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Ernest O. Lawrence  
Radiation Laboratory

BETA-RAY SPECTROMETER  
WITH REDUCED SPHERICAL ABERRATION

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# BETA-RAY SPECTROMETER WITH REDUCED SPHERICAL ABERRATION†

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Modern  $\beta$ -ray spectrometers are based upon the concept, first introduced by Svartholm and Siegbahn in 1946, of focusing in both the radial and vertical directions<sup>1</sup>). The theory of axially symmetric devices has been carefully studied by a large number of workers<sup>2</sup>), culminating in the analysis, in 1956, of Lee-Whiting and Taylor<sup>3</sup>). These last authors calculate aberrations through the sixth order and show that by appropriate choice of the magnetic field a spectrometer can be designed with a relatively large transmission and a high resolution. The acceptable transmission is remarkable because the second-order "spherical" aberration in the median plane of the image cannot be made to vanish identically, and consequently the design is forced to a tall thin aperture (or a slightly less advantageous short wide aperture) which a priori would seem to imply a low transmission.

It is the purpose of this communication to show that if the arbitrary restriction to axially symmetric fields is removed, then both the radial and the vertical contributions to the "spherical" aberration can be made to vanish in second order. That azimuthally varying fields (AVF) afford the freedom to accomplish this end may well be suspected in view of the technological revolution that the concept has brought to other particle-handling devices.

A general treatment of optics in arbitrary static electric and magnetic fields has been given by Sturrock<sup>4</sup>), and our calculations can be considered a special case of his work. We give a self-contained analysis for an AVF spectrometer, and obtain explicit formulas for the "spherical" aberration through second order, and then demonstrate, by example, that the aberration can be made to vanish.

Restricting attention to devices with median-plane symmetry and a central trajectory which lies on a circle, one can show that the most general field in the (source-free) region of the optic circle has cylindrical components

$$\frac{B_z}{B_0} = 1 + a_1(\theta)x + a_2(\theta)x^2 - \left[ \frac{a_1(\theta)}{2} + a_2(\theta) \right] y^2 + \dots ,$$

$$\frac{B_r}{B_0} = a_1(\theta)y + 2a_2(\theta)xy + \dots , \quad (1)$$

$$\frac{E_\theta}{B_0} = \frac{da_1(\theta)}{d\theta} xy + \dots ,$$

where  $r_0$  is the optic circle radius,

$$x = \frac{r - r_0}{r_0} , \quad (2)$$

$$y = \frac{z}{r_0} ,$$

and  $a_1(\theta)$  and  $a_2(\theta)$  are two arbitrary functions. The path of an electron of momentum  $p_0$ , where

$$p_0 = -\frac{e r B_0}{c} , \quad (3)$$

is described in the neighborhood of the optic circle by the equations (valid through second order):

$$x'' + [1 + a_1(\theta)] x = - [1 + 2a_1(\theta) + a_2(\theta)] x^2 + \frac{x'^2}{2} - \frac{y'^2}{2} + \left[ \frac{a_1(\theta)}{2} + a_2(\theta) \right] y^2 , \quad (4)$$

$$y'' - a_1(\theta)y = 2 [a_1(\theta) + a_2(\theta)] xy + x'y' ,$$

where the primes denote differentiation with respect to  $\theta$  . An electron of slightly different momentum

$$p = p_0(1 + \epsilon) , \quad (5)$$

has radial motion which in first order is determined by

$$x'' + [1 + a_1(\theta)]x = \epsilon . \quad (6)$$

Explicit formulas for the second-order aberration coefficients can be given in terms of the independent solutions of the linearized approximations to eqs. (4). A convenient set of such solutions,  $x_{L1}$  and  $x_{L2}$  , may be specified by the boundary conditions

$$x_{L1}(0) = 0 , \quad x'_{L1}(0) = 1 ; \quad (7)$$

$$x_{L2}(0) = 1 , \quad x'_{L2}(0) = 0 ;$$

and similarly for the y-equation. Solutions of eqs. (4) which are correct through second order and satisfy the boundary conditions

