

Topological symmetry, spin liquids and CFT duals of Polyakov model with massless fermions

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ABSTRACT: We prove the absence of a mass gap and confinement in the Polyakov model with massless complex fermions in any representation of the gauge group. A $U(1)_*$ topological shift symmetry protects the masslessness of one dual photon. This symmetry emerges in the IR as a consequence of the Callias index theorem and abelian duality. For matter in the fundamental representation, the infrared limits of this class of theories interpolate between weakly and strongly coupled conformal field theory (CFT) depending on the number of flavors, and provide an infinite class of CFTs in $d = 3$ dimensions. The long distance physics of the model is same as certain stable spin liquids. Altering the topology of the adjoint Higgs field by turning it into a compact scalar does not change the long distance dynamics in perturbation theory, however, non-perturbative effects lead to a mass gap for the gauge fluctuations. This provides conceptual clarity to many subtle issues about compact QED₃ discussed in the context of quantum magnets, spin liquids and phase fluctuation models in cuprate superconductors. These constructions also provide new insights into zero temperature gauge theory dynamics on $\mathbb{R}^{2,1}$ and $\mathbb{R}^{2,1} \times S^1$. The confined versus deconfined long distance dynamics is characterized by a discrete versus continuous topological symmetry.

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1. Introduction

The Polyakov model, Yang-Mills theory with an adjoint Higgs scalar on \mathbb{R}^3 , is one of the cornerstones in the study of confinement in gauge theories [1]. Abelian duality is used to show the emergence of a mass gap, and to exhibit linear confinement via the proliferation of the monopoles in the vacuum. Another theory which realizes confinement and a mass gap similarly, i.e. via the proliferation of the flux (or monopoles) is compact lattice QED₃. These are two different microscopic theories with a different set of symmetries at the cut-off scale. However, at long distances, they are gapped, and they flow to the same theory, constituting a non-perturbative long distance duality.

Although we do not know whether the Polyakov model is relevant in Nature, the lattice QED₃ with fermionic fields appears in two dimensional spin systems, in the spin liquid approach to high

T_c superconductivity and in the phase fluctuation model of the cuprate superconductors (See the reviews [2, 3] and [4].) Therefore, the issue of deconfined versus confined long distance characteristic of 2+1 dimensional lattice QED with fermionic matter is experimentally relevant. An important question in this context is the existence (or non-perturbative stability) of the spin liquids, the non-magnetic Mott insulators with no broken symmetries. In QED₃, this question translates into whether the strongly coupled fermions and gauge fluctuations remain massless in the long distances, when the non-perturbative effects (consistent with microscopic symmetries) are taken into account. If so, this implies deconfinement and stability. In the literature, a permanent confinement and instability was argued in [5–7]. Ref. [8, 9] showed that, at least in a large n_f limit where $SU(2)$ spin symmetry is generalized to $SU(n_f)$, there are some spin liquids which are stable. For small numbers of fermionic flavors, which is experimentally most interesting, this is still an unsettled matter.

In this work, we discuss a variety of related gauge theories, each of which needs to be distinguished very carefully via their microscopic symmetries. For example, consider non-compact continuum QED₃ minimally coupled to $2n_f$ flavors of fundamental fermions, and assume one wishes to incorporate the compactness of the gauge field. We show that, common bottom-up arguments which claim to account for the compactness of the gauge fields are ill-defined, due to non-uniqueness of this procedure. In the continuum, a standard way to obtain compact QED₃ is via the gauge “symmetry breaking” $SU(N) \rightarrow U(1)^{N-1}$ in a parent Yang-Mills adjoint Higgs systems. We show that there are at least two classes of parent theories which differ in the topological structure of their adjoint Higgs field (compact versus noncompact), yet both lead to the desired gauge symmetry breaking and reduce (necessarily) to continuum compact QED₃. Although indistinguishable in perturbation theory, the non-perturbative behavior of these theories are strikingly opposite: In the theory with non-compact adjoint Higgs scalar (Polyakov model with massless fermions), we demonstrate

$$\underbrace{SU(N)}_{\text{with noncompact scalar}} \xrightarrow{\text{Higgsing}} \underbrace{[U(1)]^{N-1}}_{\text{compact QED}_3} \xrightarrow{\text{nonperturbative}} \underbrace{U(1)}_{\text{CFT or free photon}}, \quad (1.1)$$

the existence of a massless photon in the long distances, and the absence of confinement. Of course, the dramatic behavior here is the appearance of a conformal field theory (CFT) in certain cases, to be discussed below. In the theory with compact adjoint Higgs field, the gauge structure reduces at longer distances as (for moderately small number of flavors)

$$\underbrace{SU(N)}_{\text{with compact scalar}} \xrightarrow{\text{Higgsing}} \underbrace{[U(1)]^{N-1}}_{\text{compact QED}_3} \xrightarrow{\text{nonperturbative}} \underbrace{\text{nothing}}_{\text{gapped gauge bosons}}. \quad (1.2)$$

The photon gains a mass, and the theory confines. As opposed to the common assertions in the literature, the presence or absence of monopoles has nothing to do with the confining or deconfining behavior of a *generic* gauge theory. (See the table.1). We introduce a sharp (topological) symmetry characterization to describe the long distance limits (deconfined versus confined, and more delicate refinements) of gauge theories on \mathbb{R}^3 and small $S^1 \times \mathbb{R}^3$.¹

¹Naively, the (1.1) seems to be in accord with Ref. [8, 9], and (1.2) seems to be coinciding with the results of [5–7]. This is not quite correct. The references [5–9] study a spin Hamiltonian which, in the π -flux state, maps into a compact lattice QED₃ with fermions. The global symmetries of this lattice theory is different (although related, see §.4.2) from the continuum discussion above. Despite these differences, we will establish precise non-perturbative long distance dualities between spin system and Polyakov model with massless fermions in certain cases.

We first discuss the question of confinement in the Polyakov model with massless fermions, either in real and complex representations. The answer is known for one real representation adjoint Dirac fermion [10]. The fermion number symmetry breaks down spontaneously, and there is a gapless Nambu-Goldstone boson (the dual photon). The masslessness of dual photon is protected by symmetry breaking order, i.e, Goldstone theorem, and the adjoint fermion acquires a mass. For complex representation fermions, the infrared is more interesting. There are strongly coupled gauge fluctuations and fermions which remain massless in the infrared. The answer entails a different mechanism to keep fermions and a boson massless. It is referred as quantum order (or non-symmetry breaking order) in condensed matter physics [11,12]. The appearance of quantum order in the Polyakov model is new. In the first application, the spontaneous breaking of a global symmetry generates and protects a massless boson, in the latter, the unbroken symmetry implies the existence of massless boson and fermions.

The main concept behind the deconfinement in the Polyakov model with massless fermions is a $U(1)_*$ **topological symmetry**. This symmetry arises in the long distance and protects only one dual photon from acquiring a mass. It relies on the Jackiw-Rebbi zero modes and the index theorem of Callias [13,14]. Due to the index theorem, a $U(1)_A$ symmetry of the high energy theory transmutes into a shift symmetry for the dual photon. For complex representation fermions, the combination of the topological symmetry and other global symmetries is very powerful, and they severely restrict any perturbative or non-perturbative relevant or marginal operators that may destabilize the masslessness of the strongly interacting photon and fermions. In particular, in theories with $N_f \geq 4$ fundamental fermions are quantum critical due to the absence of relevant or marginal operators which may destabilize their masslessness. We argue that the strong correlation physics of the fermions and gauge boson at long distance produce a scale invariant, conformal field theory (CFT). In three dimensional non-abelian gauge theories, the earlier examples of infrared strongly coupled CFTs are mostly among extended supersymmetric theories [15,16]. The nonsupersymmetric gauge theories discussed in this paper provide an infinite class of infrared CFTs which interpolate between weak and strong coupling as the number of flavors is varied, $4 \leq N_f < \infty$, with a dimensionless coupling constant $\sim \frac{1}{\sqrt{N_f}}$. The $N_f = 2$ theory turns out to be non-critical, due to the presence of a relevant, non-perturbatively generated flux operator with fermion zero mode insertion.

The existence of the continuous $U(1)_*$ topological shift symmetry is the **necessary and sufficient** condition to prove that the photon remains massless in the Polyakov model with massless complex fermions.² In fact, the fundamental distinction between the theories in (1.1) and (1.2) is that, in the latter, the continuous topological shift symmetry for the dual photon is replaced by a discrete one. As opposed to continuous shift symmetry, the discrete shift symmetries cannot prohibit the appearance of a mass term for the scalar. Thus, the photons in the latter case should acquire mass according to symmetry considerations. However, there is the possibility that the monopole fugacity may become irrelevant at large distance in the renormalization group sense. In this case, the long distance theory will exhibit an enhanced topological symmetry relative to the microscopic theory. This implies that the presence of the discrete topological symmetry is necessary, but not **sufficient** for confining behavior.

²For a real massless Majorana fermion in the adjoint representation, there is no $U(1)_*$ symmetry. Such theories on \mathbb{R}^3 do indeed confine. [10].

Finally, equipped with the understanding of the Polyakov models, we turn to the discussion of spin systems. As stated earlier, the spin systems can be mapped into lattice gauge theories in the slave fermion mean field theory. We investigate the relation between the Polyakov model and lattice QED₃, both with massless fermions, in the long distance limit. These are theories with distinct microscopic symmetries. But, perhaps the most significant distinguishing feature of the lattice QED₃ and continuum Polyakov models is the absence of an analog of the Callias index theorem in lattice QED₃ as shown by Marston [17], and the analog of a global $U(1)_A$ symmetry in the lattice model. The first is not as severe as it sounds despite the concerns raised in literature [18]. In fact, the latter is the main problem. We will show that, were the global $U(1)_A$ a symmetry of the spin Hamiltonian, the topological symmetry would indeed arise in the infrared despite the absence of an index theorem. If this were the case, we could have carried a precise analogy with the Polyakov model even at small N_f . Unfortunately, only in the sufficiently large N_f limit can we make a reliable statement about the infrared structure of the lattice theory. In particular, we are not able to improve the discussion given in [8, 9]. In the Polyakov model with massless fermions, we are able to side-step the renormalization group and large N_f analysis of Hermele et.al. [8]. In lattice QED₃, this analysis seems inevitable. Thus, there is a long distance duality between the spin liquids and Polyakov models with massless fermions in the large N_f limit where both theories flow into the same interacting CFT.

2. Gauge theories in three dimensions

We consider $SU(N)$ Yang-Mills gauge theory with a noncompact adjoint Higgs scalar on \mathbb{R}^3 (also known as Georgi-Glashow model) in the presence of massless fermions. The fermions are chosen in complex and real representations such as fundamental(F) and adjoint(adj). We will label these theories as P(F) and P(adj), respectively. Before discussing them, it is useful to review the basics of the pure Polyakov model [1] and set the notation.

2.1 Polyakov model

The action of $SU(2)$ gauge theory with an adjoint scalar is

$$S = \int_{\mathbb{R}^3} \frac{1}{g_3^2} \text{tr} \left[\frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2} (D_\mu \Phi)^2 + V[\Phi] \right] \quad (2.1)$$

Φ is a Lie algebra valued non-compact scalar Higgs field, $F_{\mu\nu}$ is the non-abelian field strength, and $\mu, \nu = 1, 2, 3$. The classical potential $V[\Phi]$ is chosen such that, at tree level, the theory is in its Higgs regime, $SU(2) \rightarrow U(1)$. At long distances, only the abelian components are operative. To all orders in perturbation theory, the infrared is a free (non-interacting) Maxwell theory.

The Gaussian fixed point is destabilized due to nonperturbative instanton (monopole) effects. This instability is easiest to see in a dual formulation where the gauge boson is dualized to a scalar, $F = *d\sigma$.³ Since an instanton has a finite action, they will proliferate due to entropic effects. This generates nonperturbative e^{-S_0} effects in the long distance Lagrangian

$$L = \frac{1}{2} (\partial\sigma)^2 - e^{-S_0} (e^{i\sigma} + e^{-i\sigma}) \quad (2.2)$$

³Our discussion mostly relies on symmetries. Therefore, to lessen the clutter of expressions, we set the dimensionful parameters (e.g. g_3) to one. These parameters will be restored if necessary.

The $\cos \sigma$ is a relevant operator which alters the IR physics drastically, and leads to a mass gap $\sim e^{-S_0/2}$.

It is worth noting that, the dual of the free Maxwell theory, i.e., in the absence of monopoles, described by $L = \frac{1}{2}(\partial\sigma)^2$, has a continuous shift symmetry

$$U(1)_{\text{flux}} : \sigma \rightarrow \sigma - \beta \quad (2.3)$$

which protects σ from acquiring mass. The current associated with the shift symmetry is $\mathcal{J}_\mu = \partial_\mu \sigma = \frac{1}{2}\epsilon_{\mu\nu\rho}F_{\nu\rho} = F_\mu$, and its divergence is zero, $\partial_\mu \mathcal{J}_\mu = \partial_\mu F_\mu = 0$, reflecting the absence of monopoles and conservation of magnetic flux, hence the name $U(1)_{\text{flux}}$.

