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A HIGH-RESOLUTION HIGH-LUMINOSITY BETA-RAY SPECTROMETER DESIGN EMPLOYING AZIMUTHALLY VARYING MAGNETIC FIELDS*

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ABSTRACT

A double-focusing magnetic field for a spectrometer of the flat type which gives radial focusing to roughly the sixth order, and which utilizes azimuthal variation of the field coefficients, has been devised. In the type of magnetic field assumed, the nonvanishing component B_z in the symmetry plane $z = 0$ may be written as

$$B_z = B_0 \left[1 + a_1(\theta) \frac{r - r_0}{r_0} + a_2(\theta) \left(\frac{r - r_0}{r_0} \right)^2 + \cdots \right] ,$$

where the coefficients $a_n(\theta)$ are assumed to be of the form

$$a_n(\theta) = D_{0n} + D_{1n} \sin\theta + D_{2n} \cos\theta + D_{3n} \sin2\theta + D_{4n} \cos2\theta + D_{5n} \sin3\theta + D_{6n} \cos3\theta$$

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A computer and some basic analytic properties of the aberration coefficients have been employed to arrive at appropriate values of the field constants D_{mn}. The focusing angle is 6 radians, within which there is one intermediate axial focus. The magnitude of the ratio of the momentum dispersion to the radial magnification is approximately 12; this means that the spectrometer, for a given resolution, will accommodate a source that is three times as broad as the source of the well-known $\pi\sqrt{2}$ type. For a point source and a solid angle of 0.9% of 4π, the momentum resolution is 0.01%. In the present field the total source height is limited to about three times the source width due to the presence of a fairly large second-order cross term between the source height parameter and the axial aperture parameter. It is found, however, that by a slightly different choice of $a_2(\theta)$ such a cross-term can be eliminated. The general possibility of control of cross-terms, as indicated by this result, could be of considerable value if a system of very high luminosity should be attempted, for instance by using an azimuthally varying magnetic field in conjunction with a recently developed electrostatic method for reducing the influence of source width resolution. Some comments are made on the problem of realizing an azimuthally varying field of the present type.

1. Introduction

In attempting to find an efficient magnetic beta-ray spectrometer design incorporating high resolution and high luminosity, one usually limits attention to the so-called "flat lay-out," i.e., designs in which the momentum selecting dispersion is confined to one direction in space. The resolution is in this case, at least in first order, independent of the source height, thus giving promise of a reasonably large luminosity (product of source area and aperture solid angle) at high resolution. In flat spectrometers with simultaneous radial and axial focusing and having a rotationally symmetric magnetic field with a circular central orbit, the magnitude of the aperture is in lowest order limited by the fact that it is impossible to cancel simultaneously both the ϕ_r^2 and ϕ_z^2 terms in the resolution determining radial aberration.

There have been a number of ways devised to overcome this difficulty. Daniel[1] departed from the condition of simultaneous axial focusing at the radial focus, and in this way provided degrees of freedom for reducing the aberrations. Lee-Whiting[2] investigated the case where the magnetic field has mirror and rotational symmetry but where the central orbit is not circular, and achieved considerable progress in reducing the aperture aberrations. Bergkvist[3] devised a system involving in addition to the magnetic field two electrostatic components which allowed considerable reduction both of the arperture defect and of the influence on the resolution of a source of finite size. Sessler[4] suggested that "strong-focusing," i.e., a departure from the rotational symmetry of the magnetic field, could reduce the aperture defect, and showed that, in principle, the simultaneous vanishing of both the ϕ_r^2 and the ϕ_z^2 terms in the radial aberrations could be achieved.

Detailed design considerations in Ref. 4 were limited to a rather unphysical field which was employed only because it allowed analytic treatment. The example did, however, provide an explicit demonstration of the removal of second-order radial aperture aberration. Subsequently, the Heidelberg group[5] has presented a physically acceptable field which has the same property.

The present work is an extension of the principles suggested in Ref. 4: We exhibit a set of physically realizable fields resulting in a spectrometer in which the radial aperture aberration has been removed to high (roughly sixth) order. In contrast to the methods used in Ref. 4, we are forced to extensive computations coupled with certain basic analytic relations between the aberration coefficients and the field parameters, in order to achieve this end.

Section 2 describes the magnetic field expansion which is employed; Section 3 discusses some general considerations leading to the design studied herein. Section 4 outlines the procedure for the determination of the field coefficients and gives some illustrations of the effect of the strong focusing or azimuthally varying field (AVF) in different aberration orders. Section 5 gives the basic data of the spectrometer field we have obtained; it yields an aperture solid angle of 0.9% of 4π at a maximum radial image width of 1.0×10^{-4} in momentum. In Section 6, the behavior of the spectrometer with respect to sources of finite size is discussed. Section 7 deals with the problem of realizing the desired azimuthally varying field, and also contains a suggested alternative approach to spectrometer design which may have advantages over that employed in this paper.

2. Magnetic Field Expansion

We employ cylindrical coordinates r, θ, z and assume that in the plane $z = 0$ the only nonvanishing field component, B_z, is expressed as

$$B_z(x,0) = B_0(1 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots) \quad , \tag{1}$$

where $x = (r - r_0)/r_0$ is the relative radial departure from the central circle $r = r_0$.

The field coefficients $a_n(\theta)$ are assumed to have an azimuthal dependence of the form

$$a_n(\theta) = D_{0n} + D_{1n}\sin\theta + D_{2n}\cos\theta + D_{3n}\sin 2\theta + D_{4n}\cos 2\theta + D_{5n}\sin 3\theta + D_{6n}\cos 3\theta \quad . \tag{2}$$

A computer program[6] employing the above field allows the calculation, at any angle θ, of the position of an orbit starting at the point x_0, $y_0(= z_0/r_0)$, $\theta = 0$ with the radial and axial direction parameters ϕ_r and ϕ_z . The equations of motion and the field components B_r , B_z , and B_θ are given in the Appendix. The computer program is accurate through sixth order.

