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Fundamental role of the retarded potential in the electrodynamics of superluminal sources: reply to comment

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Neither Eq. (6.52) of Jackson [*Classical Electrodynamics*, 3rd ed. (Wiley, 1999)], nor Hannay's derivation of that equation in the preceding Comment [J. Opt. Soc. Am. A, . . . (2009)], are applicable to a source whose distribution pattern moves faster than light *in vacuo* with nonzero acceleration. It is assumed in Hannay's derivation that the retarded distribution of the density of any moving source would be smooth and differentiable if its rest-frame distribution is. By working out an explicit example of a rotating superluminal source with a bounded and smooth density profile, we show that this assumption is erroneous. The retarded distribution of a rotating source with a moderate superluminal speed is, in general, spread over three disjoint volumes (differing in shape from each other and from the volume occupied by the source in its rest frame) whose boundaries depend on the spacetime position of the observer. Hannay overlooks the fact that the limits of integration in his expression for the retarded potential (which delineate the boundaries of the retarded distribution of the source) are not differentiable functions of the coordinates of the observer at those points on the source boundary that approach the observer, along the radiation direction, with the speed of light at the retarded time. In the superluminal regime, derivatives of the integral representing the retarded potential are well defined only as generalized functions.

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1. Introduction

The integral representation of the retarded potential [Eq. (4) below] is differentiable as a classical (as opposed to generalized) function only in the case of a moving source whose speed does not exceed that of the waves it generates. The steps, familiar from the subluminal regime, that are taken in the preceding Comment [1] to differentiate the retarded potential as a classical function are not mathematically permissible when the moving source has volume elements that approach the observer with the wave speed and zero acceleration at the retarded time. To demonstrate this, here we shall render these steps explicit by taking them in the case of a specific source distribution: a source distribution that is bounded and smooth but entails motion at speeds exceeding the speed of light *in vacuo*.

Being based on an analysis in which neither the motion of the source nor the position of the observer are specified, Hannay's argument [1] overlooks the specifically superluminal feature of the problem that gives rise to the extra contribution to the value of the field found in [2]. Whereas in the subluminal regime, the contributions to the field from the derivatives of the limits of integration in the integral representation of the retarded potential are zero when the density of the source vanishes smoothly at its boundary, here the corresponding contributions of those volume elements on the boundary of the source that approach the observer with the wave speed are divergent. Leibniz's formula for the differentiation of a definite integral (as a classical function) is not applicable if there are any points at which the limits of integration lack differentiability [3].

In the superluminal regime, the retarded potential should be written (with the aid of Dirac's delta function) as an integral over the rest-frame distribution (instead of the retarded distribution) of the source, so that its domain of integration is independent of the observer [Eq. (23) below]. Differentiation of the resulting generalized function and the regularization of its derivatives can then be rigorously handled by Hadamard's method [4, 5].

2. An explicit example of the retarded potential arising from a bounded and smooth superluminal source

2.A. The source and its retarded distribution

Let us consider a spherical source with the radius a whose center moves on a circle of radius r_c with the constant angular velocity ω, and whose density smoothly reduces from a maximum ρ_0 at its center to zero at its boundary, *e.g.,* it has the form

$$\rho(r, \hat{\varphi}, z) = \begin{cases} \rho_0 \cos^2[\pi R_s/(2a)] & \text{if } R_s \leq a,\ -\pi < \hat{\varphi} \leq \pi, \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

where

$$\hat{\varphi} \equiv \varphi - \omega t, \tag{2}$$

(r, φ, z) are the cylindrical polar coordinates based on the axis of rotation, t is time, and

$$R_s \equiv (z^2 + r^2 + {r_c}^2 - 2rr_c \cos\hat{\varphi})^{1/2} \tag{3}$$

is the distance of a point $(r, \hat{\varphi}, z)$ that is stationary in the rotating frame from the center ($r = r_c$, $\hat{\varphi} = 0$, $z = 0$) of the sphere. Note that the ranges of values of both φ and t in Eq. (1) are infinite, as in the case of a rotating point source, but the coordinate $\hat{\varphi}$, which labels each source element by its azimuthal position at $t = 0$, lies in an interval of length 2π. Note, moreover, that the localized source described by Eq. (1) does not have a sharp edge; the gradient of its density vanishes at its boundary.