In the $U(1)$ gauge theory with monopoles, the current \mathcal{J}_μ is not conserved. Its divergence is $\partial_\mu \mathcal{J}_\mu = \nabla^2 \sigma = \partial_\mu F_\mu = \rho_m(x)$ where $\rho_m(x)$ is the monopole charge density. Since the $U(1)_{\text{flux}}$ is no longer a symmetry, there is no symmetry reason for the σ field to remain massless. Indeed, σ acquires a mass as shown in (2.2).

SU(N): More generally, let the $SU(N)$ gauge symmetry be broken down to $U(1)^{N-1}$ via an adjoint Higgs vacuum expectation value

$$\langle \Phi \rangle = \text{Diag}(a_1, \dots, a_N) \quad (2.4)$$

where $a_1 < a_2 < \dots < a_N$. There are $N - 1$ photons which remain massless to all orders in perturbation theory. Let us dualize them into $(F_1, \dots, F_{N-1}) = *d(\sigma_1, \dots, \sigma_{N-1})$. Non-perturbatively, there are $N - 1$ types of elementary monopoles associated with this pattern, which we label by their magnetic charges $\{\alpha_1, \dots, \alpha_{N-1}\}$ where each α_i is an $N - 1$ vector with charges under $U(1)^{N-1}$. The antimonopoles carry opposite charges. The monopole operator in a theory without fermions is $e^{-S_0} e^{i\alpha_i \sigma}$, and the sum over all elementary monopole effects induce $e^{-S_0} \sum_{j=1}^{N-1} \cos(\alpha_j \sigma)$ rendering all $N - 1$ varieties of photons massive. ⁴

2.1.1 Introducing complex representation fermions

Our goal is to construct the non-perturbative long distance description of Polyakov models with massless fermions. The long distance effective field theory must respect all the (non-anomalous) symmetries of the underlying microscopic theory. In other words, the (perturbative or non-perturbative) operators that can be generated are severely restricted by the microscopic symmetries. Therefore, it is useful to clearly state the symmetries of the microscopic P(F) model. This will also ease the comparison of microscopic and enhanced (emergent) macroscopic global flavor and spacetime symmetries of the theory.

Consider the addition of the massless fermions in the fundamental representation of the gauge group into the Polyakov model. (The generalization to other complex representation fermions is possible.) We interchangeably use the four-component Dirac spinors or two two-component Dirac spinors ψ_1 and ψ_2 related to each other via

$$\Psi^a = \begin{pmatrix} \psi_1^a \\ \bar{\psi}_2^a \end{pmatrix}, \quad \bar{\Psi}_a = \begin{pmatrix} \psi_2^a \\ \bar{\psi}_1^a \end{pmatrix}, \quad (2.5)$$

⁴We assume, for simplicity, $S_{0,i} \equiv \frac{4\pi}{g^2} |a_{i+1} - a_i| = S_0$ for the elementary monopoles by tuning the potential. This can be relaxed if desired.

We consider the theories with $N_f = 2n_f$ two component Dirac spinors, or equivalently, n_f four component spinors. The $a = 1, \dots, n_f$ and subscripts (1,2) are flavor indices. In our conventions, the representations of the two component fermions under the $SU(N)$ gauge group are $(\psi_1^a, \bar{\psi}_2^a) \in (\square, \square)$ where \square denotes the fundamental representation. These combinations and our subsequent Dirac γ matrix choices are for later convenience, and will make the Callias index analysis slightly simpler.⁵ The fermions couple to gauge fields and adjoint scalars as

$$L_F = i\bar{\Psi}^a \left(\gamma_\mu (\partial_\mu + iA_\mu) + i\gamma_4 \Phi \right) \Psi_a \quad (2.6)$$

where the Euclidean γ matrices are given by

$$\gamma_\mu = \sigma_1 \otimes \sigma_\mu, \quad \gamma_4 = \sigma_2 \otimes I \quad \{\gamma_M, \gamma_N\} = 2\delta_{MN}, \quad M, N = 1, \dots, 4 \quad (2.7)$$

It is also convenient to define

$$\bar{\sigma}_M = (\sigma_\mu, -iI) \equiv (\sigma_\mu, \sigma_4), \quad \sigma_M = (\sigma_\mu, iI) \equiv (\sigma_\mu, -\sigma_4),$$

where σ_μ are the Pauli matrices. The explicit form of the Dirac-like operator in this basis is

$$\gamma_M D_M = \gamma_\mu D_\mu + \gamma_4 (i\Phi) = \begin{bmatrix} 0 & \sigma_\mu (\partial_\mu + iA_\mu) + \sigma_4 (i\Phi) \\ \sigma_\mu (\partial_\mu + iA_\mu) - \sigma_4 (i\Phi) & 0 \end{bmatrix} \quad (2.8)$$

and consequently,

$$L_F = i\bar{\psi}_1^a (\sigma_\mu (\partial_\mu + iA_\mu) + i\sigma_4 \Phi) \psi_1^a + i\bar{\psi}_2^a (\sigma_\mu (\partial_\mu + iA_\mu) - i\sigma_4 \Phi) \psi_2^a \quad (2.9)$$

In this representation, it is easier to see the global symmetries of the theory. Besides the $SO(3)_L$ Euclidean Lorentz symmetry and the C, P, T discrete charge conjugation, parity and (Euclidean) time reversal symmetries, the theory possesses a discrete \mathbb{Z}_2

$$\mathbb{Z}_2 : \quad \Phi \rightarrow -\Phi, \quad \psi_1 \rightarrow \bar{\psi}_2, \quad \psi_2 \rightarrow \bar{\psi}_1 \quad (2.10)$$

and the following global (flavor) symmetries

$$\begin{aligned} SU(n_f)_1 : \quad & \psi_1 \rightarrow U\psi_1, & \bar{\psi}_2 \rightarrow \bar{\psi}_2, \\ SU(n_f)_2 : \quad & \psi_1 \rightarrow \psi_1, & \bar{\psi}_2 \rightarrow V\bar{\psi}_2, \\ U(1)_V : \quad & \psi_1 \rightarrow e^{i\delta}\psi_1, & \bar{\psi}_2 \rightarrow e^{i\delta}\bar{\psi}_2 \\ U(1)_A : \quad & \psi_1 \rightarrow e^{i\beta}\psi_1, & \bar{\psi}_2 \rightarrow e^{-i\beta}\bar{\psi}_2 \end{aligned} \quad (2.11)$$

Note that the gauge covariant term possesses a larger global $SU(2n_f)$ symmetry group. Were the Yukawa's not present in the theory, the $SU(n_f)_1 \times SU(n_f)_2 \times U(1)_A$ global symmetry would enhance into the $SU(2n_f)$. However, the relative sign difference between the covariant derivative and Yukawa

⁵In Euclidean space, ψ_a and $\bar{\psi}_a$ should be viewed as independent variables. In particular, they are not related to each other by conjugation

couplings prevents this enhancement in the microscopic theory. Since there is no chiral anomaly in $d = 3$ dimensions, the $U(1)_A$ symmetry is a true symmetries of the theory. The discrete P and \mathbb{Z}_2 symmetries, and continuous flavor symmetry prohibits a fermion mass term. To summarize, the full microscopic symmetry group $\mathcal{G}_{\mathcal{M},P(F)}$ of the theory is

$$\mathcal{G}_{\mathcal{M},P(F)} = SO(3)_L \times C \times P \times T \times \mathbb{Z}_2 \times U(1)_V \times U(1)_A \times SU(n_f)_1 \times SU(n_f)_2 \quad (2.12)$$

2.1.2 Real representation fermions

We restrict attention to the adjoint representation fermion. Since the adjoint representation is real, the two component (complex) Dirac spinors is appropriate for all circumstances. Thus, $N_f = n_f$. The coupling of fermions to gauge boson and adjoint scalar is

$$L_{\text{adj}} = \text{itr} \left[\bar{\psi}_a \left(\sigma_\mu (\partial_\mu + i[A_\mu, \cdot]) + \sigma_4 [i\Phi, \cdot] \right) \psi_a \right] \quad (2.13)$$

The global flavor symmetries of the theory is given by

$$\begin{aligned} SU(n_f) : \quad \psi &\rightarrow U\psi, \\ U(1)_A : \quad \psi &\rightarrow e^{i\beta}\psi. \end{aligned} \quad (2.14)$$

Note that, in this case, $U(1)_A$ may be viewed as fermion number symmetry. However, since it does not have the same interpretation in the theories with complex representation fermions, we will not use this nomenclature. Thus, the full symmetry group $\mathcal{G}_{\mathcal{M},P(\text{adj})}$ of the microscopic theory is

$$\mathcal{G}_{\mathcal{M},P(\text{adj})} = SO(3)_L \times C \times P \times T \times U(1)_A \times SU(n_f) \quad (2.15)$$

Remark on QCD: At the classical level, the flavor symmetry group of the Polyakov models with fermions on \mathbb{R}^3 is the same as the flavor symmetries of the corresponding QCD on \mathbb{R}^4 or $S^1 \times \mathbb{R}^3$. However, in QCD in four dimensions, the analog of the symmetry that we referred as $U(1)_A$ in (2.11) and (2.14) is anomalous. In odd dimensions, there is no chiral anomaly, and the $U(1)_A$ is a true symmetry of the Polyakov model with massless fermions. In four dimensions, due to instanton effects, only a discrete \mathbb{Z}_{2h} subgroup of $U(1)_A$ survives quantization, where $2h$ is the number of fermionic zero modes in the background of a four dimensional instanton. The microscopic $U(1)_A$ symmetry will play a major role in the characterization of deconfinement in $P(\mathcal{R})$ theories.

2.1.3 Perturbative operators and flux operators

In all the $P(\mathcal{R})$ theories, we assume that the theory is always maximally Higgsed, and the long distance is dictated by the maximal abelian subgroup. There are massless bosons whose masslessness is protected to all orders in perturbation theory. Also, there are fermionic zero modes which interact with gauge fluctuations at long distances. Our interest is to determine the stability of such massless fields. There are two categories of operators which may be generated, and alter the long distance physics. These are, following [8],

- perturbative (without flux), naturally incorporated in terms of the original variables.
- nonperturbative (flux operators), or topological excitations, naturally incorporated in terms of dual photon.

For example, in the pure Polyakov model, a would-be operator of the first category is the relevant Chern-Simons term,

$$\frac{in}{4\pi} \int \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho \quad (2.16)$$

which would induce a mass term for the photon. However, this operator does not get generated at one loop order (or any order in perturbation theory), because the microscopic theory is parity invariant and the Chern-Simons term is parity odd. Thus, this type of instability does not occur.

An operator in the second category is the monopole operator. Indeed, it is allowed by all symmetries and generates the $e^{-S_0}(e^{i\sigma} + \text{c.c.})$ interaction, which, in the deep infrared, is a mass term for the dual photon. This is the type of instability that we will look for in the Polyakov models with massless fermions and some related gauge theories.

We will see that the microscopic symmetries $\mathcal{G}_{\mathcal{M}}$ and a topological shift symmetry which arises as a natural consequence of the Callias index theorem very severely restrict the types of operators that can be generated. In some circumstances, the infrared theory is quantum critical, in the sense that there exists no perturbative or nonperturbative operators which may destabilize the masslessness of photons and fermions, and some such theories become conformal field theories.

2.2 Callias index theorem and (continuous) topological symmetry

In the presence of massless (or light) fermions, the monopoles may carry fermionic zero modes attached to them [13]. The number of the fermionic insertions is determined uniquely by the Callias index theorem [14], and matter content of the theory.⁶ Let \mathcal{I}_{α_i} denote the index associated with the monopole with charge α_i . The typical form of the monopole operator in the theory with fermions is

$$e^{-S_0} e^{\pm i\alpha_i \sigma} O_{\text{fermions}}. \quad (2.17)$$

The number of fermion insertions of each flavor/type, say ψ_1^a , in O_{fermions} , is determined by the index \mathcal{I}_{α_i} , by the difference of the dimensions of the zero energy eigenstates:

$$\mathcal{I}_{\alpha_i} = (\dim \ker \mathcal{D}_{\alpha_i} - \dim \ker \overline{\mathcal{D}}_{\alpha_i}) \quad (2.18)$$

Here, $\mathcal{D}_{\alpha_i} = [\sigma_\mu(\partial_\mu + iA_\mu) + \sigma_4(i\Phi)]_{\alpha_i}$ is the Dirac-like operator in $d = 3$ dimensions in the background of the monopole α_i . In our conventions, the O_{fermions} in the monopole operator has only ψ_a insertions, and an anti-monopole operator can only have $\overline{\psi}_a$ insertions.

$$e^{-S_0} e^{+i\alpha_i \sigma} O_{\text{fermions}}(\psi), \quad e^{-S_0} e^{-i\alpha_i \sigma} O_{\text{fermions}}(\overline{\psi}) \quad (2.19)$$

This was indeed the reason for the peculiar spinor decomposition (2.5). For an adjoint fermion, the index is equal to $\mathcal{I}_{\alpha_i} = 2$. In the presence of fundamental fermions, the index is $\mathcal{I}_{\alpha_i} = \delta_{i, \hat{i}}$ where \hat{i} is the monopole that the zero mode is localized into. This is for each flavor of two component Dirac fermion. Since we have even number of fundamental fermions, the number of fermionic zero mode insertion in O_{fermions} is always even.