It appears from (1) that the assumed field is constant along the central circle $x = 0$. Although, in principle, this lack of generality of the azimuthal variation may exclude the discovery of some physically favorable field structure, it adds vastly to the simplicity and straightforwardness of the analysis of the aberrations. In particular, if the momentum p is chosen in such a way that the central circle is a possible orbit and the source is assumed to be at $x_0 = y_0 = 0$, it is then possible to divide the aperture aberrations into various orders such that an aberration feature of order n remains unaffected when field coefficients of order larger than n are varied. In contrast, in the more general case the various orders of aberrations are coupled. (This coupling could be removed by choice of a field expansion properly suited to the central ray, but the electromagnetic problem of finding the off-median-plane fields would become very much more complicated.)

3. Choice of the Basic Configuration

With an azimuthally varying field, simultaneous first-order radial and axial focusing can be obtained within a wide range of azimuthal angles θ . Very generally, as was demonstrated in Ref. 4, the possibility of reducing the higher order aberrations by means of AVF depends on the fact that the first-order radial and axial orbit solutions are made unequal. With the magnetic field varying azimuthally, there exists the possibility of independently varying the

ϕ_r^2 and ϕ_z^2 terms in the radial aberrations (and in a more complicated way, successively, the higher order aberrations). Although we can not claim any general validity to the observation, there had been indications that when the AVF coefficients in first order were chosen fairly large, the AVF coefficients of increasing order tended to grow when the higher order aberration coefficients were sucessively made equal to zero. For this reason, we decided to study a case with simultaneous first-order radial and axial focusing, where a considerable unequality between the first-order radial and axial orbit solutions could be obtained with a very moderate degree of AVF in first order.

In the azimuthally symmetric case with simultaneous axial and radial focusing at $\theta = \theta_F = \pi\sqrt{5}$, the axial orbit solution oscillates twice as fast as the radial solution. However, the focusing angle, being slightly larger than 2π, is experimentally inconvenient. By adding a relatively moderate degree of AVF, in first order, the focusing angle can be moved to an angle somewhat smaller than 2π. As the largest value of θ_F considered to be of experimental interest was $\theta_F = 6$, this was consequently chosen for further study.

4. Determination of the Field Coefficients

When the source is a point on the central circle, emitting particles with direction parameters ϕ_r, ϕ_z and of such a momentum that $r = r_0$ is a possible orbit (having also $\phi_{r_0} = \phi_{z_0} = 0$), the radial coordinate x^* of the image point can be expressed as

$$x^* = \sum_{\mu,\nu} A_{\mu\nu} \phi_r^{\ \mu} \phi_z^{\ \nu} , \tag{3}$$

where for symmetry reasons ν must be even.

The main emphasis of the present work has been on the reduction of the radial aperture aberration coefficients $A_{\mu\nu}$. Provided reasonable radial dispersion is maintained, this will ensure favorable performance for the case of a point source. When the parameters x_0 , y_0 differ from zero a variety of new terms enter in x^* , these terms will influence the performance with the use of an extended source. The coefficients for these new terms are also, in principle, adjustable by means of the AVF coefficients. Some comments on this important point, and an example, may be found in Section 6.

Analytic determination of the functions $a_i(\theta)$ that will make the aberration coefficients $A_{\mu\nu}$ zero would be extremely tedious; we have resorted to digital computation instead.

From a consideration of the general structure of the solutions, insofar as their dependence on the $a_i(\theta)$ is concerned, if follows that for $\mu + \nu = n$, $A_{\mu\nu}$ is a linear function of the coefficients D_{mn} in $a_n(\theta)$, and furthermore $A_{\mu\nu}$ is independent of $a_{n'}$, for $n' > n$.

These last observations allow, in principle, a straightforward way of determining the a_n while making the $A_{\mu\nu}$ of successively increasing order zero. Assume that the $A_{\mu\nu}$ of order $n - 1$ are zero. With the digital computer, the x^* corresponding to a suitable set of orbits is determined for a particular set of D_{mn} in a_n. Now, by graphical or algebraic means, the coefficients $A_{\mu\nu}$ of order n are evaluated. The procedure is then repeated, after making an increment in one of the D_{mn}. In this manner, the sensitivities $\partial A_{\mu\nu}/\partial D_{mn}$ are all obtained. Now, due to the linear dependence of $A_{\mu\nu}$ (of order n) on the D_{mn}, the appropriate changes ΔD_{mn} in the D_{mn} coefficients, in order to make the $A_{\mu\nu}$ zero, are obtained by solving the linear equations[7]

$$A_{\mu\nu} + \sum_m \left(\frac{\partial A_{\mu\nu}}{\partial D_{mn}}\right) \Delta D_{mn} = 0 \quad . \tag{4}$$

In practice -- because of various sources of numerical errors -- we were able to obtain about a factor of four reduction in the values of the $A_{\mu\nu}$, for each cycle of the above process.

In Figs. 1 and 2, examples are given of the efficacy of azimuthally varying fields in reducing the aberration coefficients $A_{\mu\nu}$. The figures are explained in the figure captions.

It is evident from Eq. (2) that we have included seven free parameters in each of the field coefficients $a_i(\theta)$. Even in sixth order, however, there are only four radial aberration coefficients $A_{\mu\nu}$, and hence there are extra degrees of freedom available for the reduction of other types of aberration coefficients. In Section 6 we give an example in which attention is also devoted to the cross term between the axial aperture parameter ϕ_z and the source height parameters y_0 . In the present case, with emphasis upon only the $A_{\mu\nu}$, we have chosen to employ a minimum number of D_{mn} taken so as to remove the aberrations with the smallest magnitudes of AVF .

5. The Spectrometer: Specification and Characteristics

In Table I are given the values of the AVF coefficients a_n determined in the way described. Also included are the dispersion, the radial and axial magnifications, and the solid angle at a point source resolution of 1.0×10^{-4} in momentum.

