An Infinite Branching Hierarchy of Time-Periodic Solutions of the Benjamin-Ono Equation

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Abstract

We present a new representation of solutions of the Benjamin-Ono equation that are periodic in space and time. Up to an additive constant and a Galilean transformation, each of these solutions is a previously known, multi-periodic solution; however, the new representation unifies the subset of such solutions with a fixed spatial period and a continuously varying temporal period into a single network of smooth manifolds connected together by an infinite hierarchy of bifurcations. Our representation explicitly describes the evolution of the Fourier modes of the solution as well as the particle trajectories in a meromorphic representation of these solutions; therefore, we have also solved the problem of finding periodic solutions of the ordinary differential equation governing these particles, including a description of a bifurcation mechanism for adding or removing particles without destroying periodicity. We illustrate the types of bifurcation that occur with several examples, including degenerate bifurcations not predicted by linearization about traveling waves.

Key words. Periodic solutions, Benjamin-Ono equation, non-linear waves, solitons, bifurcation, exact solution

AMS subject classifications. 35Q53, 35Q51, 37K50, 37G15

1 Introduction

The Benjamin-Ono equation is a model water wave equation for the propagation of unidirectional, weakly nonlinear internal waves in a deep, stratified fluid [7, 9, 19]. It is a non-linear,

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non-local dispersive equation that, after a suitable choice of spatial and temporal scales, may be written

$$u_t = Hu_{xx} - uu_x, \qquad Hf(x) = \frac{1}{\pi}\, PV\!\int_{-\infty}^{\infty} \frac{f(\xi)}{x-\xi}\, d\xi. \tag{1}$$

Our motivation for studying time-periodic solutions of this equation was inspired by the analysis of Plotnikov, Toland and Iooss [21, 13] using the Nash-Moser implicit function theorem to prove the existence of non-trivial time periodic solutions of the two-dimensional water wave over an irrotational, incompressible, inviscid fluid. We hope to learn more about these solutions through direct numerical simulation. As a first step, in collaboration with D. Ambrose, the author has developed a numerical continuation method [4, 3] for the computation of time-periodic solutions of non-linear PDE and used it to compute families of time-periodic solutions of the Benjamin-Ono equation, which shares many of the features of the water wave such as non-locality, but is much less expensive to compute.

Because we came to this problem from the perspective of developing numerical tools that can also be used to study the full water wave equation, we did not take advantage of the existence of solitons or complete integrability in our numerical study of the Benjamin-Ono equation. The purpose of the current paper is to bridge this connection, i.e. to show how the form of the exact solutions we deduced from numerical simulations is related to previously known, multi-periodic solutions [22, 10, 16]. Our representation is quite different, describing time-periodic solutions in terms of the trajectories of the Fourier modes, which are expressed in terms of N particles $\beta_j(t)$ moving through the unit disk of the complex plane. Thus, one of the main results of this paper is to show the relationship between the meromorphic solutions described e.g. in [8] and these multi-periodic solutions. Our representation also simplifies the computation and visualization of multi-periodic solutions. Rather than solve a system of non-linear algebraic equations at each x to find $u(x)$ as was done in [16], we represent $u(x)$ through its Fourier coefficients by finding the zeros β_j of a polynomial whose coefficients involve only a finite number of non-zero temporal Fourier modes. We find that plotting the trajectories of the particles $\beta_j(t)$ often gives more information about the solution than making movies of $u(x,t)$ directly.

