi$ ,  $w$ ,  $V$ , etc. are effective quantities but we can still define them in terms of modifications of the Friedmann equation from the general relativistic, matter-dominated case.

### A. DGP Braneworld and $H^\alpha$

We first consider the Dvali-Turner [22] modification (also see [23])

$$H^2 = 8\pi G\rho_m(a)/3 + (1 - \Omega_{m,0})H_0^2 (H/H_0)^\alpha, \quad (47)$$

where  $\Omega_{m,0} = \Omega_m(a=1)$ . This was motivated by consideration of extradimensional theories, with the index  $\alpha$  depending on boundary conditions between our four dimensional universe and a higher dimensional bulk volume. The DGP braneworld cosmology [24, 25] is the special case  $\alpha = 1$ . Following [2], this acts as an effective freezing scalar field with

$$w = -\left(1 + \frac{\alpha}{2-\alpha}\Omega_m\right)^{-1} \quad (48)$$

$$w' = 3w(1+w) \left[1 - \frac{2}{\alpha}(1+w)\right]. \quad (49)$$

This provides the opportunity to follow the flow parameter for a freezing (indeed tracking) field starting far from  $w = -1$ , rather than for a thawing field. From Eq. (43), we find the effective flow parameter

$$\begin{aligned} F(a) &= \frac{1}{3} \left[ \frac{1-w}{1-(2/\alpha)w(1+w)} \right]^2 \\ &= \frac{1}{3} \frac{(2\frac{2-\alpha}{\alpha} + \Omega_m)^2 (\frac{2-\alpha}{\alpha} + \Omega_m)^2}{\left[ \Omega_m^2 + 2(1+\alpha^{-1})\frac{2-\alpha}{\alpha}\Omega_m + (\frac{2-\alpha}{\alpha})^2 \right]^2} \end{aligned} \quad (50)$$

Considering the early time limit of  $F$ , we expect in the matter-dominated universe that  $\Omega_m \rightarrow 1$ , so  $F \rightarrow 1/3$ . That is, the flow parameter indeed agrees with the tracking value of Eq. (45). Asymptotically, the effective equation of state is constant at  $w = -1 + \alpha/2$  and one can in fact show that the effective potential has the form of an inverse power law potential  $V \sim \phi^{-n}$  with index  $n = 2\alpha/(2-\alpha)$ .

In the future, the  $H^\alpha$  term will come to dominate (assuming  $\alpha < 2$ ) so that  $\Omega_m \rightarrow 0$ . Its effective equation of state then approaches  $w = -1$ . The flow parameter  $F \rightarrow 4/3$ , which corresponds to  $w' = 3w(1+w)$ . This flow parameter is the maximum value for freezing scalar fields and the minimum value for barotropic fields (see [26] for the barotropic case). In this limit, the  $H^\alpha$  effective potential approaches a quadratic form,  $V \approx V_\infty [1 +$

$(3/8)(\phi_\infty - \phi)^2]$ , where the field has an asymptotic maximum value  $\phi_\infty$  and  $V_\infty = 3H_0^2 [1 - \Omega_m(z=0)]^{2/(2-\alpha)}$ . Note that this class of models thus effectively incorporates a cosmological constant. Figure 5 shows the full solution for the effective potential.

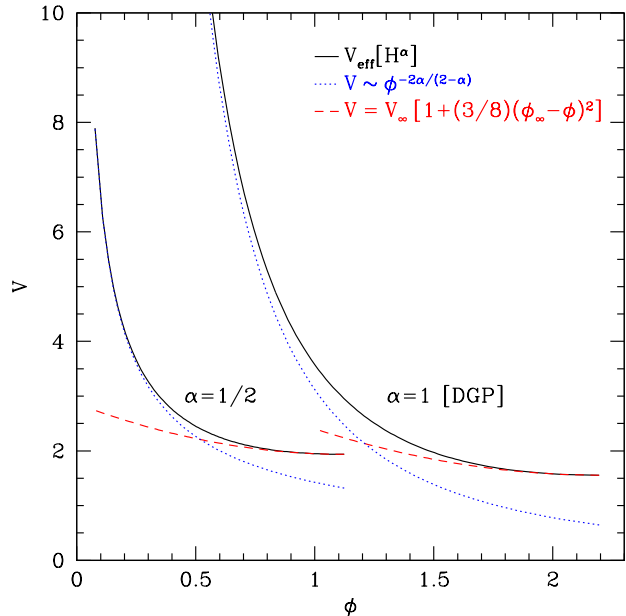


FIG. 5: The effective potential of the modified gravity case involving  $H^\alpha$  is plotted in units of  $H_0^2$  for  $\alpha = 1/2, 1$ . At high redshift (small  $\phi$ ) it possesses the flow parameter of a tracking scalar field and acts like an inverse power law potential, while in the far future it freezes to a cosmological constant state, approaching it as a quadratic potential.

