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Multidimensional First and Second Order Symmetric Strang Splitting for Hyperbolic Systems

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Abstract

We propose an algebraic basis for symmetric Strang splitting for first and second order accurate schemes for hyperbolic systems in N dimensions. Examples are given for two and three dimensions. Optimal stability is shown for symmetric systems. Lack of strong stability is shown for a non-symmetric example. Some numerical examples are presented for some Euler-like constant coefficient problems.

Key words:
Multidimensional Hyperbolic Systems, Lax-Friedrichs, Lax-Wendroff, Symmetric Splitting, Optimal Stability

1. Introduction - One Dimension

In his landmark paper [6] G. Strang introduced two splitting techniques designed to maintain second order accuracy for finite difference methods for hyperbolic systems. The second of these has an advantage over the first in that it is symmetric with respect to interchange of dimension. In this note we shall develop symmetric splitting for both the first order Lax-Friedrichs type scheme, and for the second order Lax-Wendroff type, in N space dimensions, for hyperbolic conservation laws of the form

$$\frac{\partial u}{\partial t} = \sum_{j=1}^{N} \frac{\partial f_j(u)}{\partial x_j}.$$

Here, $u = u(x,t)$, $x = (x_1, \cdots, x_N)$, and $f_j(u)$ are vector valued functions.

The difference schemes will be defined on uniform tensor product grids in terms of one-dimensional two point average and difference operators. In particular we define

$$\alpha_j \equiv \text{average over adjacent grid points in the j-th dimension}$$

and

$$\beta_j \equiv \text{difference over adjacent grid points in the j-th dimension.}$$

The form of these operators in the more familiar grid index notation will be clear later.

These operators all commute with each other.

In these terms the familiar elementary one-dimensional finite difference operators are: Lax-Friedrichs (we suppress j here),

$$S_F = \alpha + \frac{\lambda}{2}\beta f,$$

and the two-step version of Lax-Wendroff,

$$S_L = I + \lambda \beta f \circ S_F,$$

where the $\lambda = \lambda_j$ are the mesh ratios,

$$\lambda_j = \frac{\Delta t}{\Delta x_j}.$$

The one dimensional time advance in the case of Lax-Wendroff is

$$u^{n+1} = S_L u^n,$$

while in the case of Lax-Friedrichs it is

$$
\begin{aligned}
u^{n+\frac{1}{2}} &= S_F u^n \\
u^{n+1} &= S_F u^{n+\frac{1}{2}} = S_F \circ S_F u^n.
\end{aligned}
$$

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2. N-dimensional Lax-Friedrichs

We offer two versions of N-dimensional Lax-Friedrichs type schemes. The first is

$$S = \frac{1}{N!} \sum_{p \in P} \left(\alpha_{p(1)} + \frac{\lambda_{p(1)}}{2} \beta_{p(1)} f_{p(1)} \right) \circ \left(\alpha_{p(2)} + \frac{\lambda_{p(2)}}{2} \beta_{p(2)} f_{p(2)} \right) \circ \cdots$$

$$\cdots \circ \left(\alpha_{p(N)} + \frac{\lambda_{p(N)}}{2} \beta_{p(N)} f_{p(N)} \right), \quad (1)$$

where P is the set of all permutations of $\{1 \cdots N\}$.

The second version is

$$S = \prod_{l=1}^{N} \alpha_l + \sum_{k=1}^{N} \frac{\lambda_k}{2} \beta_k f_k \circ \left[\prod_{l \neq k} \alpha_l + \frac{1}{2} \sum_{j \neq k} \frac{\lambda_j}{2} \beta_j f_j \circ \left[\prod_{l \neq k, l \neq j} \alpha_l + \right. \right.$$

$$\left. \left. \frac{1}{3} \sum_{m \neq j, m \neq k} \frac{\lambda_m}{2} \beta_m f_m \circ \left[\prod_{l \neq k, l \neq j, l \neq m} \alpha_l + \frac{1}{4} \cdots \right] \right] \right].$$