$$\begin{aligned} x(0) = x_0 \quad , \quad & x'(0) = x'_0 \quad ; \\ y(0) = y_0 \quad , \quad & y'(0) = y'_0 \quad ; \end{aligned} \tag{8}$$

can be expressed in the form

$$\begin{aligned} x(\theta) = x_0 x_{L2}(\theta) + x'_0 x_{L1}(\theta) + A_{xx}(\theta) x_0^2 + A_{yy}(\theta) y_0^2 + A_{x'x'}(\theta) x'_0{}^2 \\ + A_{x'x}(\theta) x_0 x'_0 + A_{y'y'}(\theta) y'_0{}^2 + A_{y'y}(\theta) y_0 y'_0 \quad ; \end{aligned} \tag{9}$$

and similarly for  $y(\theta)$ . The condition for simultaneous vertical and radial linear focusing is that for some angle  $\theta_f$ ,

$$x_{L1}(\theta_f) = y_{L1}(\theta_f) = 0 . \tag{10}$$

For a sufficiently small source the resolution of the spectrometer is limited by the "spherical" aberration in the radial dimension of the image, namely by the magnitude of the coefficients  $A_{x'x'}(\theta_f)$  and  $A_{y'y'}(\theta_f)$ , which are given explicitly by

$$\begin{aligned} A_{x'x'}(\theta_f) = -x_{L2}(\theta_f) \int_0^{\theta_f} x_{L1}(\theta) \left\{ -[1 + 2a_1(\theta) + a_2(\theta)][x_{L1}(\theta)]^2 \right. \\ \left. + \frac{1}{2} [x'_{L1}(\theta)]^2 \right\} d\theta , \\ A_{y'y'}(\theta_f) = -x_{L2}(\theta_f) \int_0^{\theta_f} x_{L1}(\theta) \left\{ \left[ \frac{a_1(\theta)}{2} + a_2(\theta) \right] [y_{L1}(\theta)]^2 \right. \\ \left. - \frac{1}{2} [y'_{L1}(\theta)]^2 \right\} d\theta . \end{aligned} \tag{11}$$

The dispersion of the spectrometer is proportional to the displacement  $D$  of the image of an electron with  $x(0) = x'(0) = 0$  and momentum fractionally differing from  $p_0$  by  $\epsilon$  [as in eq. (5)]. From eq. (6) one finds the displacement  $D(\epsilon)$ , given to lowest order by

$$D(\epsilon) = -\epsilon x_{L2}(\theta_f) \int_0^{\theta_f} x_{L1}(\theta) d\theta . \quad (12)$$

For spectrometers in which  $a_1(\theta)$  is independent of azimuth, eq. (10) can only be satisfied with  $\theta_f < 2\pi$  by choosing  $a_1(\theta) = -1/2$ , as in the spectrometer of Svartholm and Siegbahn. In this case  $\theta_f = \sqrt{2} \pi$ , and the aberration coefficients of eq. (11) are:

$$A_{x'x'}(\theta_f) = \frac{2}{3} - 4 \int_0^{\pi} \sin^3 \psi a_2(\sqrt{2} \psi) d\psi , \quad (13)$$

$$A_{y'y'}(\theta_f) = -2 + 4 \int_0^{\pi} \sin^3 \psi a_2(\sqrt{2} \psi) d\psi .$$

It is thus clear that no choice of  $a_2(\theta)$  will allow the removal of the second-order "spherical" aberration<sup>5</sup>). To accomplish this end we must, in general, choose  $a_1$  to vary with  $\theta$ . The physical reason is clear: Maxwell's equations [see eq. (1)] relate  $B_z$  to  $B_r$  and thus relate the effect of  $a_2(\theta)$  on particles with a certain x-amplitude to the effect on particles with an equal y-amplitude. This relation, as may be seen in eq. (13), does not allow  $A_{x'x'}$  and  $A_{y'y'}$  to vanish simultaneously. In order to affect particles with x displacements differently from particles with y displacements we must break the cylindrical symmetry about the optic circle; i.e., choose  $a_1(\theta)$  different from  $-\frac{1}{2}$ , and consequently, in general, varying<sup>6</sup>) with  $\theta$ .