The circle in broken lines in Fig. (1) shows the intersection, with the plane $z = 0$, of the boundary of the above source in the $(r, \hat{\varphi}, z)$ space for $r_c = 2c/\omega$ and $a = \frac{1}{4}c/\omega$, where c is the speed of light *in vacuo.* The axes in this figure are marked in units of c/ω and the larger dash-dotted circles designate the light cyliner $r = c/\omega$ and the orbit $r = 2c/\omega$ of the center of the souce, respectively.

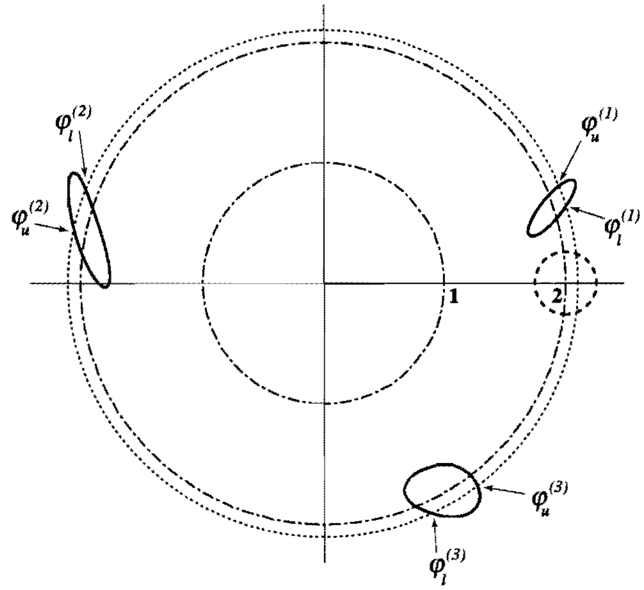

Fig. 1. The smallest circle (in broken lines) designates the boundary of the source distiribution described by Eq. (1), in its rest frame, for $r_c = 2c/\omega$ and $R_s = \frac{1}{4}c/\omega$. (The axes are marked in units of c/ω.) The two circles (in dots and dashes) with the radii 1 and 2 (in units of c/ω) represent the light cylinder and the orbit of the center of the source, respectively. The closed curves (in solid lines) show the cross section of the retarded distribution of the source with the plane of the orbit (*i.e.*, the multiple images of the circle in broken lines) for an observer who is located at $r_{\rm P} = 3c/\omega$, $\varphi_{\rm P} = \arccos(1/3) - 8^{1/2}$, $z_{\rm P} = 0$, at the observation time $t_{\rm P} = 0$. The intersections of these boundaries of the retarded source distribution with r=const., $z = 0$ (the dotted circle) specify the limits $({\varphi_l}^{(n)}, {\varphi_u}^{(n)})$ of the φ integration in the expression for the retarded potential.

The electric current arising from the rotational motion of the above charge distribution has the density $\mathbf{j} = r\omega\rho(r, \hat{\varphi}, z)\hat{\mathbf{e}}_\varphi$, where $\hat{\mathbf{e}}_\varphi$ is the unit vector along the azimuthal direction. The quantity that enters the expression used by Hannay [1] for the retarded vector potential,

$$\mathbf{A}(\mathbf{x}_\mathrm{P}, t_\mathrm{P}) = \frac{1}{c}\int \mathrm{d}^3x \, \frac{[\mathbf{j}(\mathbf{x}, t)]}{|\mathbf{x} - \mathbf{x}_\mathrm{P}|}, \tag{4}$$