In either case, these operators map functions defined on the original grid to functions on the fully staggered grid at the half time step, or they map functions defined on the fully staggered grid to the original grid. So, as in one dimension,

$$u^{n+\frac{1}{2}} = Su^n$$
$$u^{n+1} = Su^{n+\frac{1}{2}} = S \circ Su^n.$$

2.1. Linear Equivalence

The two versions are *linearly equivalent*, that is, if the functions f_j are replaced by multiplication by constant matrices then the two expressions are algebraically equivalent. To prove this, consider

$$A = \frac{1}{N!} \sum_{p \in P} (\alpha_{p(1)} + B_{p(1)})(\alpha_{p(2)} + B_{p(2)}) \cdots (\alpha_{p(N)} + B_{p(N)}),$$

where the B's are linear operators that commute with the α's. Then A has the form

$$A = \prod_{l=1}^{N} \alpha_l + c_k \sum_{k=1}^{N} (\sum_{l_i \neq l_j} B_{l_1} \cdots B_{l_k} \prod_{l \neq l_i} \alpha_l). \quad (2)$$

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There are $\binom{N}{k}$ ways to place the k factors containing k particular B's in the same order somewhere in an ordered product with the other factors, and there $(N - k)!$ arrangements of the other factors each producing the same product of α's, so

$$c_k = \frac{1}{N!} \binom{N}{k} (N - k)! = \frac{1}{k!}.$$

(Or, by symmetry, there must be $N!/k!$ terms in the sum that are the product of k chosen B's in some order.) Next, collect all terms in (2) that have B_1 as the leftmost factor, then all terms that have B_2 as the leftmost, etc, to obtain

$$A = \prod_{l=1}^{N} \alpha_l + \sum_{k=1}^{N} B_k \left[\prod_{l \neq k} \alpha_l + \frac{1}{2} \sum_{j \neq k} B_j \left[\prod_{l \neq k, l \neq j} \alpha_l + \frac{1}{3} \sum_{m \neq j, m \neq k} B_m \left[\prod_{l \neq k, l \neq j, l \neq m} \alpha_l + \frac{1}{4} \cdots \right] \right] \right],$$

which is the desired result.

2.2. Example

In [4] the authors developed a 2 and 3 dimensional Lax-Friedrichs scheme, following the idea proposed in [2] of using iterated Riemann solvers. The second scheme above is the extension to N dimensions of what is found in [4]. Consider three dimensions. In order to keep the notation consistent with [4], let the system of differential equations be

$$\frac{\partial u}{\partial t} = \frac{\partial f(u)}{\partial x} + \frac{\partial g(u)}{\partial y} + \frac{\partial h(u)}{\partial x}.$$

With standard notation, first observe that the various products of $\alpha's$ are just the standard 2, 4 or 8 point averages. For example,

$$(\alpha_1 \alpha_3 u) = \frac{1}{4}(u_{i,j,k} + u_{i+1,j,k} + u_{i,j,k+1} + u_{i+1,j,k+1}) = u_{i+\frac{1}{2},j,k+\frac{1}{2}}.$$

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Thus,

$$
\begin{aligned}
u^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} \;=\; & \frac{1}{8}\Big(u^n_{i,j,k} + u_{i+1,j,k} + u_{i,j,k+1} + u_{i+1,j,k+1} + \\
& \quad u_{i,j+1,k} + u_{i+1,j+1,k} + u_{i,j+1,k+1} + u_{i+1,j+1,k+1} \Big) \\
& + \frac{\lambda_1}{2}\left(f(u^*_{i+1,j+\frac{1}{2},k+\frac{1}{2}}) - f(u^*_{i,j+\frac{1}{2},k+\frac{1}{2}}) \right) \\
& + \frac{\lambda_2}{2}\left(g(v^*_{i+\frac{1}{2},j+1,k+\frac{1}{2}}) - g(v^*_{i+\frac{1}{2},j,k+\frac{1}{2}}) \right) \\
& + \frac{\lambda_3}{2}\left(h(w^*_{i+\frac{1}{2},j+\frac{1}{2},k+1}) - h(w^*_{i+\frac{1}{2},j+\frac{1}{2},k}) \right).
\end{aligned}
$$