The radial aberrations (for a point source) of the spectrometer field are displayed in Fig. 3. The aperture displayed in the same figure contains those aperture directions giving radial aberrations contained within a total radial image width of $10^{-3}r_0$. Since the momentum dispersion of the field is 10, this width corresponds to a momentum resolution of 1.0×10^{-4} . At that resolution the solid angle is approximately 0.9% of 4π .

No particular attention was paid to the axial aberrations except in the choice of a_1 , which was determined to give first-order focusing in both ϕ_r and ϕ_z . With the field given in Table I the size of the maximum axial aberration within the aperture amounts to $\pm$ 0.02 , and for most orbits is considerably smaller.

There are still some fifth and higher order aberrations present in the aberration pattern shown in Fig. 3. By further attention to a_5 and a_6 , these aberrations could be reduced; however, aberrations of higher than the sixth order are already becoming prominent. These high-order aberrations are not accessible for free adjustment in the present program. Therefore, only a minor (and probably spurious) improvement in the solid angle of the spectrometer of Table I can be obtained within the confines of the present computer program.

We note in Table I that the magnitude of the AVF coefficients do not much exceed unity and do not show any tendency to increase with order n.

6. Luminosity: Use of extended source

The choice of the fairly large focusing angle of 6 radians for the case described in Table I is compatible with the desire of obtaining a high dispersion and hence the ability to accommodate relatively wide sources. As is seen below, the possible source width is proportional to the magnitude of the ratio between the radial dispersion and the radial magnification. In the field given in Table I the magnitude of this ratio is 12, i.e., 3 times that of the well-known $\pi\sqrt{2}$ spectrometer and hence very favorable from the point of view of high luminosity. Apart from this concern for a good luminosity, the determination of appropriate structures for the AVF coeffieients $a_n(\theta)$ in Table I was governed entirely by the requirement that the radial aperture aberration coefficient $A_{\mu\nu}$ should vanish, implying favorable performance in the case of a point source. As was remarked earlier, with the form of $a_n(\theta)$ used in the present work there are remaining degrees of freedom available to satisfy additional requirements on the performance of the spectrometer field. In this section, we shall make some comments related to the question of whether one can use such further degrees of freedom to arrange particularly favorable performance when an extended source is used.

If x_0 and $2y_0$ are the radial and axial source dimensions, respectively, the radial image width is in lowest order given by $|M|\, x_0$, where M is the radial magnification. If ω is the desired resolution in momentum, and D the dispersion, one must choose x_0 such that

$$\frac{|M| x_0}{|D|} \leq \omega \text{ , i.e., } x_0 \leq \left|\frac{D}{M}\right| \omega \ .$$

To allow a wide source one clearly wants the ratio $|D/M|$ as large as possible. An analytic treatment reveals that the ratio D/M can be expressed as

$$\frac{D}{M} = -\int_0^{\theta_F} x_1(\theta)\, d\theta \ , \tag{5}$$

where $x_1(\theta)$ is a function such that $\phi_r \cdot x(\theta)$ is the first-order radial orbit solution for a particle leaving the source at $x_0 = y_0 = 0$ with the direction ϕ_r and with such a momentum that the central circle $x = y = 0$ becomes the orbit when $\phi_r = \phi_z = 0$. In the AVF case given in Table I, one has $|D/M| = 12$ and $\theta_F = 6$. The magnitude of $x_1(\theta)$ is such that for $\phi_r = 0.1$ the maximum radial departure of the orbit from the central circle is approximately 0.5 . It is evident that any dramatic increase in the ratio $|D/M|$ will involve either a focusing angle $\theta_F >> 6$ or such a magnitude of the function $x_1(\theta)$ that a favorable radial aperture becomes incompatible with a reasonably moderate radial spread of the orbits. In these respects the AVF case does not differ significantly from the rotationally symmetric case, and hence the source will normally have to be limited in width in much the same way.

When the source width is limited by the condition that the contribution $|M| x_0$ to the radial image width is to be tolerable, the possible source area will be determined by the permissible source height $2y_0$. This enters in lowest order through terms of the type y_0^2 and $y_0 \phi_z$ in the radial image width. Of these the y_0^2 term is easily compensated by bending the source by an appropriate amount. However, there is no similarly simple method of getting rid of a $y_0^2 \phi_z$ term. In the spectrometer field given in Table I it turns out that the magnitude of the coefficient for the $y_0 \phi_z$ term is fairly large, of the order of unity. With an axial aperture of $|\phi_z| \leq 0.3$, this means that the contribution to the radial image width from the $y_0 \phi_z$ term equals that from the source width for a total source height equal to about three times the source width. Although these source proportions are perhaps not too different from those often chosen in practice for various other reasons, it is evident that when highest possible luminosity is essential, the magnitude of the $y_0 \phi_z$ term will be a real limitation. For this reason we investigated whether the further degrees of freedom contained in the present choice of structure for $a_2(\theta)$ could be used to cancel the coefficient for the $y_0\phi_z$-term. The procedure was similar to that used when setting the $\Lambda_{\mu\nu}$ coefficients equal to zero, i.e., observing the fact that the coefficient of the $y_0\phi_z$ term must depend linearly on the D_{m2} in $a_2(\theta)$. It turns out that simultaneous vanishing of the ϕ_r^2 , ϕ_z^2 , and $y_0\phi_z$ terms can be obtained, for instance, when

$$a_2(\theta) = 1.180556 - 0.121764 \sin\theta + 1.121828 \cos\theta \quad .$$

For this value of $a_2(\theta)$ the magnitudes of the three aberration coefficients are all smaller than 0.005 . A comparison with $a_2(\theta)$ in Table I shows that the cancelling of the $y_0\phi_z$ term has been arranged by only a very minor change in the AVF structure in $a_2(\theta)$. We therefore believe that an extension to higher orders should present no divergence difficulties of the kind mentioned in Section 3.