is the retarded distirbution of this density:

$$[\mathbf{j}(\mathbf{x}, t)] \equiv \mathbf{j}(\mathbf{x}, t_\mathrm{P} - |\mathbf{x} - \mathbf{x}_\mathrm{P}|/c) = r\omega\rho(r, \hat{\varphi}_\mathrm{ret}, z)\hat{\mathbf{e}}_\varphi, \tag{5}$$

where $(\mathbf{x}, t) = (r, \varphi, z; t)$ and $(\mathbf{x}_\mathrm{P}, t_\mathrm{P}) = (r_\mathrm{P}, \varphi_\mathrm{P}, z_\mathrm{P}; t_\mathrm{P})$ stand for the spacetime positions of source points and the observation point, respectively,

$$\hat{\varphi}_\mathrm{ret} \equiv \varphi - \omega(t_\mathrm{P} - R/c), \tag{6}$$

denotes the retarded value of the variable $\hat{\varphi}$ defined in Eq. (2), and

$$R = [(z - z_\mathrm{P})^2 + r^2 + {r_\mathrm{P}}^2 - 2rr_\mathrm{P}\cos(\varphi - \varphi_\mathrm{P})]^{1/2} \tag{7}$$

is the distance $|\mathbf{x} - \mathbf{x}_\mathrm{P}|$ between the observation point and source points. In terms of the dimensionless variables

$$(\hat{r}, \hat{z}; \hat{r}_\mathrm{P}, \hat{z}_\mathrm{P}) \equiv (r\omega/c, z\omega/c; r_\mathrm{P}\omega/c, z_\mathrm{P}\omega/c), \tag{8}$$

Eq. (6) has the form

$$\hat{\varphi}_\mathrm{ret} = \hat{\varphi}_\mathrm{P} + g(r, r_\mathrm{P}, \varphi - \varphi_\mathrm{P}, z - z_\mathrm{P}), \tag{9}$$

where

$$g \equiv \varphi - \varphi_\mathrm{P} + [(\hat{z} - \hat{z}_\mathrm{P})^2 + \hat{r}^2 + \hat{r}_\mathrm{P}^2 - 2\hat{r}\hat{r}_\mathrm{P}\cos(\varphi - \varphi_\mathrm{P})]^{1/2}, \tag{10}$$

and $\hat{\varphi}_\mathrm{P} \equiv \varphi_\mathrm{P} - \omega t_\mathrm{P}$.

To insert $[\mathbf{j}(\mathbf{x}, t)]$ in Eq. (4), we need to know not only the expression in Eq. (5), but also the boundary of this retarded distribution, which specifies the limits of integration in Eq. (4). In the present case, this boundary is described by

$$R_s\Big|_{\hat{\varphi}=\hat{\varphi}_\mathrm{ret}} = [z^2 + r^2 + {r_c}^2 - 2rr_c\cos(\hat{\varphi}_\mathrm{P} + g)]^{1/2} = a \tag{11}$$

[see Eqs. (1) and (3)]. Equation (11) is satisfied by the following two expressions for g:

$$g = \phi_{b\pm}(r, z, \hat{\varphi}_\mathrm{P}), \tag{12}$$

where

$$\phi_{b\pm} \equiv \pm 2\arcsin\left\{\left[\frac{a^2 - z^2 - (r - r_c)^2}{4rr_c}\right]^{1/2}\right\} - \hat{\varphi}_\mathrm{P}. \tag{13}$$

Absolute value of the argument of the arcsin in Eq. (13) is less than or equal to unity because the (r, z) coordinates of any point on the boundary of the retarded source distribution would automatically lie within the projection, $(r - r_c)^2 + z^2 \leq a^2$, of the rest-frame source distribution onto a meridional plane. Equation (12) determines the φ coordinates of the points on the boundary of the retarded source distribution as functions of their (r, z) coordinates. It is a transcendental equation and so has to be solved numerically.