For u^* we have

$$
\begin{aligned}
u^*_{i,j+\frac{1}{2},k+\frac{1}{2}} = u_{i,j+\frac{1}{2},k+\frac{1}{2}} \;+\;& \frac{\lambda_2}{4}\left(g(v^{**}_{i,j+1,k+\frac{1}{2}}) - g(v^{**}_{i,j,k+\frac{1}{2}}) \right) \\
& + \frac{\lambda_3}{4}\left(h(w^{**}_{i,j+\frac{1}{2},k+1}) - h(w^{**}_{i,j+\frac{1}{2},k}) \right)
\end{aligned}
$$

and

$$
\begin{aligned}
v^{**}_{i,j,k+\frac{1}{2}} \;=\;& \frac{1}{2}(u_{i,j,k+1} + u_{i,j,k}) + \frac{\lambda_3}{6}(h(u_{i,j,k+1}) - h(u_{i,j,k})) \\
w^{**}_{i,j+\frac{1}{2},k} \;=\;& \frac{1}{2}(u_{i,j+1,k} + u_{i,j,k}) + \frac{\lambda_3}{6}(h(u_{i,j+1,k}) - h(u_{i,j,k})).
\end{aligned}
$$

The functions v^* and w^* are similarly defined.

2.3. Stability

If we consider now linear hyperbolic systems, that is

$$
\frac{\partial u}{\partial t} = \sum_{j=1}^{N} A_j \frac{\partial u}{\partial x_j},
$$

Hyperbolicity means that every linear combination of the matrices A_j with real coefficients has only real eigenvalues. Strict hyperbolicity means that every linear combination of the matrices A_j with real coefficients has only real eigenvalues and is diagonalizable. We are only able to establish a stability condition for *symmetric hyperbolic systems*

Using the usual Fourier analysis, the symbol of the difference operator S is

$$\hat{S} = \frac{1}{N!} \sum_{p \in P} (\cos \theta_{p(1)} + i\lambda_{p(1)} \sin \theta_{p(1)} A_{p(1)})(\cos \theta_{p(2)} + i\lambda_{p(2)} \sin \theta_{p(2)} A_{p(2)}) \cdots$$

$$\cdots (\cos \theta_{p(N)} + i\lambda_{p(N)} \sin \theta_{p(N)} A_{p(N)}) \quad (3)$$

and

$$\|\hat{S}\| \leq \prod_{j=1}^{N} \|\cos \theta_j I + i\lambda_j \sin \theta_j A_j\|.$$

A sufficient stability condition then is that

$$\max_{\theta} \max_{j} \|\cos \theta I + i\lambda_j \sin \theta A_j\| \leq 1.$$

If the A_j are *symmetric* then

$$\|\cos \theta I + i\lambda_j \sin \theta A_j\|^2 = \cos^2 \theta + \lambda_j^2 \sin^2 \theta \|A_j\|^2$$

Thus, for symmetric systems we have shown that the N-dimensional version of Lax-Friedrichs described above is stable if

$$\max \lambda_j \text{ (spectral radius of } A_j) \leq 1.$$

In other words, in this case the scheme is optimally stable since the CFL limit is 1. This is also true if each A_j is unitarily similar to a real diagonal matrix.

2.4. A counter-example

Consider the linear system

$$\frac{\partial \rho}{\partial t} = \frac{\partial u}{\partial x} + \frac{\partial \rho}{\partial x}$$
$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}$$

In this case the Lax-Friedrichs symbol is

$$\hat{S} = \cos \theta I + iA\lambda \sin \theta$$

where

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Note that neither A nor $\hat{S}$ is diagonalizable, so that this is not a strictly hyperbolic system.