The possibility of handling cross-terms between the aperture parameters ϕ_r , ϕ_z and the source parameters x_0 , y_0 must be considered an outstanding feature of the AVF approach to spectrometer design. If the electrostatic method of Ref. 3 for the reduction of the influence of the source width on the resolution is used in conjunction with an AVF spectrometer, one should have the possibility of mastering in the radial image width the (i) pure source parameter terms, (ii) the pure aperture terms, and (iii) the cross-terms. For appreciable source width and source height, and appreciable aperture solid angle one has then, in principle, the possibility of achieving an arbitrarily good resolution.

7. Realization of the Spectrometer Field

(a) Stability requirements

When relatively large aperture angles are combined with very high resolution, it is to be expected that the field shape must be realized and maintained at a very high precision in order not to adversely affect the inherent performances of the system. This precision, naturally, refers both to the particular field symmetries employed in the field expansion of Eq. (1) and to the realization of the appropriate values of the AVF coefficients a_n. Some quantitative ideas about the sensitivity of the radial focusing with respect to small departures from the proper field coefficients can be obtained by inspecting how rapidly an aberration coefficient $A_{\mu\nu}$ of order n varies for changes in the field constants D_{mn} in a_n. For the ϕ_r, ϕ_r^3, and ϕ_r^5 terms in the radial aberration the sensitivities to changes in the respective constant term coefficients D_{0n} and a_n are roughly:

$$\frac{\partial A_{10}}{\partial D_{01}} \approx -20 \; ; \quad \frac{\partial A_{30}}{\partial D_{03}} \approx -250 \; ; \quad \frac{\partial A_{50}}{\partial D_{05}} \approx -3000 \; .$$

The dependences on the azimuthally varying parts in the respective $a_n(\theta)$ are roughly the same, or smaller.

With a maximum magnitude of ϕ_r in the utilized aperture equal to 0.1, a change in the first-order coefficient A_{10} becomes most critical among the three aberration coefficients included above. If the accompanying change in higher order aberrations is neglected, a radial broadening of 10% of an image width of 0.001 is obtained for a relative departure of 3 parts in 10^5 from the proper value $D_{01} = -1.11473$ in a_1. The accompanying influence on higher order terms may increase the sensitivity further. Clearly a precision of the order of 1 part in 10^5 will be required at least in the first-order field parameters.

(b) Iron magnet

If iron of infinite permeability is assumed, and the effect of the fringing field is neglected, the appropriate shape of the pole surfaces is that of the equipotential surfaces corresponding to the field of Eq. (1). Thus, from Eq. (A2) of the Appendix, the pole surface is specified by

$$\sum_{i,j=0}^{\infty} a_{ij}(\theta) \frac{x^i y^{j+1}}{(j+1)} = \text{Constant} \quad , \tag{6}$$

where $a_{ij}(\theta)$ is obtained by comparing Eqs. (A.2) and (A.6).

Although an iron magnet is attractive from the point of view of simplicity, it seems improbable that, for instance, the desired precision of 10^{-5} in a_1 can be obtained with iron, in view of remnant field effects and the variation of iron permeability with field strength. Recent progress in the design of high-field iron magnets with digital computers might, however, alter this conclusion.[8]

(c) Air-core coil

The precision requirements on field shape can most readily be satisfied by an iron-free coil system. Stimulated by the problem encountered here, of finding a suitable coil configuration, Laslett has recently established the following result:[9] Given a specified static magnetic field and a closed surface S, there exists a current distribution on S that will produce the specified field within S. Furthermore, Laslett has given an explicit prescription for determining the current distribution. Clearly this theorem establishes the existence of an adequate coil structure, but the resulting current distribution will -- in general -- be exceedingly complicated and furthermore consist of large almost cancelling currents. One needs, in practice, to find a coil system not necessarily confined to S but only exterior to S, and so chosen as to be convenient with regard to mechanical construction and economical with regard to power consumption. Laslett's theorem allows a systematic study of the non-unique nature of the coil system, but such a study has not yet been undertaken.

(d) Alternative approach

The form of the spectrometer field which was adopted in this work [namely Eq. (1)] was motivated by the desire to determine -- as easily as possible -- whether or not AVF could lead to an interesting spectrometer design. Prior to this investigation, we had only the suggestive work of Ref. 4, and it was quite unclear whether or not cross-terms and higher order aberrations could be removed with reasonable fields. By choosing a field as in Eq. (1) we could circumvent the problem of relating coil configuration to field coefficients, and thus concentrate directly on whether AVF were advantageous. We have now demonstrated the advantages of AVF with respect to achieving certain desired imaging properties. Clearly the actual field which accomplishes such properties is not unique: already within the restrictions imposed by the kind of field expansion and field coefficients used in Eq. (1) we had different choices. The field restrictions implied in, e.g., the choice of the field structure given in Table I, may well be unsuitable with respect to coil design. In the over-all computational and technical problem of realizing in practice an AVF system, it might be more profitable to apply a criterion of simplicity not to the choice of field expansion and choice of field coefficients, but rather to the choice of general coil structure. An approach along these lines may proceed by starting with various technically feasible sets of coil configurations (each specified by a number of adjustable parameters). One would determine the fields associated with each coil configuration, and then the associated aberrations. Clearly, since in general a particular coil will produce fields of all orders, the process of removing aberrations will be much more involved than that reported here. Of particular convenience would be some way of specifying "linearly independent" coil configurations; in the absence of such insight the procedure would appear to be feasible, but exceedingly time consuming.

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APPENDIX

This Appendix contains the mathematical specification of the equations of motion and field expansion, briefly described in Section 2. Further details of the computer program, which was written to solve the equations, may be found in Ref. 6.