For a fixed observation point (r_P, φ_P, z_P), g is an oscillatory function of φ when the coordinates (r, z) of the boundary point are such that

$$\Delta \equiv (\hat{r}_P^2 - 1)(\hat{r}^2 - 1) - (\hat{z}_P - \hat{z})^2 \tag{14}$$

is positive, a condition that holds for an observer in the far zone only if the source moves superluminally. Given $\Delta > 0$, the extrema of $g(\varphi)$ occur at

$$\varphi_\pm \equiv \varphi_P + 2\pi - \arccos\left(\frac{1 \mp \Delta^{1/2}}{\hat{r}_P \hat{r}}\right), \tag{15}$$

(and at $\varphi_\pm$ plus integral multiples of 2π), with $g(\varphi_\pm) \equiv \phi_\pm$ [see Fig. 2(a)]. The neighboring extrema of $g(\varphi)$ coalesce if the boundary point lies on the curve

$$\Delta = 0, \quad \varphi = \varphi_P + 2\pi - \arccos[1/(\hat{r}\hat{r}_P)] \tag{16}$$

[see Fig. 2(b)]. For $\Delta < 0$, the function $g(\varphi)$ is monotonic as in Fig. 2(c). The turning points of g occur at the locus of source points for which $\mathrm{d}R/\mathrm{d}t = -c$ at a fixed value of $\hat{\varphi}$, *i.e.*, the source points that approach the observer, along the radiation direction, with the speed of light at the retarded time. The inflection point of g occurs at the locus of source points that approach the observer not only with the wave speed but also with zero acceleration, *i.e.*, for which both $\mathrm{d}R/\mathrm{d}t = -c$ and $\mathrm{d}^2R/\mathrm{d}t^2 = 0$ at a fixed $\hat{\varphi}$.

It can be seen from Fig. 2(a) that the two equations $g = \phi_{b-}$ and $g = \phi_{b+}$ each have three solutions when $\Delta > 0$ and $\phi_- < \phi_{b\pm} < \phi_+$: each of the horizontal lines $g = \phi_{b\pm}$ would intersect curve a in Fig. 2 at three points in this case. In other words, the boundary of the retarded source distribution consists of three disjoint images of the original boundary of the source distribution in its rest frame (Fig. 1). Let us denote these six solutions of Eq. (12) by ${\varphi_l}^{(n)}$ and ${\varphi_u}^{(n)}$ with $n = 1, 2, 3$. These solutions correspond to the six intersections, shown in Fig. 1, of a circle r =const., $z = 0$, with the three images of the source.

2.B. The retarded potential and the indifferentiability of the limits of integration in its classical representation

The integral in the classical representation of the retarded potential, Eq. (4), has to be performed over the three distinct volumes shown in Fig. 1 which constitute the retarded

