Problem Proposed for the *American Mathematical Monthly*

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Problem: Define

$$P(x) \quad := \quad \sum_{k=1}^{\infty} \arctan\left(\frac{x-1}{(k+x+1)\,\sqrt{k+1}+(k+2)\,\sqrt{k+x}}\right) \ . \tag{1}$$

(a) Find explicit, finite-expression evaluations of $P(n)$ for all integers $n \geq 0$.

(b) Show $\tau := \lim_{x \to -1^+} P(x)$ exists, and find an explicit evaluation for τ.

(c) Are there a more general closed forms for P, say at half-integers?

Solution. With the abbreviations

$$r := \sqrt{k+1}, \quad s := \sqrt{k+x}$$

the argument of arctan in (1) becomes

$$\frac{s^2-r^2}{(s^2+1)\,r+(r^2+1)\,s} = \frac{s-r}{r\,s+1} = \frac{\dfrac{1}{r}-\dfrac{1}{s}}{1+\dfrac{1}{r}\,\dfrac{1}{s}} \ .$$

Therefore, by using the addition theorem ot the tangent function, the definition (1) may be written in the more convenient form

$$P(x) \quad = \quad \sum_{k=1}^{\infty}\left(\arctan\frac{1}{\sqrt{k+1}} - \arctan\frac{1}{\sqrt{k+x}}\right) \tag{2}$$

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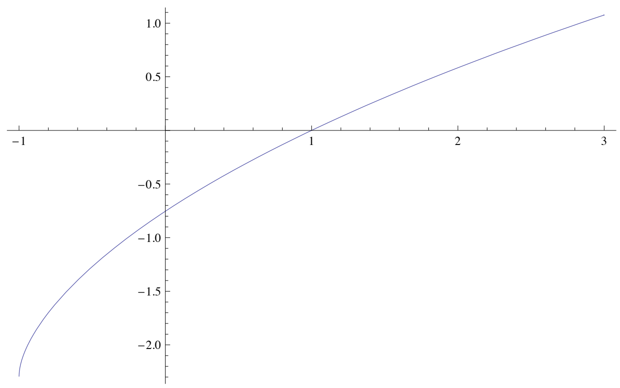

Figure 1: Plot of $P(x)$

Now define

$$A(x) \quad = \quad \arctan \frac{1}{\sqrt{x+1}}$$

and note that a telescoping sum argument gives

$$P(x) + A(x) \quad = \quad P(x+1). \tag{3}$$

It is easy to see that the series defining $P(x)$ is absolutely convergent by the Weierstrass M-test, and to verify that $P(x)$ is increasing for $x > -1$, as shown in Figure 1. Thus, τ exists.

(a). First observe that since $P(1) = 0$, the identity (3) establishes that $P(0) = -A(0) = -\pi/4$, which we had computationally observed. By iteratively applying (3) and applying induction, we establish that

$$\begin{aligned} P(2) &= \arctan \frac{1}{\sqrt{2}} \\ P(3) &= \arctan \frac{1}{\sqrt{2}} + \arctan \frac{1}{\sqrt{3}} \\ P(4) &= \arctan \frac{1}{\sqrt{2}} + \arctan \frac{1}{\sqrt{3}} + \arctan \frac{1}{2}, \end{aligned}$$

and indeed by induction we have, for all $n \geq 2$,

$$P(n) \quad = \quad \sum_{k=2}^{n} \arctan \frac{1}{\sqrt{k}}.$$

(b). We computationally discovered that to 13-digit accuracy $\tau = \lim_{x \to -1^+} P(x) = -3\pi/4$. This can be rigorously established by noting that

$$\lim_{x \to -1^+} P(x) + \frac{\pi}{2} = \lim_{x \to -1^+} P(x) + \lim_{x \to -1^+} A(x) = \lim_{x \to -1^+} P(x+1) = P(0) = \frac{-\pi}{4}.$$