As an idealized example, consider a spectrometer in which

$$\begin{aligned} a_1(\theta) &= -1, & a_2(\theta) &= a_{20}; & 0 \leq \theta \leq \theta_f/2 \\ a_1(\theta) &= 0, & a_2(\theta) &= a_{21}; & \theta_f/2 \leq \theta \leq \theta_f. \end{aligned} \quad (14)$$

Simultaneous linear focusing may be obtained if

$$\tan \theta_f/2 = -\theta_f/2, \quad (15)$$

which implies  $\theta_f = 4.0576$  (corresponding to about  $232^\circ$ ). The linear x-solutions are easily seen to be

$$x_{L1}(\theta) = \begin{cases} \theta, & 0 \leq \theta \leq \theta_f/2 \\ \frac{\sin(\theta - \theta_f/2)}{\cos \theta_f/2}, & \theta_f/2 \leq \theta \leq \theta_f; \end{cases} \quad (16)$$

$$x_{L2}(\theta) = \begin{cases} 1, & 0 \leq \theta \leq \theta_f/2 \\ \cos(\theta - \theta_f/2), & \theta_f/2 \leq \theta \leq \theta_f; \end{cases}$$

with quite similar solutions of the y-equation. From eq. (12), the dispersion is

$$\frac{D(\epsilon)}{\epsilon} = 1 + \frac{2}{S}(1 + \theta_f^2/8), \quad (17)$$

where

$$S = (4 + \theta_f^2)^{1/2}. \quad (18)$$



Numerical evaluation yields<sup>7)</sup>  $D(\epsilon)/\epsilon = 2.35$  . The aberration coefficients are

$$A_{x'x'}(\theta_f) = \frac{1}{S} \left[ \frac{\theta_f^4}{32} - \frac{\theta_f^2}{8} (3 + S) - \frac{S}{2} - 1 \right] - \frac{\theta_f^4}{32S} a_{20} - \frac{a_{21}}{S} \left[ \frac{\theta_f^2}{2} \left( \frac{S}{3} + 1 \right) + \frac{2}{3} S + \frac{4}{3} \right] , \quad (19)$$

$$A_{y'y'}(\theta_f) = \frac{-1}{S^3} \left[ \frac{\theta_f^4}{8} + \frac{\theta_f^2}{2} + 4 + 2S \right] + \frac{a_{20}\theta_f^2}{S^3} \left[ 2 + \frac{\theta_f^2}{8} \right] - \frac{4a_{21}}{S^3} \left[ 2S + 4 - \frac{3}{2} \theta_f^2 \right] ,$$

which can both be made zero by choosing  $a_{20} = 1.46$  , and  $a_{21} = -0.91$  .

A design study is now being undertaken at this laboratory with the aim of extending these concepts to higher orders as well as of realizing them in an iron-free configuration. In this way it is hoped to obtain a high-precision spectrometer with good resolution and a large transmission.

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FOOTNOTES AND REFERENCES

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5. If  $a_2(\theta)$  is taken independent of  $\theta$  then eqs. (13) become

$$A_{x'x'} = \frac{2}{3} (1 - 8a_2) ,$$

$$A_{y'y'} = \frac{2}{3} (-3 + 8a_2) .$$

The choices  $a_2 = \frac{1}{8}$  and  $a_2 = \frac{3}{8}$ , correspond respectively to the "wide aperture" and "high aperture" spectrometers of Lee-Whiting and Taylor.

6. If  $a_1$  is taken independent of  $\theta$  but chosen so that the  $x$  and  $y$  frequencies are unequal (although commensurate), then suitable choice of  $a_2(\theta)$  will make  $A_{x'x'}$  and  $A_{y'y'}$  simultaneously vanish. However,  $\theta_F$  will necessarily be larger than  $2\pi$ .
7. For this particular example the value of  $D(\epsilon)/\epsilon$  is approximately half of that associated with the axially symmetric spectrometer of Svartholm and Siegbahn. However, it can be seen from eq. (12) that there is no reason, in general, to expect the dispersion to be poorer in an AVF spectrometer than in an axially symmetric spectrometer.

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