A key difference in our setup of the problem is that we wish to fix the spatial period once and for all (using e.g. 2π) and describe all families of time-periodic solutions in which the temporal period depends continuously on the parameters of the family. This framework may be perceived as awkward and overly restrictive by some readers, and we agree that the most natural "periodic" generalization of the N-soliton solutions [15, 17] of an integrable system such as Benjamin-Ono are the N-phase quasi-periodic (or multi-periodic) solutions [12, 22, 10, 16]. However, our goal in this paper is not to study the behavior of these solutions in the long wave-length limit, but rather to understand how all these families of solutions are connected (continuously) together through a hierarchy of bifurcations. The additional restriction of exact periodicity makes the bifurcation problem harder for integrable problems, but easier for other problems that can only be studied numerically. To our knowledge, bifurcation between levels in the hierarchy of multi-periodic solutions of Benjamin-Ono has not previously been discussed. Indeed, with the exception of [10], previous representations of these solutions are missing a key degree of freedom, the mean, which must vary in order for these solutions to connect with each other. The most interesting result of this paper is that counting dimensions of nullspaces of the linearized problem does not predict certain degenerate bifurcations that allow for immediate jumps across several levels of the infinite hierarchy of time-periodic solutions. As a consequence, in our numerical studies, we found bifurcations from traveling waves to the second level of the hierarchy, and interior bifurcations from these solutions to the third level of the hierarchy, but never saw bifurcations from traveling waves directly to the third level of the hierarchy (as we did not know where to look for them). This will be important to keep in mind in problems such as the water wave, where exact solutions are not expected to be found.

We believe we have accounted for all time-periodic solutions of the Benjamin-Ono equation with a fixed spatial period, but do not know how to prove this. Even for the simplest case of a traveling wave, it is surprisingly difficult to prove that the solitary and periodic waves found by Benjamin [7] are the only possibilities; see [5] and Appendix B. For the closely related KdV equation [1, 18, 23], substitution of $u(x,t) = u_0(x - ct)$ into the equation leads to an ordinary differential equation for $u_0(x)$ with periodic solutions involving Jacobi elliptic functions; see e.g. [20]. However, for Benjamin-Ono, the equation for the traveling wave shape is non-local due to the Hilbert transform. Nevertheless, Amick and Toland [5] have shown that any traveling wave solution of Benjamin-Ono can be extended to the upper half-plane to agree with the real part of a bounded, holomorphic function satisfying a complex ODE; thus, in spite of non-locality, we are able to obtain uniqueness by solving an initial value problem for $u_0(x)$. Interestingly, these traveling wave shapes are rational functions of e^{ix}, which are simpler than the cnoidal solutions of KdV. This analysis of traveling waves via holomorphic extension to the upper half-plane is similar in spirit to the analysis of rapidly decreasing solutions of Benjamin-Ono via the inverse scattering transform [11, 14]. Thus, it may be possible to prove that we have accounted for all periodic solutions of Benjamin-Ono by developing a spatially periodic version of the IST that is analogous to the study of Bloch eigenfunctions and Riemann surfaces for the periodic KdV equation [18, 6], but this has not yet been carried out.

This paper is organized as follows. In Section 2, we describe meromorphic solutions [8] of the Benjamin-Ono equation, which are a class of solutions represented by a system of N particles evolving in the upper half-plane (or, in our representation, the unit disk) according to a completely integrable ODE. We also show the relationship between the elementary symmetric functions of these particles and the Fourier coefficients of the solution, which were observed in numerical experiments to have trajectories in the complex plane consisting of epicycles involving a finite number of circular orbits. In Section 3, we summarize the results of [3] in the form of a theorem (not proved in [3]) enumerating all bifurcations from traveling waves to the second level of the hierarchy of time-periodic solutions. The new idea that allows us to prove the theorem is to show that all the zeros of a certain polynomial lie inside the unit circle using Rouché's theorem. In Section 4, we state a theorem that parametrizes solutions at level M of the hierarchy through an explicit description of the particle trajectories $\beta_j(t)$. This theorem also describes the way in which different levels of the hierarchy are connected together through bifurcation. The proof of this theorem shows the relationship with previous studies of multi-periodic solutions. In Section 5, we give several examples of degenerate and non-degenerate bifurcations between various levels of the hierarchy. We also use these examples to illustrate some of the topological changes that occur in the particle trajectories along the paths of solutions between bifurcation states. Finally, in Appendix A, we give a direct proof that the particles $\beta_j(t)$ in our formulas lie inside the unit disk in the complex plane; in Appendix B, we discuss uniqueness of traveling wave solutions.