Let us define

$$z = \cos\theta + i\lambda\sin\theta,\ 0 < \lambda \le 1.$$

Then, the n-th power of the Lax-Friedrichs symbol can be expressed as

$$\hat{S}^n = \begin{pmatrix} z^n & i\,n\,z^{n-1}\,\sin\theta \\ 0 & z^n \end{pmatrix}.$$

The scheme is weakly stable, as guaranteed by the von Neumann condition (see [5]), but not strongly stable. To see this, set

$$|z|^2 = 1 - \tau\sin^2\theta,\ \ \tau = 1 - \lambda^2 \ge 0$$

and choose the sequence θ_n such that

$$\sin^2\theta_n = \frac{1}{n}.$$

Then

$$|nz^{n-1}\sin\theta|^2 = (1 - \frac{\tau}{n})^n n|z|^{-2}$$

which is unbounded as $n \to \infty$.

3. N-dimensional 2nd Order Scheme

In order to develop an N-dimensional symmetric scheme of Lax-Wendroff type, as in [3], we need only replace the factors in (1) by

$$I + \lambda\beta q$$

where

$$q = f(\alpha + \frac{\lambda}{2}\beta f)$$

or

$$S = I + \sum_{k=1}^{N} \lambda_k \beta_k q_k \circ \left[I + \frac{1}{2} \sum_{j \neq k} \lambda_j \beta_j q_j \circ \left[I + \right.\right.$$

$$\left.\left. \frac{1}{3} \sum_{m \neq j, m \neq k} \lambda_m \beta_m q_m \circ \left[I + \frac{1}{4} \cdots \quad \right]\right]\right]. \quad (4)$$

3.1. Accuracy

S, as defined in (4) provides a second-order accurate difference scheme for the hyperbolic system. We give the basics of the proof. First, we can drop all terms after the factor $\frac{1}{3}$ since they are $O(\Delta t)^3$. Next, the central averaging operators can be replaced by the identity, and the central differences can be replaced by derivatives. since the errors made are of order $(\Delta x)^2 \Delta t$. Now from

$$\frac{\partial u}{\partial t} = \sum_{k=1}^{N} \frac{\partial f_k(u)}{\partial x_k}.$$

we have

$$\frac{\partial^2 u}{\partial t^2} = \sum_{k=1}^{N} \frac{\partial}{\partial x_k} \left[f_k'(u) \sum_{j=1}^{N} \frac{\partial f_j(u)}{\partial x_j} \right].$$

Let

$$u^{k*} = u + \frac{\Delta t}{2} \sum_{j \neq k} \frac{\partial}{\partial x_j} \left[f_j \left(u + \frac{dt}{2} \frac{\partial}{\partial x_j} f_j(u) \right) \right].$$

We need to show that for any smooth function u,

$$\sum_{k=1}^{N} \frac{\partial}{\partial x_k} \left[f_k \left(u^{k*} + \frac{\Delta t}{2} \frac{\partial}{\partial x_k} f_k(u^{k*}) \right) \right] =$$

$$\sum_{k=1}^{N} \frac{\partial f_k(u)}{\partial x_k} + \frac{\Delta t}{2} \sum_{k=1}^{N} \frac{\partial}{\partial x_k} \left[f_k'(u) \sum_{j=1}^{N} \frac{\partial f_j(u)}{\partial x_j} \right] + O(\Delta^2). \quad (5)$$

But

$$f_k \left(u^{k*} + \frac{\Delta t}{2} \frac{\partial}{\partial x_k} f_k(u^{k*}) \right) = f_k(u^{k*}) + \frac{\Delta t}{2} f_k'(u) \frac{\partial}{\partial x_k} f_k(u) + O(\Delta t)^2,$$

and

$$f_k(u^{k*}) = f_k(u) + \frac{\Delta t}{2} f_k'(u) \sum_{j \neq k} \frac{\partial}{\partial x_j} f_j(u) + O(\Delta t)^2,$$

from which (5) follows.