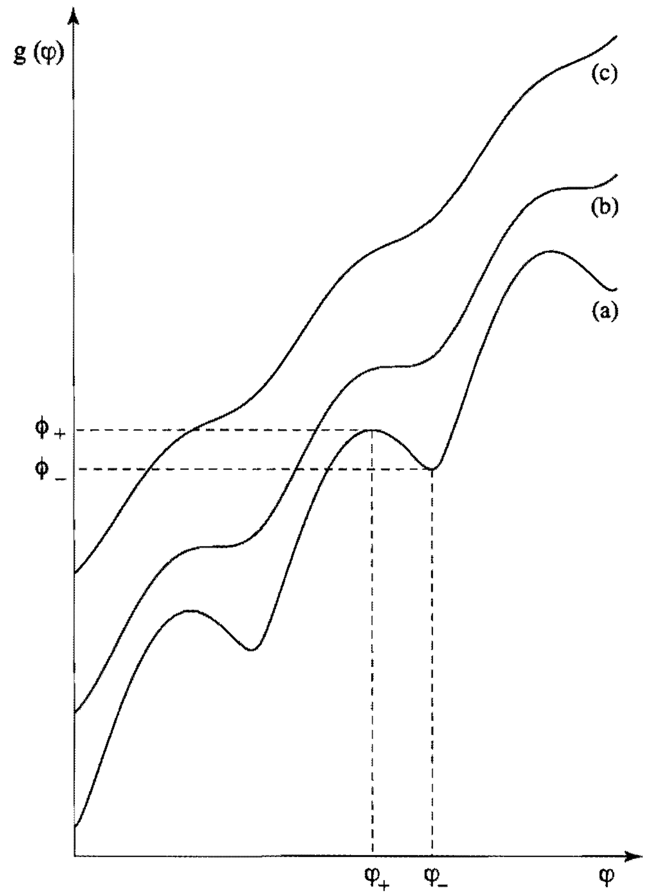

Fig. 2. The curves representing the function $g(\varphi)$, defined in Eq. (10), for $\varphi_{\rm P} = 0$, $r_{\rm P} = 3c/\omega$, $r = 2c/\omega$, and (a) $\Delta > 0$, (b) $\Delta = 0$, and (c) $\Delta < 0$. The marked adjacent turning points of curve (a) have the coordinates $(\varphi_\pm, \phi_\pm)$.

distribution of the source. With the aid of Eqs. (5), (7) and (9), the volume integral in Eq. (4) can therefore be written as

$$\mathbf{A} = \frac{\omega}{c}\rho_0 \sum_{n=1}^{3} \int_{-a}^{a} \mathrm{d}z \int_{r_c-(a^2-z^2)^{1/2}}^{r_c+(a^2-z^2)^{1/2}} \mathrm{d}r \int_{\varphi_l{}^{(n)}}^{\varphi_u{}^{(n)}} \mathrm{d}\varphi \, \frac{r^2}{R} \cos^2\left(\frac{\pi}{2a} R_s\big|_{\hat{\varphi}=\hat{\varphi}_{\rm ret}}\right) \hat{\mathbf{e}}_\varphi. \tag{17}$$

Note that the variable φ enters the integrand of Eq. (17) in only the combination $\varphi - \varphi_{\rm P}$ [see also Eqs. (10) and (11)]. Changing this integration variable to $\sigma \equiv \varphi - \varphi_{\rm P}$, and writing out the expressions for R and $R_s|_{\hat{\varphi}=\hat{\varphi}_{\rm ret}}$, we obtain

$$\begin{aligned} \mathbf{A}(r_{\rm P}, \hat{\varphi}_P, z_P) &= \frac{\omega}{c}\rho_0 \sum_{n=1}^{3} \int_{-a}^{a} \mathrm{d}z \int_{r_c-(a^2-z^2)^{1/2}}^{r_c+(a^2-z^2)^{1/2}} \mathrm{d}r \, r^2 \\ &\times \int_{\sigma_l{}^{(n)}}^{\sigma_u{}^{(n)}} \mathrm{d}\sigma \, \frac{\cos^2\{\pi[z^2 + r^2 + r_c{}^2 - 2rr_c\cos(\hat{\varphi}_{\rm P} + g)]^{1/2}/(2a)\}}{[(z - z_{\rm P})^2 + r^2 + r_{\rm P}{}^2 - 2rr_{\rm P}\cos\sigma]^{1/2}} \hat{\mathbf{e}}_\varphi, \end{aligned} \tag{18}$$

in which

$$g = \sigma + [(\hat{z} - \hat{z}_{\rm P})^2 + \hat{r}^2 + \hat{r}_{\rm P}^2 - 2\hat{r}\hat{r}_{\rm P}\cos\sigma]^{1/2} \tag{19}$$

is the same as the function defined in Eq. (10), but expressed in terms of σ, and $\sigma_l^{(n)} = \varphi_l^{(n)} - \varphi_{\rm P}$ and $\sigma_u^{(n)} = \varphi_u^{(n)} - \varphi_{\rm P}$.