⁶For the relation between the more familiar Atiyah-Singer index theorem and Callias index theorem in QCD-like gauge theories on small $S^1 \times \mathbb{R}^3$, see page.37 of [19].

More precisely, for fermions in complex representations, we have two Dirac-like operators as seen in (2.9) and two conjugates,

$$\begin{aligned} \mathcal{D}^{(1)} &= \bar{\sigma}_M D_M = \sigma_\mu(\partial_\mu + iA_\mu) + \sigma_4(i\Phi), & \overline{\mathcal{D}}^{(1)} &= \sigma_M D_M = \sigma_\mu(\partial_\mu - iA_\mu) + \sigma_4(i\Phi), \\ \mathcal{D}^{(2)} &= \bar{\sigma}_M D_M = \sigma_\mu(\partial_\mu - iA_\mu) - \sigma_4(i\Phi), & \overline{\mathcal{D}}^{(2)} &= \sigma_M D_M = \sigma_\mu(\partial_\mu + iA_\mu) - \sigma_4(i\Phi), \end{aligned} \quad (2.20)$$

The total number of fermion zero modes associated with a monopole α_i is $n_f(\mathcal{I}_{\alpha_i}^{(1)} + \mathcal{I}_{\alpha_i}^{(2)}) = 2n_f\mathcal{I}_{\alpha_i}$.

Symmetry transmutation: The microscopic Polyakov Lagrangian with massless fermions has a global $U(1)_A$ symmetry given in (2.11) and (2.14) regardless of whether fermions are in a real or complex representation. Since it is a non-anomalous symmetry, it must be a symmetry of the long distance theory. The $U(1)_A$ transformation,

$$\psi \rightarrow e^{i\beta}\psi, \quad \bar{\psi} \rightarrow e^{-i\beta}\bar{\psi} \quad (2.21)$$

implies $O_{\text{fermions}} \rightarrow e^{iN_f\mathcal{I}_{\alpha_i}\beta}O_{\text{fermions}}$. Therefore, the invariance of the monopole operator under (2.21) necessitates a continuous shift for the dual photons:

$$U(1)_* : \alpha_i\sigma \rightarrow \alpha_i\sigma - N_f\mathcal{I}_{\alpha_i}\beta \quad (2.22)$$

Since this symmetry originates from the topological index theorem, we will call it a topological shift symmetry, or simply, **topological symmetry** and refer to it as $U(1)_*$. Just like the abelian duality transform [1], the topological shift symmetry requires going to sufficiently long distances. In the IR, the $U(1)_A$ symmetry of the original theory intertwines with the shift symmetry for the dual photons (2.3). This phenomena pervades the physics of all $P(\mathcal{R})$ theories.

More precisely, recall that in the absence of fermions and monopoles, the free Maxwell theory is dual to a free scalar theory with a continuous shift symmetry $U(1)_{\text{flux}}$ (2.3). The presence of monopoles (in the absence of fermions) spoils this symmetry completely. However, in the presence of fermions, the $U(1)_*$ linear combination of the $U(1)_A$ and $U(1)_{\text{flux}}$

$$U(1)_* : U(1)_A - N_f\mathcal{I}_{\alpha_i}U(1)_{\text{flux}} \quad (2.23)$$

remains a true symmetry of the theory. ⁷

A continuous shift symmetry can protect a scalar from acquiring a mass. Since there is only one parameter in the transformation (2.22), only one dual photon is protected by the topological symmetry. At a conceptual level, this shows that one gauge degree of freedom remains massless in the IR of the $P(\mathcal{R})$ theory regardless of any other detail, so long as the microscopic theory possesses the $U(1)_A$ symmetry. We may call this phase deconfined, since a gauge boson remains infinite ranged. Although this is true, it is a crude characterization. A more refined categorization of the deconfined phases, which can distinguish a free infrared theory (free photon), and a strongly or weakly coupled conformal field theory (CFT) is needed, and will be discussed.

⁷If there was no dual photon field to soak-up the phase of the fermionic zero modes, this would indeed imply that $U(1)_A$ must be anomalous, which is incorrect on \mathbb{R}^3 . Compare this with one flavor QCD on \mathbb{R}^4 . The instanton vertex also has two fermion insertion and no extra structure to soak-up the $U(1)_A$ chiral rotation. Indeed, there is a chiral anomaly on \mathbb{R}^4 and the $U(1)_A$ is anomalous. Only a \mathbb{Z}_2 subgroup of it is anomaly-free.

2.3 Revisiting P(adj): Dual scalar as a Nambu-Goldstone boson

Consider the $SU(2)$ one flavor P(Adj). (Below is a review and slight refinement of Affleck et.al. [10]). We assume the long distance gauge structure reduces down to $U(1)$. Perturbatively, we have a photon and a neutral fermion, described by

$$L = \frac{1}{4g_3^2} F_{\mu\nu}^2 + i\bar{\psi}\sigma_\mu\partial_\mu\psi \quad (2.24)$$

a free field theory. Parity forbids relevant perturbative operators such as $\bar{\psi}\psi$ from being generated [10]. Nonperturbatively, there is only one type of elementary monopole (and its anti-monopole.) The index $\mathcal{I}_{\alpha_1} = 2$ for adjoint fermions. Thus, by (2.21) and (2.22), we have

$$\psi \rightarrow e^{i\beta}\psi, \quad \sigma \rightarrow \sigma - N_f \mathcal{I}_{\alpha_1} \beta = \sigma - 2\beta. \quad (2.25)$$

There is only one combination of the relevant $\mathcal{G}_{\mathcal{M},P(\text{adj})}$ singlet that one can construct, and which gets induced nonperturbatively:

$$\Delta L^{\text{non-pert.}} = e^{-S_0} e^{i\sigma} \psi\psi + e^{-S_0} e^{-i\sigma} \bar{\psi}\bar{\psi} \quad (2.26)$$

There is also a large class of $\mathcal{G}_{\mathcal{M},P(\text{adj})}$ singlet, but irrelevant multi-monopole operators of the form $(e^{-S_0} e^{i\sigma} \psi\psi)^k$ where k is some integer. The continuous shift symmetry (2.25) forbids any kind of potential (such as $e^{i\sigma} + \text{c.c.}$), the mass term for the dual photon. This proves that the photon must remain massless nonperturbatively. Affleck et.al. showed that, by expanding the σ fields around, say, zero, the $U(1)_*$ symmetry is spontaneously broken and the photon is the Nambu-Goldstone boson. The fermion acquires mass $\sim e^{-S_0}$ due to $U(1)_*$ breaking. This is the conventional way to have gapless scalars in a gauge field theory. For a fuller discussion, see ref. [10, 20]. For $SU(N)$ and multi-flavor generalizations, see [21].

It is useful to think of the Noether current associated with the symmetry (2.25) in the n_f flavor theory. It is

$$K_\mu = \bar{\psi}\sigma_\mu\psi - n_f \mathcal{I}_{\alpha_1} \partial_\mu\sigma = \bar{\psi}\sigma_\mu\psi - n_f \mathcal{I}_{\alpha_1} \mathcal{J}_\mu = j_\mu - n_f \mathcal{I}_{\alpha_1} \mathcal{J}_\mu \quad (2.27)$$

Recall from §.2.1 that the current associated with $U(1)_{\text{flux}}$ satisfies $\mathcal{J}_\mu = \partial_\mu\sigma = \frac{1}{2}\epsilon_{\mu\nu\rho} F^{\nu\rho} = F_\mu$ where F_μ is the magnetic field. Using $\partial_\mu F_\mu = \nabla^2\sigma = \rho_m(x)$ where $\rho_m(x)$ is the magnetic charge density, the local current conservation can be re-expressed as

$$\partial_\mu K_\mu = \partial_\mu(j_\mu - n_f \mathcal{I}_{\alpha_1} \mathcal{J}_\mu) = 0 \implies \partial_\mu j_\mu(x) = n_f \mathcal{I}_{\alpha_1} \rho_m(x) \quad (2.28)$$

which implies the conservation of the $U(1)_*$ current as stated in (2.23). The final form is the local version of the Callias index theorem, which ties the $U(1)_A$ charge with the $U(1)_{\text{flux}}$ charge. Namely, in the presence of n_f adjoint fermions,

$$\begin{aligned} Q_* &= Q_A - n_f \mathcal{I}_{\alpha_1} Q_{\text{flux}} \\ &= N_\psi - N_{\bar{\psi}} - n_f \mathcal{I}_{\alpha_1} (N_{\text{monopole}} - N_{\text{anti-monopoles}}) \end{aligned} \quad (2.29)$$

is a conserved charge, where N_X counts the number of the X excitations. This means, any perturbative or non-perturbative interaction vertex in the long distance theory preserves Q_* . However, the $U(1)_*$ is spontaneously broken by the vacuum, and the photon is a Goldstone boson.

SU(N): It is also useful to review the $SU(N)$ generalization of this theory since it carries important lessons on the interplay of symmetry and dynamics. Due to gauge symmetry breaking down to $U(1)^{N-1}$, there exist $N - 1$ photons and $N - 1$ massless fermions, the components along the Cartan subalgebra. The infrared Lagrangian in perturbation theory is, therefore,

$$L^{\text{pert.theory}} = \frac{1}{2}(\partial\sigma)^2 + i\bar{\psi}\sigma_\mu\partial_\mu\psi, \quad \sigma \equiv (\sigma_1, \dots, \sigma_{N-1}), \quad \psi \equiv (\psi_1, \dots, \psi_{N-1}), \quad (2.30)$$

The simplicity of this system relative to the complex representation fermions to be studied in the subsequent section is the electric neutrality of the zero mode fermions. In perturbation theory, there are no relevant or marginal operators which respect the underlying symmetries of the original theory and which may be generated perturbatively. Thus, the Gaussian fixed point is stable to all orders in perturbation theory.

However, there exist a plethora of relevant nonperturbative (flux) operators. The index is $\mathcal{I}_{\alpha_i} = 2$ for all $i = 1, \dots, N - 1$. The $N - 1$ monopole operators are $e^{-S_0} e^{i\alpha_i\sigma} \alpha_i \psi \alpha_i \psi$ none of which generates a mass term for the dual photons. Notice that each term is manifestly invariant under the topological $U(1)_*$ symmetry (2.21), (2.22). In the e^{-S_0} expansion, at order e^{-2S_0} , there are $N - 2$ linearly independent relevant operators, $e^{-2S_0} e^{i(\alpha_j - \alpha_{j+1})\sigma}$ which get generated. Even though there is no fermion zero mode attached to these topological objects, since they are essentially the bound states of a monopole (with charge α_i) and anti-monopole (with charge $-\alpha_{i\pm 1}$), their invariance under the $U(1)_*$ topological symmetry is also manifest.⁸ Thus, the combined nonperturbative effects up to order e^{-3S_0} is given by

$$\Delta L^{\text{non-pert.}} = e^{-S_0} \sum_{j=1}^{N-1} e^{i\alpha_j\sigma} \alpha_j \psi \alpha_j \psi + e^{-2S_0} \sum_{j=1}^{N-2} e^{i(\alpha_j - \alpha_{j+1})\sigma} + (\text{conjugates}) \quad (2.31)$$

This renders $N - 2$ varieties of the photons massive leaving the one which is protected by the shift symmetry. As in the $N = 2$ case, the $U(1)_*$ breaks down spontaneously and there exist only one Goldstone. The higher order terms in the e^{-S_0} do not alter this conclusion.

This application shows that the existence of $U(1)_*$ symmetry provides a characterization for the absence of mass gap in gauge sector and the absence of confinement. The $U(1)_*$ does not imply the absence of monopoles or the irrelevance of monopole operators. And neither the presence of elementary monopoles or magnetically charged bound states of the monopoles implies confinement.

⁸A monopole and antimonopole in the presence of massless adjoint fermions interacts logarithmically at large distances in Euclidean \mathbb{R}^3 , rather than the Coulomb's law. (Also see [17, 22] for $U(1)$ QED, but one needs to be really careful here. See formula (3.8) and the discussion around it.) The $\log|x - y|$ marginally binds a monopole into its antimonopole. The combined state is magnetically neutral, and cannot lead to Debye screening. (A monopole-antimonopole pair is a dipole, and in the long distance, the dipole-dipole interaction is $1/r^3$, hence the absence of the Debye screening.) In P(adj) with $N \geq 3$, the presence of the fermion zero modes also leads to $N - 2$ bound states of a monopole with charge α_j and antimonopole with charge $-\alpha_{j\pm 1}$. The combined topological excitation has a nonzero magnetic charge $\alpha_j - \alpha_{j\pm 1}$ and at large distances interacts via Coulomb potential, $1/r$. These excitations are referred to as magnetic bions [21]. The magnetic bions render $N - 2$ varieties of $N - 1$ photons massive. In QCD(adj) on $S^1 \times \mathbb{R}^3$ discussed in Ref. [21], due to an extra elementary monopole, one can form $N - 1$ magnetic bions, and the gauge sector is fully gapped. This also has a nice symmetry interpretation. The $U(1)_*$ continuous topological shift symmetry turns into a $(\mathbb{Z}_N)_*$ discrete shift symmetry on small $S^1 \times \mathbb{R}^3$. The discrete shift symmetry cannot prohibit mass term for scalars.