3.2. Example

For example, in two dimensions the scheme is

$$u_{i,j}^{n+1} = u_{i,j}^n + \lambda_1 \left(f(u_{i+\frac{1}{2},j}^*) - f(u_{i-\frac{1}{2},j}^*) \right) + \lambda_2 \left(g(v_{i,j+\frac{1}{2}}^*) - g(v_{i,j-\frac{1}{2}}^*) \right),$$

where

$$u_{i+\frac{1}{2},j}^* = \frac{1}{2}(u_{i+1,j}^{**} + u_{i,j}^{**}) + \frac{\lambda_1}{2} \left(f(u_{i+1,j}^{**}) - f(u_{i,j}^{**}) \right),$$

and

$$u_{i,j}^{**} = u_{i,j} + \frac{1}{2}\lambda_2 \left(g(u_{i,j+\frac{1}{2}}^{***}) - g(u_{i,j-\frac{1}{2}}^{***}) \right),$$

where

$$u_{i,j+\frac{1}{2}}^{***} = \frac{1}{2}(u_{i,j+1} + u_{i,j}) + \frac{1}{2}\lambda_2 \left(g(u_{i,j+1} - g(u_{i,j})) \right).$$

Similar expressions hold for v^*.

This agrees exactly with the 2D scheme in [3]. The 3D version obtained from (4) is not exactly the same as the one in [3].

3.3. Stability

Just as for Lax-Friedrichs, we can now establish the stability of the multi-dimensional scheme if the f_j are *symmetric matrices* A_j, for then for the symbol of (4) we have

$$\|\hat{S}\| \leq \prod \|I + 2i\lambda_j \sin\theta_j A_j(\cos\theta_j + i\lambda \sin\theta_j A_j)\|,$$

where

$$\|I + 2i\lambda \sin\theta A(\cos\theta + i\lambda \sin\theta A)\|^2$$
$$= \sup_{\|u\|=1} \left((I - 4\lambda^2 A^2 \sin^2\theta + 4\lambda^4 A^4 \sin^4\theta + 4\lambda^2 A^2 \sin^2\theta \cos^2\theta)u, u \right)$$
$$= 1 + (4\lambda^2 \sin^4\theta) \sup_{\|u\|=1} \left((\lambda^2 A^4 - A^2)u, u \right). \quad (6)$$

So we have stability if (4) holds, that is, the scheme is optimally stable.

REMARK: A simpler but non-optimally stable scheme is the following extension of the Richtmyer two-step scheme to N dimensions.

$$u^{n+\frac{1}{2}} = \left(\frac{1}{N} \sum_{j=1}^{N} \alpha_j + \frac{1}{2} \sum_{j=1}^{N} \lambda_j \beta_j f_j \right) \circ u^n$$

$$u^{n+1} = u^n + \left(\sum_{j=1}^{N} \lambda_j \beta_j f_j \right) \circ u^{n+\frac{1}{2}}$$

It is shown in [7] that the CFL limit for this scheme is $1/\sqrt{N}$.

4. Euler Equations

We know that both 3D Lax-Friedrichs and the second order scheme are optimally stable for symmetric systems. Here we examine these features for the non-symmetric system of Euler equations. Our approach is based on numerical sampling of each scheme's amplification matrix, both as regards its eigenvalues and powers.