To find the components of the generated magnetic field $\mathbf{B} = \nabla_{\rm P} \times \mathbf{A}$, we need to calculate the derivatives of the right-hand side of Eq. (19) with respect to $r_{\rm P}$, $\varphi_{\rm P}$ and $z_{\rm P}$, using Leibniz's formula for the differentiation of a definite integral:

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_{\alpha(x)}^{\beta(x)} f(x, \xi)\, \mathrm{d}\xi = \int_{\alpha(x)}^{\beta(x)} \frac{\partial f}{\partial x} \mathrm{d}\xi - f(x, \alpha)\frac{\mathrm{d}\alpha}{\mathrm{d}x} + f(x, \beta)\frac{\mathrm{d}\beta}{\mathrm{d}x} \tag{20}$$

(see, *e.g.*, [3]). There are two different types of contribution toward the value of each derivative of $\mathbf{A}$: one from the derivative of the integrand in Eq. (18), and another from the derivatives of the limits $\sigma_l{}^{(n)}$ and $\sigma_u{}^{(n)}$ of integration. The contributions arising from the differentiation of the limits of integration in Eq. (18) entail the quantities $\nabla_{\rm P}\sigma_l^{(n)}$ and $\nabla_{\rm P}\sigma_u^{(n)}$, which need to be calculated next.

Since, according to Eqs. (12), (13) and (19), $\sigma_l^{(n)}$ and $\sigma_u^{(n)}$ are solutions of

$$g = \sigma + [(\hat{z} - \hat{z}_{\rm P})^2 + \hat{r}^2 + \hat{r}_{\rm P}^2 - 2\hat{r}\hat{r}_{\rm P}\cos\sigma]^{1/2} = \phi_{b\pm}(\hat{r}, \hat{z}, \hat{\varphi}_{\rm P}), \tag{21}$$

the quantities $\nabla_{\rm P}\sigma_l^{(n)}$ and $\nabla_{\rm P}\sigma_u^{(n)}$ may be found by applying $\nabla_{\rm P}$ to both sides of Eq. (21); the result is

$$\begin{aligned} \nabla_{\rm P}\sigma &= -\frac{1}{\partial g/\partial\sigma}\left(\frac{\partial g}{\partial r_{\rm P}}\hat{\mathbf{e}}_{r_{\rm P}} + \frac{\partial g}{\partial z_{\rm P}}\hat{\mathbf{e}}_{z_{\rm P}} - \frac{1}{r_{\rm P}}\frac{\partial \phi_{b\pm}}{\partial \varphi_{\rm P}}\hat{\mathbf{e}}_{\varphi_{\rm P}}\right) \\ &= -\frac{\omega}{c}\frac{[r_{\rm P} - r\cos(\varphi - \varphi_{\rm P})]\hat{\mathbf{e}}_{r_{\rm P}} + (z_{\rm P} - z)\hat{\mathbf{e}}_{z_{\rm P}} + (R/\hat{r}_{\rm P})\hat{\mathbf{e}}_{\varphi_{\rm P}}}{R + 2r\hat{r}_{\rm P}\sin(\varphi - \varphi_{\rm P})}, \end{aligned} \tag{22}$$

where we have replaced σ by $\varphi - \varphi_{\rm P}$ in the second line.

The denominator in Eq. (22) vanishes at the turning points $\varphi = \varphi_\pm$ of the function g [see Eq. (15) and Fig. 2]. These are the points across which the number of images of the source change (from three to one or from one to three). An example of a retarded source distribution for which the number of images of the source depends on the radial coordinate r of the source points is shown in Fig. 3. The intersection of the sphere representing the distribution of the source, in its rest frame, with the plane of rotation is shown by broken lines, and that of the retarded distribution of the source by the closed solid curve.