2.4 Complex representation fermions, masslessness and quantum criticality

Let us consider an $SU(2)$ Yang-Mills noncompact adjoint Higgs system with $N_f = 2n_f$ two component fundamental Dirac fermions on \mathbb{R}^3 , the P(F) theory. The theory possess the symmetries (2.12). As always, we assume the $SU(2)$ gauge structure reduces down to $U(1)$ at long distances. The off-diagonal gauge degrees of freedom (W -bosons) and one component of the fermions in the $SU(2)$ gauge symmetry doublet, and the scalars acquire masses and decouple from the long distance physics. In perturbation theory, the infrared theory is described by the abelian QED₃ action

$$S_{\text{pert.}}^{\text{P(F)}} = \int_{\mathbb{R}^3} \left[\frac{1}{4g_3^2} F_{\mu\nu}^2 + i\bar{\Psi}^a \gamma_\mu (\partial_\mu + iA_\mu) \Psi_a \right] \quad (2.32)$$

The action possesses an enhanced (accidental) $SU(2n_f)$ flavor symmetry group, and a $U(1)_V$ symmetry which is the global part of the gauge symmetry. This enhancement is expected in perturbation theory, because the Higgs scalar acquires mass and disappears from the long distance description. Since the disparity between the gauge-kinetic term and Yukawa term in (2.9) was the source of the lower symmetry, and since there are no Yukawa's in the long distance limit, there is an enhanced symmetry in perturbation theory.

The non-perturbative effects may in principle be aware of the lower symmetry of the high energy theory, and indeed, they are. Let us first take $N_f = 2$. As in P(Adj), there is one type of monopole. The index theorem tells us that for each fundamental flavor, the monopole has $\mathcal{I}_{\alpha_1} = 1$ zero mode. There is one relevant $\mathcal{G}_{\mathcal{M}}$ singlet operator which is induced nonperturbatively:

$$\text{Relevant } \mathcal{G}_{\mathcal{M}} \text{ singlets : } e^{-S_0} e^{i\sigma} \psi_1 \psi_2 + e^{-S_0} e^{-i\sigma} \bar{\psi}_1 \bar{\psi}_2 \quad (2.33)$$

The two fermions and the dual photon transform under $U(1)_*$ as

$$U(1)_* : \psi_1 \rightarrow e^{i\beta} \psi_1, \quad \psi_2 \rightarrow e^{i\beta} \psi_2, \quad \sigma \rightarrow \sigma - 2\beta. \quad (2.34)$$

The continuous shift symmetry forbids any kind of mass term for the dual photon. In particular, it forbids the $e^{-S_0} (e^{i\sigma} + e^{-i\sigma})$ operator. Thus, the photon must remain massless nonperturbatively.

In the multi-flavor case $N_f = 2n_f \geq 4$, the simplest monopole operator has $2n_f$ insertion of the fermionic zero modes,

$$e^{-S_0} e^{i\sigma} [(\psi_1^1 \psi_2^1) \dots (\psi_1^{n_f} \psi_2^{n_f}) + \text{permutations}]$$

The equality of the number of ψ_1^a insertion with the ψ_2^a insertion is a consequence of the Callias index theorem and $U(1)_V$ symmetry, i.e, electric charge neutrality. Making the $SU(n_f)_1 \times SU(n_f)_2$ symmetry of the monopole operator manifest gives

$$\mathcal{G}_{\mathcal{M}} \text{ singlets : } e^{-S_0} e^{i\sigma} \det_{a,b} \psi_1^a \psi_2^b + e^{-S_0} e^{-i\sigma} \det_{a,b} \bar{\psi}_1^a \bar{\psi}_2^b \quad (2.35)$$

where $a, b = 1, \dots, n_f$ are flavor indices. The invariance of the vertex under $U(1)_A$ symmetry necessitates the dual photon to transform as $\sigma \rightarrow \sigma - 2n_f \beta$ under $U(1)_*$.

We identified a distinction between the behavior of $N_f = 2$ and $N_f \geq 4$ theories. In the e^{-S_0} expansion, the leading non-perturbatively generated flux operator is classically relevant in the $N_f = 2$ case, and irrelevant in the $N_f \geq 4$ cases. Therefore, the latter class of theories are quantum critical,

and will exhibit enhanced $SU(2n_f)$ symmetry at long distance. For the $N_f = 2$ case, there is one relevant direction and no enhancement of flavor symmetry takes place.

It is again useful to study the Noether currents in the effective long distance theory. Unlike P(Adj), there are two types of conserved $U(1)$ currents in the Polyakov model with n_f complex representation fermions. One is associated with $U(1)_V$ symmetry, and the latter is a linear combination of $U(1)_A$ and $U(1)_{\text{flux}}$. These are, in the conventions of §2.1.1,

$$\begin{aligned} J_\mu &= j_{1,\mu} + j_{2,\mu} = \bar{\psi}_1^a \sigma_\mu \psi_{1,a} + \psi_2^a \sigma_\mu \bar{\psi}_{2,a} \\ K_\mu &= j_{1,\mu} - j_{2,\mu} - 2n_f \mathcal{I}_{\alpha_1} \partial_\mu \sigma = \bar{\psi}_1^a \sigma_\mu \psi_{1,a} - \psi_2^a \sigma_\mu \bar{\psi}_{2,a} - 2n_f \mathcal{I}_{\alpha_1} \partial_\mu \sigma \end{aligned} \quad (2.36)$$

The conserved charge associated with the $U(1)_V$ current J_μ is

$$(N_{\psi_1} - N_{\bar{\psi}_1}) + (N_{\bar{\psi}_2} - N_{\psi_2}) \quad (2.37)$$

and the conserved charge associated with $U(1)_*$ is

$$\begin{aligned} Q_* &= Q_A - 2n_f \mathcal{I}_{\alpha_1} Q_{\text{flux}} \\ &= (N_{\psi_1} - N_{\bar{\psi}_1}) - (N_{\bar{\psi}_2} - N_{\psi_2}) - 2n_f \mathcal{I}_{\alpha_1} (N_{\text{monopole}} - N_{\text{anti-monopoles}}) \end{aligned} \quad (2.38)$$

Clearly, these symmetries are in accord with the monopole operators and their zero mode structures. In fact, the conservation of the $U(1)_*$ current, $\partial_\mu K_\mu = 0$ is the local re-incarnation of the Callias index theorem. We will discuss the infrared limit of these theories after generalizing the basic essentials to $SU(N)$ gauge theory.

SU(N): The difference of long distance physics between $N_f = 2$ and $N_f \geq 4$ is not special to the $SU(2)$ P(F) theory. The infrared limit of $N \geq 3$ $SU(N)$ gauge theory with N_f massless fermion flavors turns out to be rather similar to the N_f flavor $SU(2)$ theory, as a consequence of the non-perturbative dynamics.

We assume the gauge structure reduces into $SU(N) \rightarrow [U(1)]^{N-1}$ at long distances. In perturbation theory, the infrared has $N - 1$ types of the massless photons, and $2n_f$ massless fermions. The other fields acquire masses and decouple from the long distance physics. There are $N - 1$ varieties of elementary monopoles. Their Callias indices are given by $\mathcal{I}_{\alpha_i} = \delta_{i,1} = (1, 0, \dots, 0)_i$, $i = 1, \dots, N - 1$ where without loss of generality, we assumed that the fermion zero mode is localized into the monopole with charge α_1 . Thus, the $U(1)_*$ shift symmetry reads

$$\begin{aligned} \alpha_1 \sigma &\rightarrow \alpha_1 \sigma - (2n_f) \beta, \\ \alpha_j \sigma &\rightarrow \alpha_j \sigma, \quad j = 2, \dots, N - 1 \end{aligned} \quad (2.39)$$

The symmetries do not forbid the $N - 2$ types of monopole operators which do not carry any fermionic zero modes. The first monopole has $2n_f$ fermion insertions and is irrelevant for $2n_f \geq 4$. The list of all the flux operators invariant under the symmetries of the microscopic theory up to order e^{-2S_0} is

$$\mathcal{G}_{\mathcal{M}} \text{ singlets} : \left\{ e^{-S_0} e^{i\alpha_1 \sigma} \det_{a,b} \psi_1^a \psi_2^b, e^{-S_0} e^{i\alpha_2 \sigma}, \dots, e^{-S_0} e^{i\alpha_{N-1} \sigma} \right\} + \text{c.c.} \quad (2.40)$$

Hence, $N - 2$ out of $N - 1$ photons acquire mass due to relevant monopole induced effects. Thus, the $SU(N)$ P(F) theory undergoes changes in its gauge structure as we consider longer and longer length

scales. The first change is perturbative $SU(N) \rightarrow [U(1)]^{N-1}$ and the latter is non-perturbative $[U(1)]^{N-1} \rightarrow U(1)$ as shown in (1.1). The very long distance $U(1)$ theory is quantum critical due to the absence of any relevant or marginal perturbations which may destabilize its masslessness. We will comment on the effects of strong (non-compact) gauge fluctuations in the next section.

Note that regardless of the value of the rank N in the original gauge theory, the deep IR of the P(F) theory always reduces to an abelian $U(1)$ QED₃ theory with $2n_f$ flavors. Below, we discuss the long distance limit of this theory.

2.5 Conformal field theories (CFTs) at long distances

$2n_f \geq 4$: The $U(1)_*$ topological symmetry combined with symmetries such as parity, Lorentz and flavor symmetries forbids any relevant instability that may occur in the infrared limit of our theory. The monopole operators such as $e^{i\sigma}$, or $e^{i\sigma}$ (fermion bilinears), where σ is the dual of the final $U(1)$ factor, are forbidden. This means, in the compact continuum QED₃ theory obtained as described above, **there are no relevant flux (monopole) operators in the original “electric” theory.** Thus, the non-perturbative lagrangian is the same as the perturbative one,

$$S_{\text{nonpert.}}^{\text{P(F)}} = S_{\text{pert.}}^{\text{P(F)}} + \dots \quad (2.41)$$

where ellipsis stands for irrelevant perturbations consistent with the microscopic symmetries of the underlying theory. This is QED₃ with charged massless fermions, and with an enhanced (accidental) $SU(2n_f)$ flavor symmetry.

The theory (2.41) has no dimensionless coupling constant. The expansion parameter is $\frac{g_3^2}{k}$ where k is some euclidean momentum scale. Thus, perturbative techniques are not useful at low energies. The low energy limit is a strongly correlated system of fermions and gauge fluctuations whose masslessness is protected by $U(1)_*$. A logical possibility for the infrared theory is a weakly or strongly coupled conformal field theory (CFT) depending on the number of flavors. In order to see this, let us calculate the correction to the photon propagator at one loop order in perturbation theory. Partially integrating out fermions produce the non-analytic correction to the gauge kinetic term

$$\frac{1}{g_3^2} F_{\mu\nu}^2 \rightarrow \frac{1}{g_3^2} \left(F_{\mu\nu}^2 + \frac{g_3^2 n_f}{8} F_{\mu\nu} \frac{1}{\sqrt{\square}} F_{\mu\nu} \right). \quad (2.42)$$

In the large n_f limit, the higher order effects in perturbation theory are suppressed by powers of $1/n_f$ and the one loop result becomes reliable [23]. The low energy limit is the same as taking g_3^2 to ∞ . These changes in the photon propagator can be summarized as

$$\frac{g_3^2}{k^2} \xrightarrow{\text{one-loop}} \frac{g_3^2}{k^2 + \frac{g_3^2 n_f k}{8}} \xrightarrow{\text{low energy}} \frac{8}{n_f k} \quad (2.43)$$

Thus, we are left with a theory without any scale in the IR with gauge boson propagator $\sim \frac{1}{k}$. Using the canonical normalization for the gauge kinetic term, the Lagrangian can be expressed as

$$L \sim F_{\mu\nu} \square^{-1/2} F_{\mu\nu} + i \bar{\Psi}^a \gamma_\mu (\partial_\mu + i \frac{1}{\sqrt{n_f}} A_\mu) \Psi_a \quad (2.44)$$

with a dimensionless expansion parameter $1/\sqrt{n_f}$. This is a remarkable change in the dynamics.

To appreciate this, let us measure the potential between two external electric charges located at $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$. The Coulomb potential between the two test charges is $V_{\text{Coulomb}}(|\mathbf{x} - \mathbf{y}|) = \log |\mathbf{x} - \mathbf{y}|$, in two spatial dimensions, hence marginally confining. The non-perturbative dynamics of the pure Polyakov model alters this potential into a linearly confining one. In the infrared of the theory with massless fundamental fermions, the potential is dictated by conformal behavior. Thus,

$$V_{\text{non-pert.}}(|\mathbf{x} - \mathbf{y}|) \sim \begin{cases} |\mathbf{x} - \mathbf{y}| & \text{pure Polyakov or with heavy fermions} \\ |\mathbf{x} - \mathbf{y}|^{-1} & \text{with massless fundamental fermions,} \\ \log |\mathbf{x} - \mathbf{y}| & \text{with massless adjoint fermions,} \end{cases} \quad (2.45)$$

In some sense, the long distance behavior of the Polyakov model with massless fermions is more drastic than the Polyakov model *per se*. This example also shows that the presence of a single massless fermion can completely alter the confining property of the gauge theory! However, the main concept here is not really the presence or absence of a fermionic species. Rather, it is the nature (continuous versus discrete) of the topological symmetry, as we will discuss in more detail, especially in connection with QCD* theory.