We consider the 3D Euler equations in the form

$$\mathbf{U}_t = \mathbf{f}(\mathbf{U})_x + \mathbf{g}(\mathbf{U})_y + \mathbf{h}(\mathbf{U})_z \,,$$

where

$$\mathbf{U} = \begin{pmatrix} \rho \\ \rho\, u \\ \rho\, v \\ \rho\, w \\ E \end{pmatrix} \,,$$

and

$$\mathbf{f}(\mathbf{U}) = \begin{pmatrix} \rho\, u \\ \rho\, u^2 + p \\ \rho\, u\, v \\ \rho\, u\, w \\ u\,(E + p) \end{pmatrix} \,, \quad \mathbf{g}(\mathbf{U}) = \begin{pmatrix} \rho\, v \\ \rho\, u\, v \\ \rho\, v^2 + p \\ \rho\, v\, w \\ v\,(E + p) \end{pmatrix} \,, \quad \mathbf{h}(\mathbf{U}) = \begin{pmatrix} \rho\, w \\ \rho\, u\, w \\ \rho\, v\, w \\ \rho\, w^2 + p \\ w\,(E + p) \end{pmatrix} \,.$$

Here, ρ stands for fluid density, u, v, and w are components of velocity in each coordinate $\mathbf{w} = (u, v, w)$, E is a specific total energy defined by the

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ideal gas equation of state $E = \frac{p}{\gamma-1} + \frac{1}{2}\rho\|\mathbf{w}\|^2$, p stands for pressure, and γ is the gas adiabatic index.

This system can be rewritten in a linearized form

$$\mathbf{U}_t = A\,\mathbf{U}_x + B\,\mathbf{U}_y + C\,\mathbf{U}_z\,,$$

where A, B, and C are Jacobians of the flux functions, for example

$$A = \frac{\partial \mathbf{f}(\mathbf{U})}{\partial \mathbf{U}}$$

$$= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{\gamma-1}{2}\|\mathbf{w}\|^2 - u^2 & (3-\gamma)\,u & (1-\gamma)\,v & (1-\gamma)\,w & \gamma-1 \\ -u\,v & v & u & 0 & 0 \\ -u\,w & w & 0 & u & 0 \\ \frac{2-3\gamma+\gamma^2}{2(\gamma-1)}u\left(\|\mathbf{w}\|^2 - c^2\right) & \frac{\|\mathbf{w}\|^2}{2} + \frac{(1-\gamma)^2 u^2 - c^2}{1-\gamma} & (1-\gamma)\,u\,v & (1-\gamma)\,u\,w & \gamma\,u \end{pmatrix}$$

and similarly for B and C. Note that the eigenvalues of this matrix are u, $u+c$, and $u-c$, where $c = \sqrt{\gamma\,p/\rho}$ is the fluid sound speed.

We have constructed the symbol (amplification matrix) of the 3D Lax-Friedrichs and the second order scheme.

We only verify the von Neumann necessary stability condition, namely, that the maximum absolute value of the eigenvalues of the symbol is ≤ 1 for CFL≤ 1. If the maximal eigenvalue is greater than 1 the scheme is unstable. We have chosen several values of interest for each quantity to be used in sampling. For each velocity component, we use all values from the following: 0, ± 1, ± 2, ± 3, and ± 4, for sound speed we used values from: 0.1, 1, 2, 2.5, 5, and 10. It means that we sampled values for Mach number between 0 and $40\sqrt{3} \approx 69.3$. As for CFL, we used values $1/10$, $1/4$, $1/2$, $1-1/10$, $1-1/100$, $1-10^{-5}$, 1, $1+10^{-5}$, $1+1/100$, $1+1/10$, and 2. Fourier angles were chosen to cover all quarters and thirds of π in the whole 2π range, and monatomic gas constant $\gamma = 5/3$ was used.

The maximal eigenvalue for each scheme as a function of CFL is presented in Table 1. For computation of eigenvalues the LAPACK [1] package was used.

This verifies, at least for a supposedly typical set of arguments of the symbol, that Lax-Friedrichs and our second order scheme are optimally (weakly) stable even for non-symmetric systems.

4.1. Zero Sound Speed

Included in the results of Table 1 the smallest value of the sound speed is $c = .1$. We consider here what happens if we include $c = 0$.