2 Meromorphic Solutions and Particle Trajectories

In this section, we consider spatially periodic solutions of the Benjamin-Ono equation,

$$u_t = Hu_{xx} - uu_x. \tag{2}$$

Here H is the Hilbert transform defined in (1), which has the symbol $\hat{H}(k) = -i\,\mathrm{sgn}(k)$. It is well known [8] that meromorphic solutions of (2) of the form

$$u(x,t) = 2\,\mathrm{Re}\left\{\sum_{l=1}^{N} \frac{2k}{e^{ik[x+kt-x_l(t)]}-1}\right\} \tag{3}$$

exist, where k is a real wave number and the particles $x_l(t)$ evolve in the upper half of the complex plane according to the equation

$$\frac{dx_l}{dt} = \sum_{\substack{m=1\\ m\neq l}}^{N} \frac{2k}{e^{-ik(x_m-x_l)}-1} + \sum_{m=1}^{N} \frac{2k}{e^{-ik(x_l-\bar{x}_m)}-1}, \qquad (1 \le l \le N). \tag{4}$$

This representation is useful for studying the dynamics of solitons of (2) over $\mathbb{R}$, which may be obtained from (3) in the long wave-length limit $k \to 0$. However, over a fixed periodic domain $\mathbb{R}/2\pi\mathbb{Z}$, we have found it more convenient to work with particles $\beta_l(t)$ evolving in the unit disk of the complex plane,

$$\beta_l = e^{-i\bar{x}_l} \in \Delta := \{z \,:\, |z| < 1\}, \tag{5}$$

where a bar denotes complex conjugation. Up to an additive constant and a Galilean transformation, the solution u in (3) may then be written

$$u(x,t) = \alpha_0 + \sum_{l=1}^{N} u_{\beta_l(t)}(x), \tag{6}$$

where we have included the mean α_0 as an additional parameter of the solution and $u_\beta(x)$ is defined via

$$u_\beta(x) = \frac{4|\beta|\{\cos(x-\theta) - |\beta|\}}{1 + |\beta|^2 - 2|\beta|\cos(x-\theta)}, \qquad \big(\beta = |\beta|e^{-i\theta}\big). \tag{7}$$

From (4) or direct substitution into (2), the β_l are readily shown to satisfy

$$\dot{\beta}_l = \sum_{\substack{m=1 \\ m\neq l}}^{N} \frac{2i}{\beta_l^{-1} - \beta_m^{-1}} + \sum_{m=1}^{N} \frac{2i\beta_l^2}{\beta_l - \bar{\beta}_m^{-1}} + i(1-\alpha_0)\beta_l, \qquad (1 \le l \le N). \tag{8}$$

It is awkward to work with $u_\beta(x)$ in physical space; however, in Fourier space, it takes the simple form

$$\hat{u}_{\beta,k} = \begin{cases} 2\bar{\beta}^{|k|}, & k < 0, \\ 0, & k = 0, \\ 2\beta^k, & k > 0. \end{cases} \tag{9}$$

As a result, the Fourier coefficients $c_k(t)$ of $u(x,t)$ in (6) are simply power sums of the particle trajectories,

$$c_k(t) = \begin{cases} \alpha_0 & k = 0, \\ 2\big[\beta_1^k(t) + \cdots + \beta_N^k(t)\big], & k > 0, \end{cases} \tag{10}$$

where $c_k = \bar{c}_{-k}$ for $k < 0$. In [3], it was found numerically that although the β_l often execute very complicated periodic orbits, the elementary symmetric functions σ_j defined via

$$\sigma_0 = 1, \qquad \sigma_j = \sum_{l_1 < \cdots < l_j} \beta_{l_1} \cdots \beta_{l_j}, \qquad (j = 1, \ldots, N) \tag{11}$$