The microscopic symmetries of the P(F) theory given in (2.12) enhances and transmutes into

$$\mathcal{G}_{\text{IR,P(F)}} \sim (\text{conformal symmetry}) \times C \times P \times T \times U(1)_V \times U(1)_{\text{flux}} \times SU(2n_f) \quad (2.46)$$

in the long distances. In the $2n_f \geq 4$ cases, the relevant $U(1)_*$ respecting operators also individually respects $U(1)_A$ and $U(1)_{\text{flux}}$. The $U(1)_A$ is part of $SU(2n_f)$, and $U(1)_{\text{flux}}$ is the symmetry associated with conservation of magnetic flux. In the $2n_f = 2$ case, only the $U(1)_*$ combination is a symmetry.

Eq.(2.46) is indeed the symmetry group of the algebraic spin liquid discussed in [9]. The P(F) theory, just like the spin liquids, undergoes enormous space-time and flavor symmetry enhancement. [Compare the long distance symmetries with the short distance ones, (2.12).] Interestingly, very different microscopic theories (one is lattice spin system in the π -flux or staggered flux state and the other is continuum P(F) theory) both flow to the identical long distance interacting CFT.⁹ Thus, the multi-flavor QED₃ theories which descend from the Polyakov model are generically quantum critical. A recent work discusses the finite temperature limit of this class of CFTs [24].

It is not completely clear what occurs for fewer flavors. A logical possibility is that the weakly coupled CFT may interpolate into a strongly coupled CFT. For $2n_f \geq 4$, there is some evidence from the large scale lattice studies that no chiral symmetry breaking occurs in this theory [25]. These lattice simulations of *non-compact* QED₃ are relevant to our discussion only because the effect of compactness of the gauge boson in our theories with $n_f \geq 2$ is irrelevant in the renormalization group sense. Also,

⁹Recently, the gauge/string (AdS/CFT) correspondences are receiving much attention to model QCD-like gauge theories in 4d and lower dimensional condensed matter systems. Although there is currently no complete matching which captures both microscopic and macroscopic aspects of the most interesting gauge theories (such as the ones appearing in Nature), it certainly makes sense to model the infrared CFTs or whatever infrared behavior of the strongly coupled system by using a gravitational dual. Such constructions has computational utility at strong coupling. It may be useful to construct the gravitational duals of the spin liquids.

the inequality in Ref. [26] suggests that the $SU(2n_f)$ global symmetry should be unbroken for $n_f \geq 2$. Ref. [26] also argues that an earlier bound for a larger values ($3 < n_f < 4$) [23] is an overestimation of the truncated Schwinger-Dyson equations.

2n_f = 2 : In the $n_f = 1$ case, the nonperturbative infrared Lagrangian of P(F) is

$$L_{\text{nonpert.}}^{\text{P(F)}} = L_{\text{pert}}^{\text{P(F)}} + e^{-S_0} e^{i\sigma} \psi_1 \psi_2 + e^{-S_0} e^{-i\sigma} \bar{\psi}_1 \bar{\psi}_2 + \dots \quad (2.47)$$

where ellipsis again refer to perturbations such as $(e^{-S_0} e^{i\sigma} \psi_1 \psi_2)^k$ with $k \geq 2$ which are allowed by symmetries, but irrelevant in the renormalization group sense.

In this case, it is not possible to consult the Monte-Carlo studies for the noncompact lattice QED₃, because the effect of compactness is a relevant perturbation of non-compact QED₃ dynamics. However, it is certain that, due to topological $U(1)_*$ symmetry, the photon remains gapless. The strong coupling dynamics in the IR combined with the existence of a relevant monopole operator make the determination of the long distance physics hard, and this is left as an open problem.

To conclude, in $n_f \geq 2$, the combination of the topological symmetry and the irrelevance of operators which may lead to the breaking of the global symmetries not only protects the dual photon (scalar) from acquiring mass, it also protects the fermions. The mechanism of gaplessness is different from the Nambu-Goldstone mechanism. In particular, it relies on unbroken symmetry. Protection of masslessness due to unbroken symmetry appeared previously in the context of strongly coupled gauge theories, (see chapter 6 of [27] for a review, and references therein). More recently, refinements and generalization of this idea appeared in condensed matter context as quantum order [11, 12]. The appearance of quantum order in Polyakov model with fermions in new, and is one of the main results of this work.

3. Topology of adjoint Higgs field and QCD*

There is a way to trick the Polyakov model with massless fermions, and get confinement! In particular, we will present gauge theories which reduce to (2.32) in perturbation theory, but are gapped non-perturbatively.

Let us consider the “identical” looking action as in (2.1), however, alter the topology of the field space into a compact one. Let Φ be a *compact* adjoint Higgs field, with a vacuum expectation value $\langle \Phi \rangle = \text{Diag}(a_1, \dots, a_N)$. These eigenvalues are on the circle (rather than a line) and a_N is the nearest neighbor of a_1 . (See figure 1). Naturally enough, this vacuum expectation value will induce the very same gauge symmetry breaking as in the previous sections $SU(N) \rightarrow U(1)^{N-1}$. However, due to the change in the topology of the field space, there will be an extra elementary monopole other than the ones previously mentioned $\{\alpha_1, \dots, \alpha_{N-1}\}$. The extra monopole stems from the fact that the eigenvalues a_N and a_1 are now nearest neighbors, and if they become degenerate in real space, that corresponds to the extra monopole with charge $\alpha_N = -\sum_{i=1}^{N-1} \alpha_i$. The α_N is the affine root of the $SU(N)$ algebra. This monopole is on the same footing with the rest of the elementary monopoles, in particular, for $\langle \Phi \rangle$ backgrounds with cyclic \mathbb{Z}_N symmetry, the extra monopole has the same action $S_{0,N} = S_{0,i} = S_0$, as the rest.

Leaving these secondary issues aside, let us pose the main question: What did we really change? First, we turned a perturbatively superrenormalizable theory (the case with non-compact adjoint

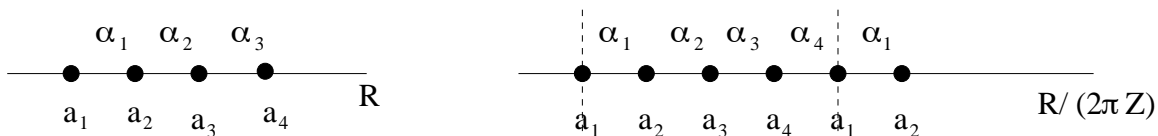


Figure 1: The vacuum expectation value of the noncompact versus compact adjoint Higgs field. Both lead to $SU(4) \rightarrow U(1)^3$ gauge symmetry breaking. On \mathbb{R}^3 , when two nearest neighbor eigenvalues become degenerate, the gauge symmetry restore partially and there is an associated elementary monopole α_i . The theory with compact field space topology has an extra elementary monopole, α_4 , which “moves in” from infinity as the compactification radius is reduced. When $|a_1(\text{image}) - a_4|$ separation becomes comparable with the other eigenvalue separation, the extra monopole gains equal fugacity (or action) with the rest of the monopoles, and contributes equally to the dynamics. The fundamental distinction between the two models is more pronounced in the presence of massless fermions, as a continuous versus discrete topological symmetry.

Higgs) into a nonrenormalizable field theory. The latter is in need of a UV completion. And there indeed exist such UV completions, but these are locally four dimensional QCD-like theories on small $S^1 \times \mathbb{R}^3$. We assert that, all the Yang-Mills compact adjoint Higgs theories with or without fermions on $\mathbb{R}^{1,2}$ have their UV completion in QCD-like gauge theories (with judiciously chosen matter content) in small $S^1 \times \mathbb{R}^{1,2}$. Here, however, without concerning ourselves with the UV completion, we will only state that the Yang-Mills *compact* Higgs system on \mathbb{R}^3 can be obtained by adding a center stabilizing deformation potential into the YM action in the small S^1 regime,

$$\begin{aligned}
S^{\text{YM}^*} &= S^{\text{YM}} + \int_{\mathbb{R}^3 \times S^1} P[\Phi] \\
&= \int_{\mathbb{R}^3 \times S^1} \left[\frac{1}{4g^2} \text{tr} F_{MN}^2 + \frac{1}{L^4} \sum_{n=1}^{[N/2]} a_n |\text{tr} U^n(\mathbf{x})|^2 \right]
\end{aligned} \tag{3.1}$$

and considering the low energy dynamics of the resultant theory.¹⁰ Here, $\Phi(\mathbf{x}) \equiv A_4(\mathbf{x})$ is the reduction of the gauge field along the short direction, which is periodic by construction. If we label the holonomy along S^1 as $U(\mathbf{x}) = P e^{i \int_0^L A_4(\mathbf{x}, x_4) dx_4} \approx e^{iL\Phi(\mathbf{x})}$ where the last equality is correct for smooth fields, the resulting theory can be brought into the form (2.1). Here, $[N/2]$ is the integer part of the half rank of the $SU(N)$ gauge group. The deformation terms with sufficiently large coefficients a_n where n goes all the way to $[N/2]$ are necessary to have maximal gauge symmetry breaking. Similarly, in theories with fermions, this procedure will produce a QCD* theory, whose action is

$$S^{\text{QCD}^*} = S^{\text{QCD}} + \int_{\mathbb{R}^3 \times S^1} P[\Phi] \tag{3.2}$$

¹⁰In condensed matter language, this center stabilizing double trace deformation may be viewed as a *frustration* of the Polyakov loop. Without the deformation, in the small S^1 regime of YM theory, $\langle \text{tr} U \rangle \neq 0$. At sufficiently large deformation, $\langle \text{tr} U \rangle = 0$ even at arbitrarily small S^1 . A perfect analogy in spin systems is an anti-ferromagnet which upon frustration become a paramagnet, as in (4.3). For fruitful applications of this idea into YM and QCD, see [19, 28]. In QCD(adj) formulated on $\mathbb{R}^{2,1} \times S^1$ where S^1 is a spatial circle endowed with periodic spin connection for fermions, these deformations are not necessary, because the quantum fluctuations (as opposed to thermal fluctuations, which are absent in this setup) prefers a center symmetric vacuum [20, 21]. This theory is the motivation behind the double trace deformations.

Study of such deformations of YM and QCD-like theories is relatively recent. The main advantage of this construction is that, some of these deformed theories are solvable in the same sense as the Polyakov model. For example, in YM*, the existence of the mass gap and linear confinement can be shown analytically [28], despite being locally four dimensional.

The relevance of this class of theories to our discussion is that, in perturbation theory, the long distance limits of these theories are indistinguishable from the appropriate Polyakov models, and reduce to the $[U(1)]^{N-1}$ QED₃ on \mathbb{R}^3 . Thus, they constitute an alternative way to embed compact QED₃ into a continuum gauge theory, different from Polyakov’s original constructions [1]. Remarkably, the non-perturbative aspects of some of these theories are opposite of the Polyakov model with massless fermions. Their gauge sectors are gapped as shown in the pattern (1.2) for moderate numbers of flavors.

3.1 Discrete topological symmetry and mass gap

How can such a “small” change in the topology of the field space alter the IR properties so drastically? The simplest reason is, as always, through symmetries. As explained above, the compact adjoint Higgs theories descend from locally 4d QCD-like theories. As it is well known, there are chiral anomalies in locally $d = 4$ dimensional theories, and since anomalies are a short distance property, it will clearly distinguish a theory whose base space is \mathbb{R}^3 from another one whose base space is secretly $S^1 \times \mathbb{R}^3$ (even if its lagrangian is expressed on \mathbb{R}^3 .)

Thus, the true symmetry structure of the $P(\mathcal{R})$ theory must be different from the $QCD(\mathcal{R})^*$. In, for example, one flavor theories, the $U(1)_A$ symmetry of the $P(\mathcal{R})$ theories is replaced by \mathbb{Z}_{2h} discrete chiral symmetry of locally four dimensional theory. Here, $h = 1$ for a fundamental and $h = N$ for adjoint fermion. The Callias index theorem is still valid in the formulation on small $S^1 \times \mathbb{R}^3$, and its precise relation to the Atiyah-Singer index theorem is well understood [19]. In the presence of massless fermions, the \mathbb{Z}_{2h} which is the discrete chiral symmetry of the microscopic theory, intertwines with the \mathbb{Z}_h discrete subgroup of the $U(1)_{\text{flux}}$, schematically as

$$\psi \rightarrow e^{i\frac{2\pi}{2h}} \psi, \quad \sigma \rightarrow \sigma - \frac{2\pi}{h} \quad (3.3)$$

such that the monopole operators (e.g. $e^{i\sigma}\psi\psi$) remains invariant. Clearly, this discrete symmetry does not forbid operators such as $e^{ih\sigma}$ from being generated, but forbids $e^{ih'\sigma}$ if $h' \neq 0 \pmod{h}$.