The equations of motion of a particle having momentum p are

$$x'' = \frac{Q^{1/2}}{(1+\varepsilon)(1+x)}\left[(1+x)y'\, b_\theta + x'y'b_r - (Q-y'^2)b_y\right] + \frac{x'^2 - y'^2 + Q}{(1+x)} ,$$

$$y'' = \frac{Q^{1/2}}{(1+\varepsilon)(1+x)}\left[-(1+x)x'b_\theta - x'y'b_y + (Q-x'^2)b_r\right] + \frac{2x'y'}{1+x} , \quad \text{(A.1)}$$

where

$$x = (r - r_0)/r_0 , \quad y = z/r_0 ,$$

r is the particle radius, z is the axial displacement, r_0 is the central circle, b_z, b_r, and b_θ are the normalized components of the magnetic field as specified below, $\varepsilon = \dfrac{p - p_0}{p_0}$,

$$p_0 = \frac{e\, r_0 B_0}{c} , \quad Q = x'^2 + y'^2 + (1+x)^2 ,$$

and primes denote derivatives with respect to θ .

The magnetic field components may be expanded in power series about the central circle $r = r_0$, $z = 0$:

$$b_z = \frac{B_z}{B_0} = \sum_{i,j=0}^{\infty} a_{ij}(\theta)\, x^i y^j \quad , \tag{A.2}$$

$$b_r = \frac{B_r}{B_0} = \sum_{i,j=0}^{\infty} b_{ij}(\theta)\, x^i y^j \quad , \tag{A.3}$$

$$b_\theta = \frac{B_\theta}{B_0} = \sum_{i,j=0}^{\infty} c_{ij}(\theta)\, x^i y^j \quad . \tag{A.4}$$

Fields symmetric about the median plane, and satisfying Maxwell's equations for a source-free region, namely

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{B} = 0 \quad ,$$

$$\underset{\sim}{\nabla} \times \underset{\sim}{B} = 0 \quad , \tag{A.5}$$

may be expressed in terms of $a_i = a_{i0}(\theta)$. In particular, through sixth order:

$$\begin{aligned} b_z = {} & a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \\ & - \left(\frac{1}{2} a_1 + a_2\right) y^2 + \left(\frac{1}{2} a_1 - a_2 - 3a_3 - \frac{1}{2} a_1''\right) xy^2 \\ & + \left(- \frac{1}{2} a_1 + a_2 - \frac{3}{2} a_3 - 6a_4 + a_1'' - \frac{1}{2} a_2''\right) x^2 y^2 \\ & + \left(\frac{1}{24} a_1 - \frac{1}{12} a_2 + \frac{1}{2} a_3 + a_4 + \frac{1}{6} a_2'' - \frac{1}{12} a_1''\right) y^4 \end{aligned}$$

$$+\left(\frac{1}{2}a_1 - a_2 + \frac{3}{2}a_3 - 2a_4 - 10a_5 - \frac{1}{2}a_3 - a_2'' - \frac{3}{2}a_1''\right)x^3y^2$$

$$+\left(-\frac{1}{8}a_1 + \frac{1}{4}a_2 - \frac{3}{4}a_3 + 2a_4 + 5a_5 + \frac{1}{2}a_3'' - \frac{1}{2}a_2'' + \frac{5}{12}a_1'' + \frac{1}{24}a_1^{(4)}\right)xy^4$$

$$+\left(-\frac{1}{2}a_1 + a_2 - \frac{3}{2}a_3 + 2a_4 - \frac{5}{2}a_5 - 15a_6 - \frac{1}{2}a_4'' + a_3'' - \frac{3}{2}a_2'' + 2a_1''\right)x^4y^2$$

$$+\left(\frac{1}{4}a_1 - \frac{1}{2}a_2 + \frac{9}{8}a_3 - \frac{5}{2}a_4 + 5a_5 + 15a_6 + a_4'' - \frac{5}{4}a_3'' + \frac{7}{6}a_2'' - \frac{7}{6}a_1'' + \frac{1}{24}a_2^{(4)} - \frac{1}{6}a_1^{(4)}\right)x^2y^4$$

$$+\left(-\frac{1}{80}a_1 + \frac{1}{40}a_2 - \frac{1}{20}a_3 + \frac{1}{10}a_4 - \frac{1}{2}a_5 - a_6 - \frac{1}{10}a_4'' + \frac{1}{20}a_3'' - \frac{7}{120}a_2'' + \frac{1}{16}a_1'' - \frac{1}{120}a_2^{(4)} + \frac{1}{80}a_1^{(4)}\right)y^6$$

$$-\frac{a_0''y^2}{(1+x)^2} + \frac{y^4}{6(1+x)^4}\left(a_0'' + \frac{1}{4}a_0^{(4)}\right) - \frac{y^6}{90(1+x)^6}\left(8a_0'' + 5a_0^{(4)} + \frac{1}{8}a_0^{(5)}\right) , \qquad (A.6)$$

$$b_r = a_1y + 2a_2xy + 3a_3x^2y + 4a_4x^3y + 5a_5x^4y + 6a_6x^5y$$

$$+\left(\frac{1}{6}a_1 - \frac{1}{3}a_2 - a_3 - \frac{1}{6}a_1''\right)y^3 + \left(-\frac{1}{3}a_1 + \frac{2}{3}a_2 - a_3 - 4a_4 - \frac{1}{3}a_2'' + \frac{2}{3}a_1''\right)xy^3$$

$$+\left(-\frac{1}{40}a_1 + \frac{1}{20}a_2 - \frac{3}{20}a_3 + \frac{2}{5}a_4 + a_5 + \frac{1}{10}a_3'' - \frac{1}{10}a_2'' + \frac{1}{12}a_1'' + \frac{1}{120}a_1^{(4)}\right)y^5$$

$$+\left(\frac{1}{2}a_1 - a_2 + \frac{3}{2}a_3 - 2a_4 - 10a_5 - \frac{1}{2}a_3'' + a_2'' - \frac{3}{2}a_1''\right)x^2y^3$$

$$+ \left(\frac{1}{10} a_1 - \frac{1}{5} a_2 + \frac{9}{20} a_3 - a_4 + 2a_5 + 6a_6 + \frac{2}{5} a_4'' - \frac{1}{2} a_3'' + \frac{7}{15} a_2'' - \frac{7}{15} a_1'' + \frac{1}{60} a_2^{(4)} - \frac{1}{15} a_1^{(4)}\right) xy^5$$