have orbits that are circles (or at worst, epicycles involving a finite number of non-zero temporal Fourier coefficients) in the complex plane. As a consequence, the spatial Fourier coefficients $c_k = 2\,\mathrm{tr}(\Sigma^k)$, where $k \geq 1$ and Σ is the companion matrix of the polynomial $P(z) = \prod(z - \beta_j)$, also have trajectories that are epicycles, which we noticed immediately in our numerical simulations. All the solutions in this paper will be of the form

$$\prod_{l=1}^{N}[z - \beta_l(t)] = \sum_{j=0}^{N}(-1)^j \sigma_j(t) z^{N-j} = P(z, e^{-i\omega t}), \tag{12}$$

where

$$P(z,\lambda) = \sum_{j=0}^{N}(-1)^j \tilde{\sigma}_j(\lambda) z^{N-j} \tag{13}$$

is a monic polynomial in z with coefficients $\tilde{\sigma}_j$ that are Laurent polynomials in λ, and such that for any λ on the unit circle S^1 in the complex plane, the roots $\beta_1, \ldots, \beta_N$ of $P(\cdot, \lambda)$ lie inside the unit disk.

We may express the solution u in (6) directly in terms of P as follows:

$$\begin{aligned} u(x,t) &= \alpha_0 + \sum_{l=1}^{N} u_{\beta_l(t)}(x) = \alpha_0 + \sum_{l=1}^{N} 4\,\mathrm{Re}\left\{\sum_{k=1}^{\infty} \beta_l(t)^k e^{ikx}\right\} \\ &= \alpha_0 + \sum_{l=1}^{N} 4\,\mathrm{Re}\left\{\frac{z}{z - \beta_l(t)} - 1\right\} = \alpha_0 + 4\,\mathrm{Re}\left\{\frac{z\partial_z P(z,\lambda)}{P(z,\lambda)} - N\right\}, \quad \begin{pmatrix} z = e^{-ix} \\ \lambda = e^{-i\omega t} \end{pmatrix}. \end{aligned} \tag{14}$$

Note that $Hu = 4\,\mathrm{Re}\{-i\,[(z\partial zP)/P - N]\}$. The choice $z = e^{-ix}$ (as opposed to e^{+ix}) follows from the decision in (9) to have Fourier coefficients with positive wave numbers carry powers of β rather than $\bar{\beta}$, while the choice $\lambda = e^{-i\omega t}$ leads to a natural sign convention when we interpret the exponents in the Laurent polynomials $\tilde{\sigma}_j(\lambda)$ in (13) as measures of the direction and velocity of the traveling waves obtained in certain limits. It was shown in [3] that (14) is a solution of (2) if there is a constant $\gamma \in \mathbb{R}$ such that

$$\begin{aligned} &\gamma P_{00}\bar{P}_{00} + \bar{P}_{00}\big[P_{20} + \omega P_{01} + (\alpha_0 - 4N)P_{10}\big] \\ &\qquad + P_{00}\big[\bar{P}_{20} + \omega \bar{P}_{01} + (\alpha_0 - 4N)\bar{P}_{10}\big] + 2P_{10}\bar{P}_{10} = 0, \end{aligned} \tag{15}$$

where

$$P_{jk} = (z\partial_z)^j (\lambda\partial_\lambda)^k P(z,\lambda)\Big|_{\substack{z=e^{-ix} \\ \lambda=e^{-i\omega t}}}. \tag{16}$$

The goal of this paper is to find explicit formulas for the solutions $P(z,\lambda)$ of (15), show how they fit in with the previously known families of multi-periodic solutions described in [22, 10, 16], and determine how these families are connected together through bifurcation.

3 Paths connecting arbitrary traveling waves

In [3], a classification of bifurcations from traveling waves was proposed after all time-periodic solutions of the linearization of (2) about traveling waves were found in closed form.