The topological $U(1)_*$ symmetries of $P(\mathcal{R})$ theories reduce into a discrete symmetry in $QCD(\mathcal{R})^*$,

$$\underbrace{U(1)_*}_{\text{non-compact Higgs or } P(\mathcal{R})} \longrightarrow \underbrace{(\mathbb{Z}_h)_*}_{\text{compact Higgs or } QCD(\mathcal{R})^*} . \quad (3.4)$$

We identified the fundamental distinction between non-compact and compact adjoint Higgs systems as a change in their microscopic, and consequently, topological symmetry:

As emphasized in the discussion of $P(\mathcal{R})$, $U(1)_*$ continuous shift symmetry is able to prohibit mass term for one variety of the dual photon. On the other hand, the $(\mathbb{Z}_h)_*$ symmetry which is a **discrete topological symmetry** is incapable of forbidding a mass term for the dual photon.¹¹ The discrete

¹¹The discrete topological shift symmetry has a representation dependence. It is \mathbb{Z}_N for adjoint, \mathbb{Z}_{N+2} for symmetric, \mathbb{Z}_{N-2} for anti-symmetric and trivial group \mathbb{Z}_1 for fundamental fermions. These are also valid for multi-flavor cases.

symmetries can at best postpone the emergence of the mass term in the e^{-S_0} expansion [20, 21], but can never forbid it. Thus, in the theory with compact adjoint Higgs field, there is no *symmetry* reason for the photon to remain massless and a mass term is generated.

At this stage we are conceptually done. But in order to come to a full circle with the first paragraph of the (§.3) which had an emphasis on the topological structure of the field space, let us discuss an example, given in [19].

3.2 Application: QCD(F)* with $n_f = 1$

Consider the analog of the gauge theory in (§.2.4), let $2n_f = 2$. Due to the change in topology of the field space, there are two types of elementary monopoles. Their magnetic and topological charges $\left(\int_{S^2} F, \int_{\mathbb{R}^3 \times S^1} F\tilde{F}\right)$ are given by $\mathcal{M}_1 : (+1, \frac{1}{2})$, $\mathcal{M}_2 : (-1, \frac{1}{2})$ for monopoles, and $\overline{\mathcal{M}}_1 : (-1, +\frac{1}{2})$, $\overline{\mathcal{M}}_2 : (+1, -\frac{1}{2})$ for the anti-monopoles. The Callias index for the monopole with quantum number $(+1, \frac{1}{2})$ is one and the index for the $(-1, \frac{1}{2})$ monopole is zero. Note that the product $\mathcal{M}_1\mathcal{M}_2$ is the four dimensional instanton vertex and the monopoles can be viewed as constituents of the 4d instanton. The zero modes localizes into one of the constituent monopoles following the ‘‘Higgs regime’’ criteria in the statement of Callias’s theorem [14]. For a nice lattice realization of the localization property, see Bruckmann et. al. [29]. The monopole operators are

$$\begin{aligned}\mathcal{M}_1(x) &= e^{-S_0} e^{i\sigma} \psi_1 \psi_2, & \mathcal{M}_2(x) &= e^{-S_0} e^{-i\sigma} \\ \overline{\mathcal{M}}_1(x) &= e^{-S_0} e^{-i\sigma} \bar{\psi}_1 \bar{\psi}_2, & \overline{\mathcal{M}}_2(x) &= e^{-S_0} e^{+i\sigma}\end{aligned}\quad (3.5)$$

which only respects $U(1)_V \times (\mathbb{Z}_2)_A$ symmetries of the microscopic QCD(F)* theory. Unlike the case with non-compact adjoint Higgs fields (2.47), the dynamics and symmetries of the compact Higgs theory admits a relevant monopole operator without a fermion zero mode insertion:

$$\Delta L^{\text{nonpert.}} \sim e^{-S_0} \cos \sigma + e^{-S_0} e^{i\sigma} \psi_1 \psi_2 + e^{-i\sigma} \bar{\psi}_1 \bar{\psi}_2 \quad (3.6)$$

The mass for the dual scalar is $\sim e^{-S_0/2}$ and is there due to the extra $\mathcal{M}_2(x) + \overline{\mathcal{M}}_2(x)$ monopole effect $e^{-S_0} \cos \sigma$. This potential pins the scalar at the bottom of the potential. Expanding $\mathcal{M}_1(x) + \overline{\mathcal{M}}_1(x)$ at the minimum of σ yields $e^{-S_0}(\psi_1 \psi_2 + \bar{\psi}_1 \bar{\psi}_2)$, a mass for the fermion proportional to e^{-S_0} , much smaller than the photon mass. Thus, the dynamical patterns of the theory is

$$SU(2) \xrightarrow[\text{Higgsing}]{} U(1) \xrightarrow[\text{nonperturbative}]{} \text{no massless modes} \quad (3.7)$$

which is a special case of (1.2). For a fuller discussion of one-flavor QCD-like theories with two index representation fermions, we refer the reader to a joint work with M. Shifman [19].

Note that the important conceptual distinction relative to the P(F) theory discussed in §.2.4 is the absence of $U(1)_*$ symmetry in the QCD(F)*. In P(F), $U(1)_*$ forbids the appearance of all the flux operators without fermion zero mode insertions, such as $(e^{-S_0} e^{i\sigma})^k$ for any integer k . In QCD(F)*, such operators are allowed by symmetries.

A consequence of the presence versus absence of a continuous topological symmetry is reflected in the interactions between topological excitations. In P(F) on Euclidean \mathbb{R}^3 , the long distance interactions of monopoles with anti-monopoles are necessarily logarithmic, whereas in QCD(F)*, the $\mathcal{M}_1(x)$

and $\overline{\mathcal{M}}_1(y)$ interaction is logarithmic, but $\mathcal{M}_2(x)$ and $\overline{\mathcal{M}}_2(y)$ interacts according to Coulomb's law as it can be seen by inspecting the leading connected correlator of the monopole operators:

$$\begin{aligned} V_{1\overline{1}}(x-y) &= -\log\langle\mathcal{M}_1(x)\overline{\mathcal{M}}_1(y)\rangle \sim 4\log|x-y| - \frac{1}{|x-y|}, \\ V_{2\overline{2}}(x-y) &= -\log\langle\mathcal{M}_2(x)\overline{\mathcal{M}}_2(y)\rangle \sim -\frac{1}{|x-y|}, \end{aligned} \tag{3.8}$$

The $\mathcal{M}_2(x), \overline{\mathcal{M}}_2(y)$ type monopoles in $\text{QCD}(\text{F})^*$ are sufficient to have the usual Debye mechanism, and generate a mass gap for the dual photon.

3.3 Remark on accidental continuous topological symmetry

Evidently, the presence of a discrete $(\mathbb{Z}_h)_*$ topological symmetry is a necessary criteria for the the presence of a mass gap in the gauge sector. If a mass term for dual photon is not protected by a symmetry, surely, it will get generated. However, it is also possible that a term allowed by all the symmetries may be irrelevant in the renormalization group sense. Thus, the presence of discrete topological symmetry is not sufficient to conclude that the theory has a mass gap and confines.

Consider the $\text{QCD}(\text{F})^*$, a theory defined on small $S^1 \times \mathbb{R}^3$ by construction, as a function of the number of flavors. Assume the number of flavor is large, but not very large so that the four dimensional coupling at the compactification scale is small. Indeed, a monopole operator is allowed, and hence is generated. However, a monopole operator may become irrelevant if there are a sufficiently large number of flavors. The classical scaling dimension of the monopole fugacity is +3. The presence of the massless fermions alters the quantum scaling dimension for the monopole operator in a significant way for large numbers of flavors ($\sim n_f$) as shown in [30]. The continuum analysis for such $\text{QCD}(\text{F})^*$ mimics the analysis of Hermele et.al. for lattice QED_3 at large n_f [8]. In both cases, the monopole operator does scale down to zero at long distances [8] due to large scaling dimension, showing the irrelevance of monopoles and emergence of an accidental $U(1)_{\text{flux}}$ symmetry associated with the conservation of gauge flux. Strictly speaking, there are some important differences between lattice QED_3 and $\text{QCD}(\text{F})^*$ to be explained after the discussion of spin liquids. However, those are immaterial for the above argument. Thus, in $\text{QCD}(\text{F})^*$ theory, there must be a critical window for the number of flavors for which the theory is a three dimensional interacting conformal field theory. It is desirable to understand the relation between these fixed points and the perturbative Banks-Zaks fixed point [31]. Plausibly, they may be smoothly connected within $\text{QCD}(\text{F})^*$.

4. Compact lattice QED_3 and $U(1)$ spin liquids

We have arrived at a very interesting situation. There are at least two ways to obtain ‘‘compact QED with fermions’’ in $d = 3$ dimensions by using **continuum** field theories. We referred to the theories with non-compact adjoint Higgs as $\text{P}(\mathcal{R})$ and the one with compact adjoint Higgs fields as $\text{QCD}(\mathcal{R})^*$. In both case, the resultant QED_3 is compact by necessity, because both are realized via gauge symmetry breaking down to the (compact) maximal torus $[U(1)]^{N-1}$.

It is well-known that pure compact QED_3 confines even at arbitrarily weak coupling. A controversial question is what happens to confinement if one introduces massless fermions. This question is

of practical importance in the context of the stability of the $U(1)$ -spin liquids in two dimensions, a phase which *may be* neighbor with the d -wave superconducting phase in cuprates.¹² Regardless of the relevance of spin liquids for cuprates, the stability of the spin liquid is associated with the concept of fractionalization, which does not arise in any naive way from a collection of electrons, but which may exist due to strong-correlation physics. Therefore, this is a conceptually interesting and experimentally relevant question. Ref. [5–7] argued that the monopole effects always render the $U(1)$ spin liquids unstable. Ref. [8] showed that there are at least some spin liquids, with gapless fermions and $U(1)$ gauge fluctuations. These works refers to a particular “3d lattice QED with massless fermions”, with a specific set of microscopic symmetries (sometimes called projective symmetry group (PSG)). In the large n_f limit, Ref. [8] exhibits by relying on the microscopic symmetries of the lattice theory and a sophisticated RG analysis which addresses the light electric and magnetic degrees of freedom simultaneously that there are no relevant perturbative or non-perturbative instabilities which may render the photon and fermions massive.

Our work shows that the compact QED₃ with fermions may arise in at least two different ways as in (1.1) and (1.2), via non-compact versus compact adjoint Higgs field. (Moreover, it can also arise from a compact lattice formulation.) The change in the topological structure of the field space produces drastically distinct physics in the IR, gapless versus gapped gauge sectors in some cases. Thus, the question of the presence or absence of a deconfined phase in compact QED₃ in the *continuum* formulation is an **ill-defined** question unless one states the symmetries of the cut-off scale (microscopic) theories clearly. (The importance of symmetries is also emphasized in lattice formulations Ref. [8].)

The analysis of Ref. [8] carefully incorporates all possible symmetry singlet operators that can be generated perturbatively, or nonperturbatively via flux (monopole) operators, in a continuum language, by remaining loyal to the symmetries of the microscopic theory. This is a basic principle in any effective field theory construction as stated in §.2.1.3, either in the continuum limit of lattice gauge theory or the long distance description of a gauge theory in which gauge structure changes over length scales. By a careful renormalization group analysis, Ref. [8] shows that in the large n_f limit, the quantum effects turn the monopole operator, which has engineering dimension +3, into an irrelevant operator. The essence of this argument, is that at the IR fixed point, the quantum scaling dimension for the monopole operator is large $\sim n_f$ [30] and forces the monopole operator to scale down to zero at long distances. The irrelevance of monopoles is the same as conservation of magnetic flux, and there is an emergent topological $U(1)_{\text{flux}}$ symmetry which characterizes the deconfined nature of this fixed point. (For the details, see Ref. [8].)

In our analysis of continuum QED₃ which descends from the Polyakov model P(F), we did not need such a renormalization group analysis to show the irrelevance of flux (monopole) operators such as $e^{-S_0} e^{iq\sigma}$ with $q \geq 1$ because they are forbidden to begin with, due to $U(1)_*$ topological symmetry. Since this symmetry is independent of the rank N and the number of flavors n_f , the assertion that P(F) theory is always in the deconfined phase did not require a large n_f limit either.

In the next sections, we will discuss whether P(F) theory or QCD(F)* theory has anything to do with the $U(1)$ compact lattice QED₃ with massless fermions. The lattice theory of interest is the one

¹²It is not certain that spin liquids play a role in cuprates. However, the question of whether doping a spin liquid by charge generates a d -wave superconductor is sensible and interesting, and its answer may give insights into the structure of the pseudo-gap regime.

which arise in the $SU(n_f)$ spin systems, which we review next.