$$+ \left(- \frac{2}{3} a_1 + \frac{4}{3} a_2 - 2a_3 + \frac{8}{3} a_4 - \frac{10}{3} a_5 - 20a_6 - \frac{2}{3} a_4'' + \frac{4}{3} a_3'' - 2a_2'' + \frac{8}{3} a_1''\right) x^3y^3$$

$$+ \frac{a_0''y^3}{3(1+x)^3} - \frac{y^5}{15(1+x)^5} \left(2a_0'' + \frac{1}{2} a_0^{(4)}\right) , \qquad (A.7)$$

$$b_\theta = a_1'xy + \left(- a_1' - a_2'\right)x^2y + \left(- \frac{1}{6} a_1' - \frac{1}{3} a_2'\right)y^3 + \left(\frac{1}{3} a_1' - a_3' - \frac{1}{6} a_1'''\right)xy^3$$

$$+ \left(a_1' - a_2' + a_3'\right)x^3y + \left(- a_1' + a_2' - a_3' + a_4'\right)x^4y$$

$$+ \left(- \frac{1}{2} a_1' + \frac{1}{3} a_2' + \frac{1}{2} a_3' - 2a_4' - \frac{1}{6} a_2''' + \frac{1}{2} a_1'''\right)x^2y^3$$

$$+ \left(\frac{1}{120} a_1' - \frac{1}{60} a_2' + \frac{1}{10} a_3' + \frac{1}{5} a_4' + \frac{1}{30} a_2''' - \frac{1}{60} a_1'''\right)y^5$$

$$+ \left(- \frac{1}{30} a_1' + \frac{1}{15} a_2' - \frac{1}{4} a_3' + \frac{1}{5} a_4' + a_5' + \frac{1}{10} a_3''' - \frac{2}{15} a_2''' + \frac{1}{10} a_1''' + \frac{1}{120} a_1^{(5)}\right)xy^5$$

$$+ \left(\frac{2}{3} a_1' - \frac{2}{3} a_2' + \frac{4}{3} a_4' - \frac{10}{3} a_5' - \frac{1}{6} a_3''' + \frac{1}{2} a_2''' - a_1'''\right)x^3y^3$$

$$+ \left(a_1' - a_2' + a_3' - a_4' + a_5'\right)x^5y + \frac{a_0'y}{(1+x)} - \frac{a_0'''y^3}{6(1+x)^3}$$

$$+ \frac{y^5}{60(1+x)^5} \left(2a_0''' + \frac{1}{2} a_0^{(5)}\right) . \qquad (A.8)$$

The azimuthal dependence of a_1 was taken as indicated in Eq. (2).

FOOTNOTES AND REFERENCES

1. H. Daniel, Rev. Sci. Instr. 31, 249 (1960).

2. G. E. Lee-Whiting, Canad. J. of Phys. 41, 496 (1963).

3. K.-E. Bergkvist, Arkiv Fysik 27, 383 (1964).

4. A. M. Sessler, Nucl. Instr. and Meth. 23, 165 (1963).

5. F. Schmutzler, H. Daniels, and L. Tauscher, Nucl. Instr. and Meth. 35, 176 (1965).

6. H. Owens, ISO--A Fortran Program for Utilizing the Strong-Focusing Principle in Beta-Ray Spectrometer Design, Lawrence Radiation Laboratory Report UCRL-16471, Oct. 21, 1965 (unpublished).

7. This straight-forward procedure could fail in two different ways. Firstly, it is possible that $\det(\partial\Lambda_{\mu\nu}/\partial D_{mn}) = 0$ so the set of equations for ΔD_{mn} cannot be solved. This would imply that some of the $\dot{\Lambda}_{\mu\nu}$ cannot be made zero. Secondly, it is possible that the field coefficients D_{mn}, which are obtained from Eq. (4), become larger in each successive order so that the expansion of Eq. (1) is nonconvergent. Although we have seen cases in which one or the other of these difficulties appear, we have also -- as detailed in Section 5 -- found cases in which neither difficulty appears.

8. 200 BeV Accelerator Design Study, Volume I, Lawrence Radiation Laboratory Report, June 1965, Chapter V (unpublished).

9. L. Jackson Laslett, An Equivalent Distribution of Surface Currents: For the Generation of a Prescribed Static Magnetic Field Within the Enclosed Volume, Lawrence Radiation Laboratory Report UCRL-16587, Dec. 20, 1965 (to be published in Journ. of Appl. Phys.).

Table I. Data for an azimuthally varying magnetic field giving simultaneous radial and axial focusing.

Focusing angle: 6 radians

Solid Angle at 0.01% point source resolution: 0.9% of 4π

Dispersion $(\Delta r/r : \Delta p/p)$: 10.0

Radial magnification: - 0.81

Axial magnification: 0.90

Field coefficients:

$$a_1 = -1.11473 - 0.659 \cos\theta$$
$$a_2 = 1.223 + 1.1837 \cos\theta$$
$$a_3 = -1.1335 - 1.355 \cos\theta$$
$$a_4 = 0.8865 + 1.277 \cos\theta - 0.0066 \sin 2\theta$$
$$a_5 = 0.09 + 0.04 \cos\theta + 0.548 \cos 2\theta$$
$$a_6 = -0.22$$

FIGURE CAPTIONS

Fig. 1. The coefficient $a_1(\theta)$ has been determined to give first-order radial and axial focusing at $\theta = 6$. If the radial aberration term $\phi_z^{\,2}$ is made zero ($A_{02} = 0$) without introducing AVF in $a_2(\theta)$ ($D_{m2} = 0$ for $m \neq 0$), one inevitably obtains a large $\phi_r^{\,2}$ term (broken lines) in the radial aberration. When AVF is introduced in a_2 it is possible to simultaneously cancel both the $\phi_z^{\,2}$ and $\phi_r^{\,2}$ term. This is illustrated by the solid line which has been obtained by taking $a_2 = 1.223 + 1.1837 \cos\theta$.