CFL	Lax-Friedrichs	2nd order scheme
$1 - 1/10$	1.0000000000	1.0000000000
$1 - 1/100$	1.0000000000	1.0000000000
$1 - 10^{-5}$	1.0000000000	1.0000000000
1	1.0000000000	1.0000000000
$1 + 10^{-5}$	1.0000100000	1.0001200054
$1 + 1/100$	1.0100000000	1.1255130848
$1 + 1/10$	1.2898723387	2.8632880000

Table 1: Maximal eigenvalue of amplification matrix for 3D Lax-Friedrichs and the second order scheme for several CFL numbers.

First, for the same sampling set, the schemes are optimally weakly stable.

Second, for CFL=1, the maximum component of the n-th power of the symbol samples increases linearly with n. This was tested for powers $n = (2, 4, \cdots, 2^{15})$. The maximum is taken over the hydrodynamic data samples and Fourier angles.

Third, following section 2.4, for CFL< 1 and $\sin^2 \theta_j = 1/n$, we again observe that the powers of the symbol are not bounded, Fig. 1.

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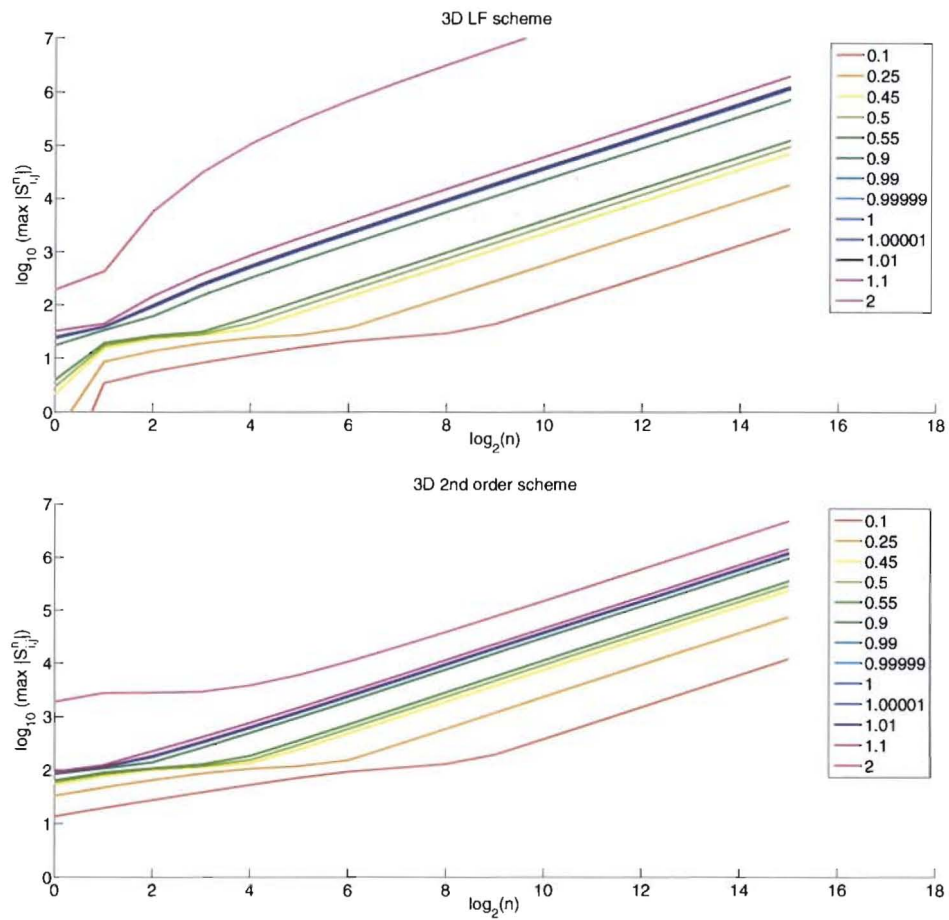
Figure 1: Growth rate (maximal component of symbol S^n) for 3D first order Lax-Friedrichs and second order Lax-Wendroff schemes for different CFL numbers and powers n.

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