4.1 From $SU(n_f)$ quantum spin model to lattice QED₃

It is useful to briefly review the route from the spin models to lattice QED₃ with massless fermions, and identify the symmetries carefully. ¹³ The Hamiltonian of a $d = 2$ dimensional spin model on a square lattice is given by

$$\begin{aligned} H &= J \sum_{\langle \mathbf{r}, \mathbf{r}' \rangle} \text{tr}[\mathbf{S}(\mathbf{r}) \cdot \mathbf{S}(\mathbf{r}')] + \dots \\ &\equiv J \sum_{a=1}^{\dim(\text{adj})} \sum_{\langle \mathbf{r}, \mathbf{r}' \rangle} S_{\mathbf{r}}^a S_{\mathbf{r}'}^a + \dots \end{aligned} \quad (4.1)$$

where $J > 0$ is the antiferromagnetic exchange, and ellipsis are higher order terms which may ease the frustration of magnetic order. This term may be due to geometric frustration or some other microscopic mechanism. Here, \mathbf{r}, \mathbf{r}' are points on a two dimensional (square) lattice and $\langle \mathbf{r}, \mathbf{r}' \rangle$ indicates the nearest neighbor interactions. The hamiltonian has a global $SU(n_f)_D$ spin rotation symmetry group acting by conjugation

$$\mathbf{S}(\mathbf{r}) \rightarrow U \mathbf{S}(\mathbf{r}) U^\dagger, \quad U \in SU(n_f)_D. \quad (4.2)$$

The subscript D stands for diagonal, due to reasons to be explained in § 4.2. The ellipsis are assumed to be singlets under the $SU(n_f)_D$ symmetry and the other symmetries of the lattice.

The description of an ordered phase in terms of the mean field approximation is well known. A more non-trivial aspect in higher dimensional systems is whether the mean field approach can be usefully applied to a phase which refuses to order. The answer to this question is relatively recent [32, 33], and eventually leads to the emergence of gauge structure (and 2+1 dimensional gauge theories) in spin systems in two spatial dimensions. A microscopic Hamiltonian which may have a non-magnetic ground state is a double-trace deformation of (4.1)

$$H = \sum_{\langle \mathbf{r}, \mathbf{r}' \rangle} \left[J \text{tr}[\mathbf{S}(\mathbf{r}) \cdot \mathbf{S}(\mathbf{r}')] + \frac{J'}{n_f} (\text{tr}[\mathbf{S}(\mathbf{r}) \cdot \mathbf{S}(\mathbf{r}')])^2 \right] \quad (4.3)$$

For sufficiently large positive J' , despite the leading anti-ferromagnetic term, no long range magnetic order will appear. The double trace deformation is same as frustration for the spin order parameter.

To see this, the local spin operators $\mathbf{S}_{\mathbf{r}}$ are expressed as a local composite of the fermionic spinon operators $f_{\mathbf{r}, \beta}$

$$S_{\mathbf{r}}^a(\mathbf{r}) = f_{\mathbf{r}, \alpha}^\dagger T_{\alpha\beta}^a f_{\mathbf{r}, \beta}, \quad \text{or} \quad \mathbf{S}_{\alpha\beta} = (S_{\mathbf{r}}^a T^a)_{\alpha\beta} = f_{\mathbf{r}, \alpha}^\dagger f_{\mathbf{r}, \beta} - \frac{1}{2n_f} \delta_{\alpha\beta} \quad (4.4)$$

Supplemented with the constraint that each site must have occupation number $n_f/2$ (with n_f even),

$$\sum_{\alpha=1}^{n_f} f_{\mathbf{r}, \alpha}^\dagger f_{\mathbf{r}, \alpha} = n_f/2, \quad (4.5)$$

¹³This section is a review of known results in quantum spin systems, see [3], and references therein.

this is an exact description of the original spin Hamiltonian. This procedure of breaking the spin into two fermionic spinons is called slave fermion mean field theory, and (4.4) should be viewed as the definition of lattice spinons, $f_{\mathbf{r},\alpha}$. The spinons obey canonical anti-commutation relations, $\{f_{\mathbf{r},\alpha}, f_{\mathbf{r}',\alpha'}^\dagger\} = \delta_{\mathbf{r}\mathbf{r}',\alpha\alpha'}$ and zero for all other anti-commutators. Clearly, the Hilbert space of the theory without the constraint is vastly larger.

There is an apparent gauge redundancy $f_{\mathbf{r},\alpha} \rightarrow e^{i\theta(\mathbf{r})} f_{\mathbf{r},\alpha}$ built-in the definition of the spinon operator. The local constraint (4.5) guarantees that the quartic Hamiltonian in terms of the spinon operators is same as the original Hamiltonian in terms of spin operators. Exploiting the gauge redundancy provides the connection between purely bosonic spin models and lattice theories with gauge fluctuations and fermions.

The spin Hamiltonian (4.1) in terms of the spinon operators is quartic. The $U(1)$ lattice QED₃ arises in describing the fluctuations of this system around the π -flux ($\pi\mathbb{F}$), and the staggered flux (sF) state [33]. Here, we only review the π -flux state. Let a mean field ansatz be denoted by

$$\bar{\chi}_{\mathbf{r}\mathbf{r}'} = \langle f_{\alpha}^\dagger(\mathbf{r}) f_{\alpha}(\mathbf{r}') \rangle. \quad (4.6)$$

The π -flux state is the configuration of $\bar{\chi}$ with flux π through each plaquette on the square lattice,

$$\prod_{\partial p} \bar{\chi}[\partial p] = e^{i\pi} = -1 \quad (4.7)$$

where p denotes an elementary plaquette and ∂p is the oriented boundary. It is clear that $\chi_{\mathbf{r}\mathbf{r}'}$ transforms gauge covariantly, as a connection on the lattice. For low energy considerations, only the phase fluctuations of the ansatz are important. Hence, the terms in Hamiltonian incorporating the fluctuations and spinon hopping term takes the form

$$H \sim J \sum_{\langle \mathbf{r}, \mathbf{r}' \rangle} \bar{\chi}_{\mathbf{r}'\mathbf{r}} f_{\mathbf{r},\alpha}^\dagger e^{i\mathbf{a}_{\mathbf{r},\mathbf{r}'}} f_{\mathbf{r}',\alpha} + \text{h.c.} \quad (4.8)$$

which is the fermionic terms in lattice QED₃ [33]. Even though the Maxwell term is not present above, it will be produced by the renormalization group, when one integrates out a thin momentum-shell of fermions. Hence, we can add it to the above Hamiltonian.¹⁴ The resulting theory is *compact* lattice QED₃ theory with minimally coupled fermionic matter.

The QED₃ also appears in the more phenomenological proposal of Franz et.al. [4,35] and [36] within the phase fluctuation model in order to describe the pseudo-gap region of cuprate superconductors. The relation between this approach and the more microscopic spin liquid approach to the underdoped cuprate superconductors, and in particular, a relation between the lattice spinons and nodal quasi-particles is currently not clear.

4.2 Reverse engineering of lattice spinons and twisting

It is useful to understand the relation between the symmetries of the compact lattice QED₃ (4.8) and continuum QED₃ with Lagrangian (2.32). In particular, considering the important role played

¹⁴The reader familiar with the staggered fermions (or Kogut-Susskind fermions) in lattice QCD will realize immediately that the spinons are the analogs of the staggered fermions [34], and by construction, we are guaranteed to get a relativistic dispersion relations, and Lorentz invariance (in a naive continuum limit.) The Dirac algebra and spinors of the continuum theory translates into the π -flux relation and Grassmann valued operators in the (reverse) Kogut-Susskind construction.

by $U(1)_A$ symmetry and Callias index theorem in the Polyakov model, it is desirable to understand whether an analog of these may arise in the lattice formulations.

For ease of presentation, we relabel the $2n_f$ fermionic continuum fields as

$$\{\psi_{1,a}, \bar{\psi}_{2,a}\} \rightarrow \{\lambda_{1,a}, \lambda_{2,a}\} \equiv \{\lambda_1, \lambda_2, \dots, \lambda_{2n_f}\}, \quad a = 1, \dots, n_f \quad (4.9)$$

where Lorentz indices are suppressed. The continuum Lagrangian in terms of λ fields reads

$$\mathcal{L} = \frac{1}{4g_3^2} F_{\mu\nu}^2 + \sum_{b=1}^{2n_f} i\bar{\lambda}^b \sigma_\mu (\partial_\mu + iA_\mu) \lambda_b \quad (4.10)$$

The continuum theory has an $U(1)_V \times SU(2n_f)$ global symmetry, where $U(1)_V$ is the global part of gauge symmetry and $SU(2n_f)$ is a global flavor symmetry.

In the Polyakov model embedding or “regularization” of the compact version of this theory, only

$$U(1)_V \times SU(n_f)_1 \times SU(n_f)_2 \times U(1)_A \subset U(1)_V \times SU(2n_f) \quad (4.11)$$

is present, where we loosely view the inverse W -boson mass as the lattice spacing.

Let us now reverse engineer the lattice QED₃ theory starting with continuum formulation. This will be useful in understanding what the lattice symmetries mean in the continuum and ease the comparison with Polyakov’s model. Consider continuum QED₃ theory in Hamiltonian formulation on $\mathbb{R}^{1,2}$ and latticeize \mathbb{R}^2 . Let us consider the $SU(2) \times SU(n_f)_D$ subgroup of the $SU(2n_f)$ flavor symmetry. Since we are in the Hamiltonian formulation, we split the Lorentz symmetry into $SO(2)$ and continuous time translations. The fermions are in two dimensional spinor representation of $SO(2)$, two dimensional spinor representation of $SU(2)$ and in the fundamental representation of $SU(n_f)_D$. Now, we wish to discuss a well defined procedure, called *twisting*, which intertwines the Lorentz and flavor symmetry such that the continuum spinors are mapped into Grassmann valued operators residing on the lattice sites (the lattice spinors). In spin systems, the $SU(n_f)_D$ corresponds to the global rotation symmetry (4.2) of the spin. It is also the diagonal subgroup of the $SU(n_f)_1 \times SU(n_f)_2$ decomposition which appeared in the Polyakov model. The $SU(n_f)_D$ will have no impact in our discussion, so we suppress it.

The fermion $\lambda_{\alpha,a}$ transforms as $\lambda \rightarrow O\lambda U^\dagger$ under $O \in SO(2)_L$ and $U \in SU(2)$ flavor. We can write every two by two matrix such as $\lambda_{\alpha,a}$ in a basis spanned by the identity and the Pauli matrices ($1, \sigma_\mu, \sigma_{\mu\nu} = i\frac{1}{2}\epsilon_{\mu\nu}\sigma_\mu\sigma_\nu$). Thus,

$$\lambda_{\alpha,a} = (f1 + f_\mu\sigma_\mu + \frac{1}{2}f_{\mu\nu}\sigma_{\mu\nu})_{\alpha,a} \quad \alpha = 1, 2, \quad a = 1, 2 \quad (4.12)$$

This is to say that under the diagonal $SO(2)_D = \text{Diag}(SO(2)_L \times SU(2))$ subgroup, the spinor becomes a collection of p -forms, one scalar, one vector and one two form anti-symmetric tensor, which we label as $(f, f_\mu, f_{\mu\nu})$. On the lattice, a p -form is naturally associated with a p -cell, zero form with sites, one form with links, and two form with faces. This twist is also sometimes referred to as the “Dirac-Kähler” construction in lattice gauge theory¹⁵ and is known to be equivalent to staggered fermions. We can

¹⁵This type of decomposition is one of the cornerstone of the recent progress in supersymmetric lattices, see for example, [37, 38].

map these fermions onto a lattice with half the spacing. The mapping takes the single component Grassmanns f, f_1, f_2, f_{12} onto the sites $(0, 0), (1, 0), (0, 1), (1, 1)$, in a unit cell, respectively. The new lattice repeats itself in amounts $(2, 0)$ and $(0, 2)$ in the x and y directions. The twisting procedure is the reverse engineering of the appendix of Ref. [8]. To see this, rewrite (4.12) in the component language:

$$(\lambda_{\alpha,a}) = \begin{pmatrix} f + f_{12} & f_1 + if_2 \\ f_1 - if_2 & f - f_{12} \end{pmatrix} \quad (4.13)$$

This is indeed the relation between the lattice spinons and continuum spinors given in Ref. [8] modulo a minor renaming of the fields.¹⁶

The discrete rotational symmetries of the QED₃ lattice action discussed in Ref. [8] are in fact the subgroup of $G_{\text{discrete}} \subset SO(2)_D = \text{Diag}(SO(2)_L \times SU(2))$. In the continuum, when the $SO(2)_D$ restores, one can always undo the twist. This reverse procedure gives the so-called emergent flavor $SU(2)$ subgroup of $SU(2n_f)$ for free. To summarize, the compact lattice QED₃ possesses

$$\mathcal{G}_{\text{QED}_3} \sim G_{\text{discrete}} \times C \times P \times T \times U(1)_V \times SU(n_f)_D \quad (4.14)$$

This is indeed the symmetry structure of spin system in the gauge theory formulation in the π -flux state. This needs to be compared with much larger microscopic symmetry (2.12) of P(F) theory.