Fig. 2. This figure shows radial aberration patterns at two intermediate stages of the determination of the field coefficients given in Table I. The field coefficients through third order have essentially already been established. However, there is some third-order contribution of the type $\phi_r\phi_z^{\,2}$ present in the pattern of Fig. 2(a), giving the slope of the $\phi_z = 0.15$ curve at $\phi_r = 0$. The finite aberration magnitude at this point is essentially due to a $\phi_z^{\,4}$ term. There are also prominent contributions of the types $\phi_r^{\,2}\phi_z^{\,2}$ and $\phi_r^{\,3}\phi_z^{\,2}$ present in the pattern. The mentioned aberration types have all been much reduced in Fig. 2(b): the third and fourth order terms by means of appropriate changes in a_3 and a_4 and the fifth order $\phi_r^{\,3}\phi_z^{\,2}$ as a happy consequence of these changes. The dominant aberration type left is of the type $\phi_r^{\,5}$.

Fig. 3. Behavior of the radial aberrations of the spectrometer field, having the parameters presented in Table I, as functions of ϕ_r for three different values of ϕ_z(0, 0.15, 0.30). Also included is the shape of an entrance aperture containing orbits which form an image with a radial width not exceeding $10^{-3}r_0$, i.e., giving a point source resolution of 1.0×10^{-4} in momentum. At this resolution the solid angle of the entrance aperture is 0.9% of 4π.