### B. $f(R)$ Gravity

Another class of theories where dark energy is an effective quantity arising from extending gravity is  $f(R)$  theories, where the action involves a function of the Ricci scalar, here considered in addition to the usual linear term (so  $f = 0$  corresponds to general relativity). [27] gives the modified Friedmann expansion equation and an expression for the effective dark energy density  $\rho_{de}$ . We consider the deviation from the early, high curvature, matter-dominated era and write the equation in terms of the effective total equation of state of universe  $w_{tot}$ ,

$$\begin{aligned} f'' + \left(\frac{1}{2} + \frac{3}{2}w_{tot} + W\right) f' \\ - \frac{3}{2}(1 - 2w_{tot} - 3w_{tot}^2 + w'_{tot}) f \\ = 24\pi G \rho_{de} (1 - 2w_{tot} - 3w_{tot}^2 + w'_{tot}), \end{aligned} \quad (51)$$

where  $W = (3 - 3w_{tot} - 15w_{tot}^2 - 9w_{tot}^3 + 5w'_{tot} + 9w_{tot}w'_{tot} - w''_{tot}) / (1 - 2w_{tot} - 3w_{tot}^2 + w'_{tot})$ .

Now we consider the evolution of the departure from general relativistic matter domination with total effective equation of state  $w_{\text{tot}} = 0$ . Since the total equation of state  $w_{\text{tot}} = w\Omega_w(a)$ , where the dimensionless effective dark energy density  $\Omega_w(a) = (8\pi G/3)\rho_{\text{de}}/H^2$ , then we have  $\Omega_w(a)$ ,  $w_{\text{tot}}$ , and its derivatives all of the same order. The dark energy density  $\Omega_w(a) \ll 1$  at the epoch considered so we can expand in this small quantity. The zeroth order solution, with  $w_{\text{tot}} = 0 = \Omega_w(a)$ , gives  $f \sim a^p$  with  $p = (-7 + \sqrt{73})/4$ , as found by [27]. This homogeneous solution acts as an initial condition that quickly becomes unimportant. The first order departure from matter domination and general relativity has

$$f_1(a) = AH^2(a)\Omega_w(a), \quad (52)$$

where  $A$  is of order unity, and shows that gravity deviates from general relativity at the same rate as the effective dark energy density evolves (cf. [18]). The proportionality quantity  $A$  does not have to be constant, in general, just of order unity (thus any time variation is on Hubble scales or longer).

If we restrict  $A$  to be constant, then we can solve the equation in terms of effective dark energy equation of state and phrase this as the flow parameter

$$F(a) = \frac{3(1-w^2)^2}{[3A^{-1} - 2 - (5/2)w]^2}. \quad (53)$$

We can find a tracking solution with  $w' = 0$ , yielding  $A = -3/[1 - (5/2)w - 3w^2]$ , so a given choice of constant  $A$  corresponds to a given constant  $w$ . Since the dark energy density evolves as  $\rho_{\text{de}} \sim a^{-3(1+w)}$  and the Ricci scalar is dominated by matter  $\rho_m \sim a^{-3}$ , in this limit the gravitational modification looks like  $f(R) \sim R^{1+w}$ . However, considerable freedom exists to choose other solutions, e.g. with  $A$  varying.

Note that the physics governing the true scalar field evolution can be very different from that operating for modified gravity, so there is no expectation that the same relations should hold. Flows unlike the well-determined quintessence behavior may provide hints of modified gravity.

## VI. CONCLUSIONS

Our universe is old, having expanded by a factor of perhaps  $10^{28}$  since the last period of acceleration during the

inflationary epoch. This strongly affects the evolution of a scalar field that may give rise to the current epoch of acceleration and determines some key properties of the dark energy. Although conventional slow roll approximations are invalid, we show that analysis in terms of an integral relation between the Hubble friction and potential driving terms and a well characterized field expansion can give insights into the evolutionary behavior.

For the case of thawing fields, the field expansion provides a clear initial track in the  $w$ - $w'$  phase space, a unification of a number of interesting special cases, and a rapid convergence in the evolution of the dark energy density. For tracking fields, the ratio between the Klein-Gordon terms reaches a constant value. In both cases, the evolution of the deviation of  $\Omega_\phi$  from zero and of the tilt  $1 + w$  scale in a manner constrained by the long matter (and radiation) dominated era. Phrased in terms of a flow parameter combining the scalings, this ratio is nearly conserved until quite recent times,  $z \approx 1 - 2$ , when the dark energy finally begins to take over.

This physical behavior means that dark energy dynamics is not random, or equally probable, e.g. in the sense of a uniform prior over  $w$ - $w'$ , but is focused – “flows” – in specific ways. We have also tested this for two modified gravity theories and found some similar behavior but also some deviations that could offer clues to the nature of the acceleration. While one can always arrange initial conditions such that the dark energy comes out of the matter-dominated era with arbitrary behavior, this involves fine tuning. The oldness of our universe does provide a natural path for dark energy dynamics.

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