In this case, the locus of points given in Eq. (16) is tangential to the inner boundary of the source at $\hat{r} = 1$, and so the three images of the source coalesce within the radial interval $1 \le \hat{r} \le 1.6$. In $1.6 < \hat{r} \le 2$, the retarded distribution of the source consists of three disjoint parts. As the integration variable $\hat{r}$ in Eq. (18) approaches either the value 1 or the value 1.6, the limits of φ-integration in this equation approach $\varphi_\pm$ and so their gradients diverge. That is to say, the gradients $\nabla_{\rm P}\sigma_l^{(n)}$ and $\nabla_{\rm P}\sigma_u^{(n)}$ are singular at those points on the boundary of the retarded source distribution that approach the observer with the speed of light along the radiation direction. Moreover, since within the framework of the theory of classical (as opposed to generalized) functions, the indifferentiability of any of the limits of integration in Eq. (20) implies the indifferentiability of the integral in that equation, irrespective of the behaviour of its integrand [3], the singularities of $\nabla_{\rm P}\sigma_l^{(n)}$ and $\nabla_{\rm P}\sigma_u^{(n)}$ imply the indifferentiability of $\mathbf{A}$.

The counterparts of the second and third terms on the right-hand side of Eq. (20), when the operator $\nabla_{\rm P}\times$ is applied to the integral in Eq. (18), consist of the products of $\nabla\sigma_l^{(n)}$ or $\nabla\sigma_l^{(n)}$ with the corresponding values of the integrand at $\sigma = \sigma_l^{(n)}$ or $\sigma = \sigma_u^{(n)}$. At those points within the (r, z) domain of integration where $\sigma_l^{(n)}$ or $\sigma_u^{(n)}$ assume one of the values $\varphi_\pm - \varphi_{\rm P}$, the factors $\nabla\sigma_l^{(n)}$ and $\nabla\sigma_l^{(n)}$ diverge while the values of the integrand at the corresponding boundary points vanish. The products of the divergent factors $\nabla\sigma_l^{(n)}$ and $\nabla\sigma_l^{(n)}$ with the vanishing values of the integrand at the source boundary constitute contributions toward the derivatives of the retarded potential that are neither infinite nor zero (the degree of divergence of the former factors is dictated by the constructive interference of the emitted waves, while the vanishing of the integrand depends on the smoothness of the source at its boundary). These products are *indeterminate* quantities, contravening the required conditions for the differentiability of the integral as a classical function, that can only be handled by means of the theory of generlized functions (see below). Nor does the fact that some of the upper and lower limits of integration in Eq. (18) approach one another, as their gradients diverge, have any bearing on this conclusion. From the standpoint of the classical theory of functions, Leibniz's theorem does not hold if any of the derivatives in Eq. (20) cease to exist [3].