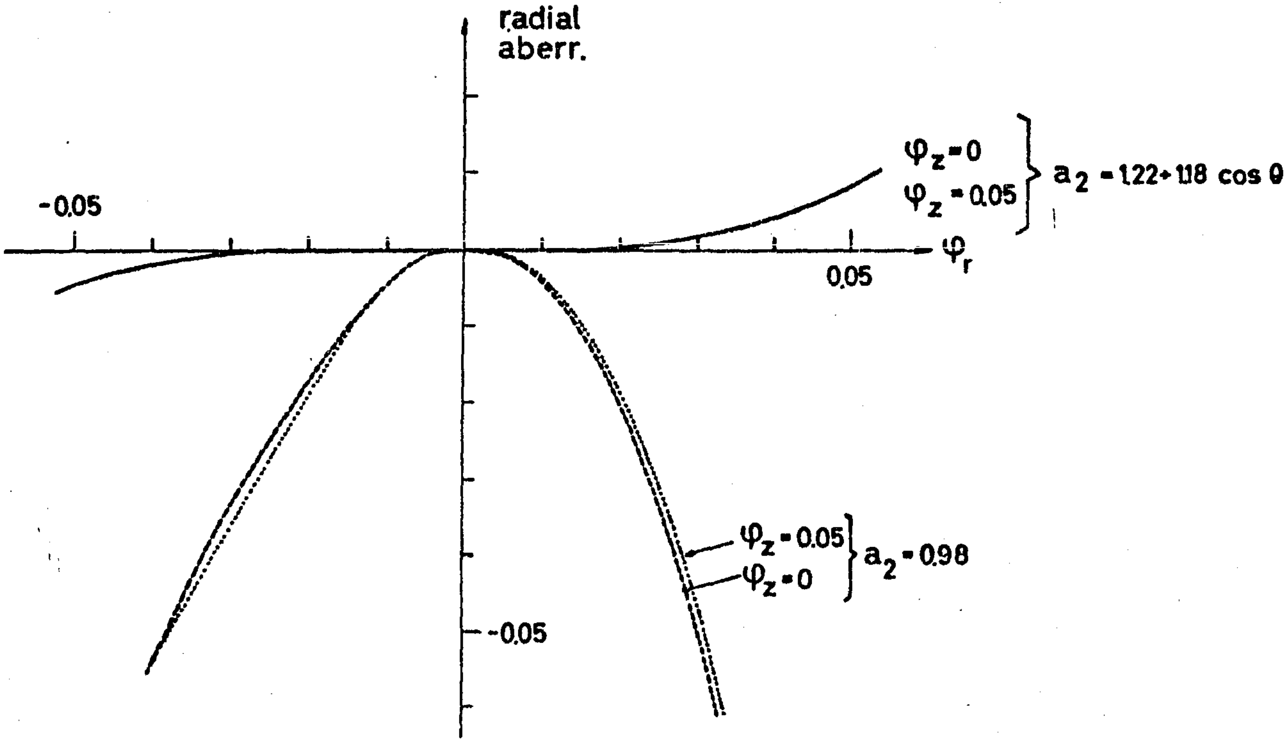

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Fig. -1

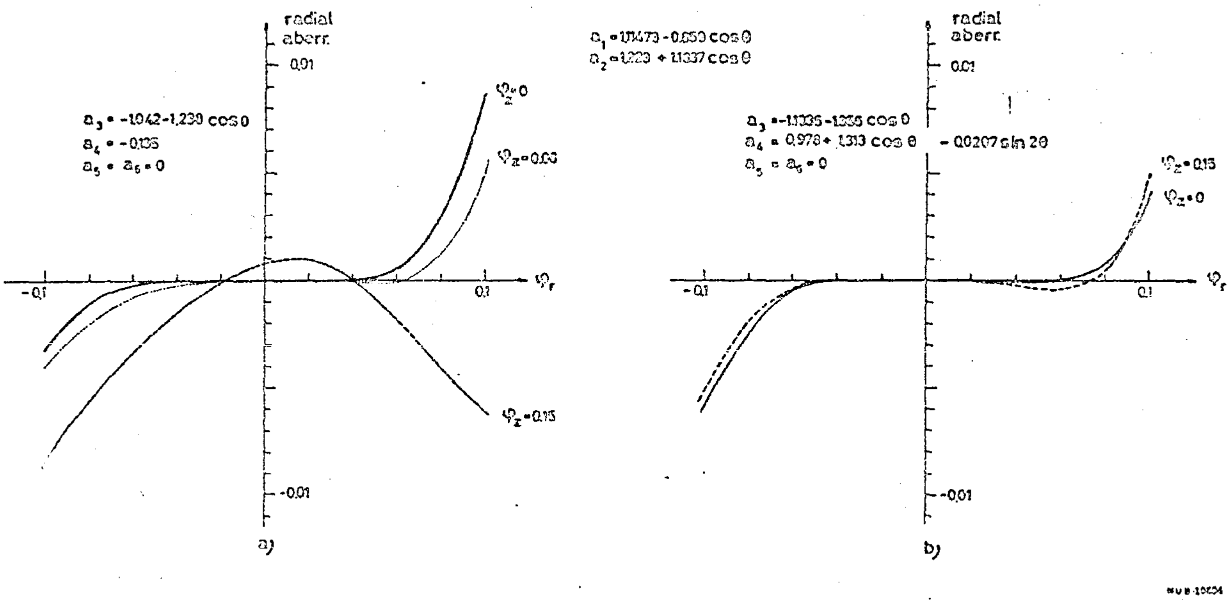
radial
aberr.
0,01
a_3 = -1,042-1,239 cos θ
a_4 = -0,135
a_5 = a_6 = 0
φ_z = 0
φ_z = 0,03
φ_r
-0,1
0,1
φ_z = 0,15
-0,01
a)
a_1 = 1,1473 - 0,650 cos θ
a_2 = 1,229 + 1,1337 cos θ
radial
aberr.
0,01
a_3 = -1,1335 - 1,356 cos θ
a_4 = 0,978 + 1,313 cos θ - 0,0207 sin 2θ
a_5 = a_6 = 0
φ_z = 0,15
φ_z = 0
φ_r
-0,1
0,1
-0,01
b)

Fig. 2

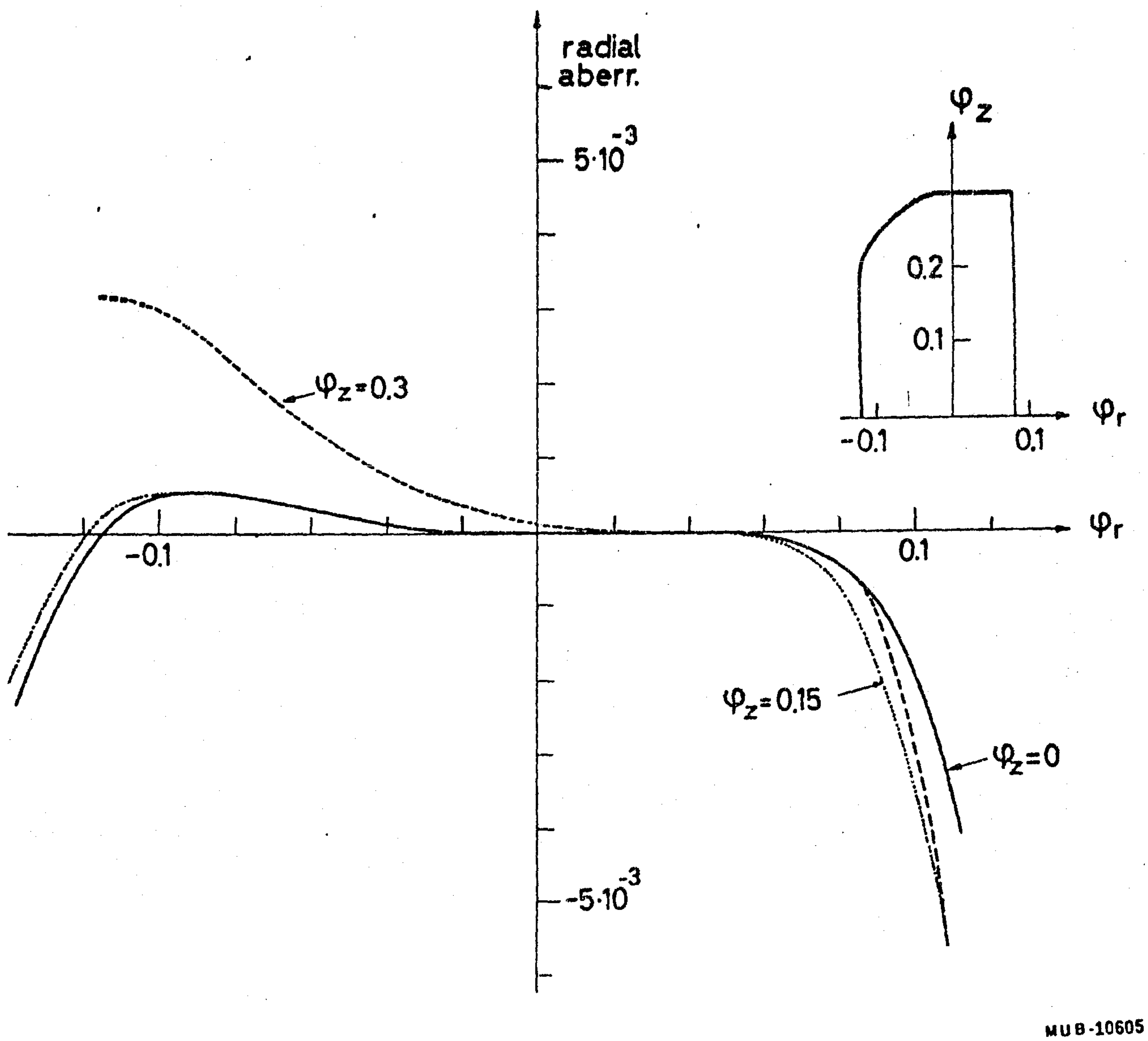
radial
aberr.
5·10^-3
-5·10^-3
φz=0.3
φz=0.15
φz=0
φr
-0.1
0.1
φz
0.2
0.1
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Fig. 3