The analog of the $U(1)_A$ symmetry in the P(F) theory is part of the $SU(2) \subset SU(2n_f)$ symmetry in the continuum of the QED₃. Unfortunately, in the π -flux state of the spin system, and in the specific lattice regularization described above, the continuous $U(1)_A$ does not survive at the cut-off scale. Only a discrete subgroup of it is hidden in G_{discrete} . However, G_{discrete} is practically useless (like any other discrete symmetry) for forbidding generic flux operators in lattice QED₃.

This is the significant difference between P(F) theory, QCD(F)* theory and lattice QED₃. The P(F) theory has $U(1)_A$ symmetry at short distances and this transmutes into a continuous topological symmetry in the IR preventing a mass term for a photon, for any number of flavors. In QCD(F)*, the short distance theory only has a \mathbb{Z}_2 discrete chiral symmetry, which again transmutes into a trivial $(\mathbb{Z}_1)_*$ topological symmetry, which cannot prohibit mass term for the dual photon. For small numbers of flavors, the theory exhibits a mass gap in gauge sector. At sufficiently large number of flavors, an accidental $U(1)_*$ may arise as discussed in (§.3.3). In the lattice versus continuum QED₃, the critical target theory has a $U(1)_A$ symmetry embedded into $SU(2n_f)$ for *any* n_f . However, the lattice Hamiltonian does not respect it. This makes this problem different and relatively harder than the previous two problems that we have discussed.

4.3 The emergent topological $U(1)_*$ symmetry

It is not *a priori* clear whether there is a relation between P(F) theory, and lattice QED₃ studied in Ref. [8]. Clearly, continuum P(F) theory is a theory with a scalar and with a larger set of symmetries

¹⁶This is also the reason why fields that transform in a single valued representation of the lattice point group symmetry maps into the double valued spinor representations under the continuum Lorentz symmetry. This clearly does not make any sense without the twisting idea, which mixes Lorentz symmetry and some global symmetry. This is in fact a recurring and fruitful theme in diverse fields of theoretical physics. It appeared initially in staggered (Kogut-Susskind) fermions [34], and most recently in supersymmetric lattices constructions [37, 38]. It also arises naturally in “topologically” twisted version of the supersymmetric theories, where under the diagonal subgroup of space-time and some flavor symmetry, the spinors decompose as p -forms, single valued representations [39]. Apparently, such structures are also ubiquitous in spin systems, in particular, the π -flux and staggered flux states [33].

than the lattice QED₃. However, the infrared physics of these two theories seems to be coincident at least in the large n_f limit. It is in principle plausible that different microscopic theories may flow to the same theory in their long distance limits.

In our opinion, the most important physical issue is associated with the topological $U(1)_*$ symmetry. In P(F), the origin of $U(1)_*$ is clear. It is a natural consequence of the the $U(1)_A$ symmetry combined with the Callias index theorem. In large n_f lattice QED₃, the $U(1)_*$ symmetry is referred as an emergent topological symmetry of the IR theory [8]. The reason it may be considered emergent is twofold: One is the analog of $U(1)_A$ is not present in the spin system and resulting lattice QED₃. The second is the analog of the Callias index theorem on lattice QED₃ does not exist as shown by Marston [17].

The result of Ref. [17] looks discouraging, as stated in [18]. However, the more severe issue is the absence of the $U(1)_A$ symmetry in lattice QED₃, or spin system. Below we will prove the following assertion: If the $U(1)_A$ is a symmetry of the cut-off (lattice) QED₃ theory, despite the absence of the Callias index theorem, the topological $U(1)_*$ symmetry will emerge in the long distances even at *small* n_f .

Let us see how this works. The result of ref. [17] does not tell us that monopole-multifermion type operators are excluded. It only states that in a monopole operator of the form $e^{i\sigma} O_{\text{fermion}}$, the structure of O_{fermion} is not dictated by an index theorem. O_{fermion} may be $\{1, (2 \text{ fermions}), (4 \text{ fermions}), \dots\}$, a plethora of (even) numbers of fermion insertions allowed by other symmetries of the lattice. Let us list a set of operators which may be induced nonperturbatively

$$\{e^{i\sigma}, e^{i\sigma} \lambda_{1,a} \bar{\lambda}_{2,a}, e^{i\sigma} (\lambda_{1,a} \bar{\lambda}_{2,a})^2, \dots, e^{2i\sigma}, e^{2i\sigma} \lambda_{1,a} \bar{\lambda}_{2,a}, e^{2i\sigma} (\lambda_{1,a} \bar{\lambda}_{2,a})^2, \dots\} \quad (4.15)$$

where we suppress Lorentz indices. This is the set of monopole operators and the composites of monopoles with the fermion fields. By assumption, the $U(1)_A$, under which $\lambda_{1,a} \bar{\lambda}_{2,a} \rightarrow e^{2i\beta} \lambda_{1,a} \bar{\lambda}_{2,a}$ is a symmetry of the cut-off theory. Our goal here is to show that the absence of an index theorem by itself does not imply that the continuous $U(1)_A$ symmetry cannot be transmuted into the dual photon as a shift symmetry.

Let $e^{i\sigma} (\lambda_{1,a} \bar{\lambda}_{2,a})^q$ be the lowest dimensional flux operator with multiple fermion insertions allowed by lattice symmetries. Since the $U(1)_A$ is a symmetry of the cut-off theory, it must be a symmetry of the long distance theory. As before, this can be accomplished by intertwining $U(1)_A$ with $U(1)_{\text{flux}}$, the shift symmetry of dual photon, in the infrared. The invariance of $e^{i\sigma} (\lambda_{1,a} \bar{\lambda}_{2,a})^q$ under $U(1)_A$ demands that the dual photon must have a shift symmetry $\sigma \rightarrow \sigma - 2q\beta$. Thus, reconciling $U(1)_A$ symmetry with the long distance physics forbids any operators in the list except $[e^{i\sigma} (\lambda_{1,a} \bar{\lambda}_{2,a})^q]^k$. Most importantly, it forbids the monopole operator $e^{i\sigma}$ and other pure flux operators such as $e^{2i\sigma}$ regardless of the value of $q \geq 1$. This implies that relevant monopole operators (which render the photon massive) may be forbidden by the *accidental* $U(1)_*$ pure flux forbidding symmetry even at small n_f if the cut-off theory has the $U(1)_A$ symmetry.

Unfortunately, the spin system does not have the analog of $U(1)_A$ symmetry. The flux operators such as $e^{i\sigma}$ which are not forbidden by symmetry will be generated. Under the given circumstances, the *only* way that such operators will not generate a mass for dual photon is if they are irrelevant in the long distances in the renormalization group sense. We reach to the conclusion that, for spin

systems in a π -flux phase, unlike the P(F) theories, the renormalization group and large n_f analysis are unavoidable [8].

5. Conclusions and prospects

Theory	Description	Topological symmetry, microscopic precursor	Gauge sector	Long distances
P [1]	noncompact Φ ,	none, none	gapped	confined
P(adj) [10]	noncompact Φ , real \mathcal{R} , complex fermions	$U(1)_*, U(1)_A$	gapless	deconfined, free photon
P(F)	noncompact Φ , complex \mathcal{R} , complex fermions	$U(1)_*, U(1)_A$	gapless	deconfined, CFT
YM* [28]	compact Φ	none, none	gapped	confined
QCD(adj)* [21]	compact Φ , real \mathcal{R} , complex fermions, n_f small	$(\mathbb{Z}_N)_*$, \mathbb{Z}_{2N} axial	gapped	confined
QCD(F)* [19]	compact Φ , complex \mathcal{R} , complex fermions, n_f small	none, \mathbb{Z}_2	gapped	confined
compact lattice QED ₃ [1]	compact gauge fluctuations	none, none	gapped	confined
compact lattice QED ₃ with fermions [8]	complex \mathcal{R} , complex fermions, $N_f \gg 1$	emergent $U(1)_*$, none	gapless	deconfined, CFT

Table 1: The role of topological symmetry in the determination of the deconfined/confined long distance behavior. It is worth emphasizing that all the theories in the list has magnetic monopoles in a semi-classically tractable regime. Thus, the presence or absence of the magnetic monopoles does not tell much about the infrared property of the theory. A more refined characterization is through the topological symmetry.

Topological symmetry and classification of gauge theories: In this paper, we discussed a large class of gauge theories formulated on \mathbb{R}^3 and $S^1 \times \mathbb{R}^3$ whose long distance gauge structure is described by abelian $U(1)^{N-1}$. Examples are $SU(N)$ continuum P(\mathcal{R}) on \mathbb{R}^3 , $SU(N)$ continuum QCD(\mathcal{R})*, and $U(1)^{N-1}$ lattice QED₃ in three dimensions. We arrived to sharp topological symmetry realizations which distinguish the zero temperature phases of such gauge theories, such as confined versus deconfined. ¹⁷

¹⁷In QCD(\mathcal{R})*, the small $S^1 \times \mathbb{R}^3$ should be viewed as a spatial (not thermal) compactification, along which fermions are endowed with periodic boundary condition. Its Minkowski space continuation is $S^1 \times \mathbb{R}^{2,1}$. If one wishes to study these gauge theories at finite temperature, a thermal circle should be formed out of the temporal direction on $\mathbb{R}^{2,1}$.

- 1) The existence of continuous $U(1)_*$ topological symmetry is the necessary and sufficient condition to demonstrate the absence of mass gap in the gauge sector and provides an unambiguous characterization of de-confinement.
 - 1.a) If the $U(1)_*$ symmetry is spontaneously broken, then there is a Goldstone boson. The infrared theory is the free scalar (which is same as a photon on \mathbb{R}^3 .)
 - 1.b) If the $U(1)_*$ symmetry is unbroken, the unbroken $U(1)_*$ protects the masslessness of the dual scalar. In some cases, the infrared theory flows into an interacting CFT.
- 2) The existence of a discrete topological symmetry is necessary, but not sufficient to exhibit confinement.
 - 2.a) If the monopole (or other flux) operators are irrelevant at large distances, then there is an emergent topological $U(1)_{\text{flux}}$ symmetry. This class of theories will deconfine, and some will flow into interacting CFTs.
 - 2.b) If the monopole (or other flux) operator is relevant at large distances, then the mass gap and confinement will occur. Showing the relevance of flux operators is the sufficient criteria to exhibit mass gap and confinement.

Some examples for these classes are tabulated in table.1 along with useful references. I wish to point out that some of these necessary and sufficient conditions are not completely novel. An example of class 1.a) was discussed long ago by Affleck, Harvey and Witten [10], and the statement of 2.a) is constructed in the work of Hermele et.al [8] on stable spin liquids, but it applies more generally to gauge theories. The totality of these criteria is new. ¹⁸

There are many interesting questions on the generalizations of these criteria. The most obvious is whether the topological symmetry characterization can be generalized to cases where the long distance dynamics is non-abelian. Another one is whether the abelian CFTs discussed in this paper has non-abelian counterparts? Assuming this is the case, are they dual to non-abelian spin liquids at large distances? Can we make use of this topological characterization towards the decompactification \mathbb{R}^4 limit of $\text{QCD}(\mathcal{R})^*$? We leave these questions for future work.

Ambiguity in defining compact QED_3 in continuum and resolution: There are at least two continuum gauge theories which produce compact QED_3 in perturbation theory via gauge symmetry breaking in $\text{P}(\mathcal{R})$ and $\text{QCD}(\mathcal{R})^*$. These flow into opposite IR theories, such as a CFT versus a theory with a mass gap in some cases, as shown in table.1.

Spin liquid and $\text{P}(\mathbf{F})$ duality: We demonstrated that the $SU(N)$ Polyakov model with $2n_f$ massless fundamental fermions and $SU(n_f)_D$ spin systems in the $n_f \gg 1$ limit flow into the same interacting conformal field theory. This is to some extent surprising due to the absence of the Callias index theorem in lattice QED_3 [17], and very distinct symmetries of the spin Hamiltonian and $\text{P}(\mathbf{F})$

¹⁸See also Refs. [40, 41] which use disorder operators to probe confinement. These works also attempt to provide a symmetry realization for confinement. An application to a QCD-like theory with adjoint fermions is given in [42]. It may be useful to perform the lattice simulations on an asymmetric lattice, which mimics $\mathbb{R}^3 \times S^1$ where S^1 is endowed with periodic spin connection for fermions. The theoretical analysis shows that the small S^1 regime must exhibit confinement without chiral symmetry breaking [20, 21]. It would be interesting to test this on lattice.

model. Both theories are quantum critical in the sense that there are no relevant perturbative or non-perturbative operators consistent with the symmetries of the microscopic theory. Thus, these theories flow into interacting conformal field theories at long distances. As the number of flavors is reduced, the long distance limit of $2n_f \geq 4$ P(F) theory interpolate in between the weakly and strongly coupled CFT's. What happens with lattice QED₃ at small number of flavors is still ambiguous.

Given the long distance duality between the spin liquids and P(F) gauge theory, a sensible question is the meaning of the doping of spin liquids by holons on the gauge theory side. Clearly, compactification of the field space brings in new excitations (flux operators) from infinity, and generates a QCD* type of theory, with a mass gap in its gauge sector. It is desirable to understand the relation, if any, between the QCD* theories and d -wave superconducting phase of high T_c cuprates.

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