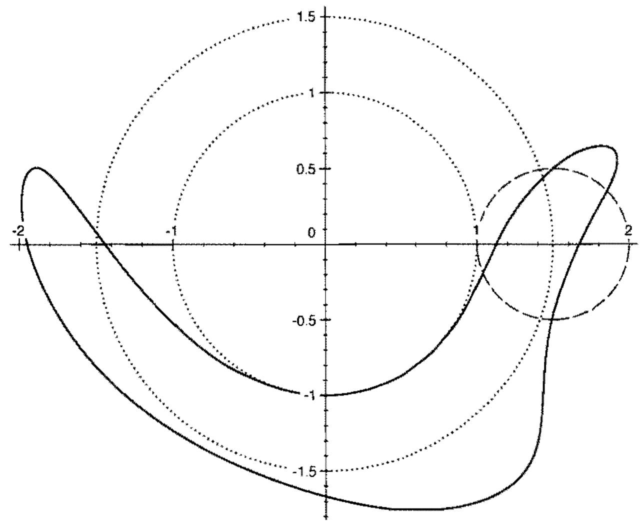

Fig. 3. The smallest circle (in broken lines) designates the boundary of the source distiribution described by Eq. (1) in its rest frame for $r_c = \frac{3}{2}c/\omega$ and $R_s = \frac{1}{2}c/\omega$. (The axes are marked in units of c/ω.) The two dotted circles with the radii 1 and 3/2 (in units of c/ω) represent the light cylinder and the orbit of the center of the source, respectively. The closed solid curve shows the cross section of the retarded distribution of the the source with the plane of the orbit for an observer who is located at $r_{\rm P} = \frac{5}{2}c/\omega$, $\varphi_{\rm P} = \arccos(2/5) - (21)^{1/2}/2$, $z_{\rm P} = 0$, at the observation time $t_{\rm P} = 0$.

3. Differentiation of the retarded potential as a generalized function

We have seen that, because the retarded time is a multivalued function of the observation time in the case of a source that moves faster than its own waves (Fig. 2 and [7]), the retarded distribution of the density of a superluminal source (such as that whose contour $\rho = 0$ is shown in Fig 3) can lack differentiability even when its original distribution is smooth. The contribution from the boundary of the retarded distribution of a smooth source toward the derivative of the potential it generates is zero, as in Eq. (6.52) of Jackson [6], only in the familiar subluminal regime where the gradients of the limits of integration in the classical representaion of the retarded potential [Eq. (4)] are singularity-free. In the case of a superluminal source, where the limits of integration delineating the boundary of the source in Eq. (4) lack differentiability, the integral itself is not differentiable as a classial function (see, *e.g.*, [3]).

In the superluminal regime, Leibniz's rule may be applied to the retarded potential if the integral representation in Eq. (4) is written as

$$\mathbf{A}(\mathbf{x}_{\rm P}, t_{\rm P}) = \frac{1}{c}\int \mathrm{d}^3x\,\mathrm{d}t\,\mathbf{j}(\mathbf{x},t)\frac{\delta(t_{\rm P} - t - |\mathbf{x}_{\rm P} - \mathbf{x}|/c)}{|\mathbf{x}_{\rm P} - \mathbf{x}|}, \tag{23}$$

so that the spatial integration in this representation extends over the rest-frame (instead of the retarded) distribution of the source, *i.e.*, over a volume that is independent of the spacetime coordinates of the observer. In this way, an expression is found for the field

$$\nabla_{\rm P}\times\mathbf{A} = \frac{1}{c}\int \mathrm{d}^3x\,\mathrm{d}t\,\mathbf{j}\times\frac{\mathbf{x}_{\rm P} - \mathbf{x}}{|\mathbf{x}_{\rm P} - \mathbf{x}|^2}\left[\frac{1}{c}\delta'(t_{\rm P} - t - |\mathbf{x}_{\rm P} - \mathbf{x}|/c) + \frac{\delta(t_{\rm P} - t - |\mathbf{x}_{\rm P} - \mathbf{x}|/c)}{|\mathbf{x}_{\rm P} - \mathbf{x}|}\right] \tag{24}$$

that, in contrast to that given in Eq. (18), is well defined as a generalized function. In the case of the above example, where $\mathbf{j}$ depends on φ and t in only the combination $\varphi - \omega t \equiv \hat{\varphi}$, one of the integrations in Eq. (24) can be performed independently of the source density [by adopting $(r, \hat{\varphi}, z; t)$ as the variables of integration] to obtain a representation of $\nabla_{\rm P}\times\mathbf{A}$ in terms of the Liénard-Wiechert fields of the constituent volume elements of the source [7]. The indifferentiability of the limits of integration that is encountered in the classical representation is thus found to be reflected in the nonintegrability of the singularities of the superposed Liénard-Wiechert fields in this alternative representation [8]. However, such singularities can be rigorously handled by means of the Hadamard regularization technique [4, 5].

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