A numerical continuation method was then developed to follow paths of non-trivial time-periodic solutions beyond the realm of validity of the linearization until another traveling wave was reached (or until the solution blows up as the bifurcation parameter approaches a critical value). Through extensive data fitting of the numerical solutions, the exact form of the solutions on this path was deduced. In this section, we give an alternative formula for these exact solutions that unifies the three cases described in [3] and makes it possible to show that the roots β_l of the polynomial $P(\cdot, \lambda)$ are inside the unit circle for $\lambda \in S^1$.

An N-hump traveling wave is uniquely determined by the mean, α_0, a complex number $\beta \in \Delta$, and a positive integer, N:

$$u_{\alpha_0,N,\beta}(x,t) = \alpha_0 + \sum_{l=1}^{N} u_{\beta_l(t)}(x), \qquad \beta_l(t) = \sqrt[N]{\beta} e^{-ict}, \qquad c = \alpha_0 - N\frac{1-3|\beta|^2}{1-|\beta|^2}. \tag{17}$$

Here β_l ranges over all Nth roots of β. This solution may also be written

$$u_{\alpha_0,N,\beta}(x,t) = u_{N,\beta}(x-ct) + c, \qquad u_{N,\beta} = N\frac{1-3|\beta|^2}{1-|\beta|^2} + Nu_\beta(Nx), \tag{18}$$

where $u_{N,\beta}(x) = Nu_{1,\beta}(Nx)$ is the N-hump stationary solution; hence, the traveling wave moves to the right if $c > 0$. We can solve for c and α_0 in terms of the period $T > 0$ and a speed index $\nu \in \mathbb{Z}$ indicating how many increments of $\frac{2\pi}{N}$ the wave moves to the right in one period:

$$cT = \frac{2\pi\nu}{N}, \qquad \alpha_0 = c + N\frac{1-3|\beta|^2}{1-|\beta|^2}. \tag{19}$$

In order to bifurcate to a non-trivial time-periodic solution, the period T must be related to an eigenvalue

$$\omega_{N,n} = \begin{cases} (n)(N-n), & 1 \le n \le N-1, \\ (n+1-N)\Big[n+1+N\Big(1-\frac{1-3|\beta|^2}{1-|\beta|^2}\Big)\Big], & n \ge N \end{cases} \tag{20}$$

of the linear operator [3] governing the evolution of solutions of the linearization about the N-hump stationary solution:

$$\omega_{N,n}T = \frac{2\pi m}{N}, \qquad 1 \le m \in \begin{cases} n\nu + N\mathbb{Z}, & 1 \le n < N, \\ (n+1)\nu + N\mathbb{Z}, & n \ge N. \end{cases} \tag{21}$$

This requirement on the oscillation index m enforces the condition that the linearized solution over the stationary solution return to a phase shift of itself to account for the fact that the traveling wave has moved during this time; see [3]. Here we have used the fact that if $u(x,t) = u_{N,\beta}(x)$ is a stationary solution and

$$U(x,t) = u(x-ct,t) + c \tag{22}$$

is a traveling wave, then the solutions v and V of the linearizations about u and U, respectively, satisfy $V(x,t) = v(x-ct,t)$. The parameter β together with the four integers (N,ν,n,m) enumerate the bifurcations from traveling waves, which comprise the first level of the hierarchy of time-periodic solutions of the Benjamin-Ono equation, to the second level of this hierarchy. We will see later that other bifurcations from traveling waves to higher levels of the hierarchy also exist, which is interesting as they are not predicted from counting dimensions of nullspaces in the linearization.

After (numerically) mapping out which bifurcations (N,ν,n,m) and (N',ν',n',m') were connected by paths of non-trivial solutions, it was found that N, N', ν and ν' can be chosen independently as long as

$$N' < N, \qquad \nu' > \frac{N'}{N}\nu. \tag{23}$$

The other parameters are then given by

$$m = m' = N\nu' - N'\nu > 0, \qquad n = N - N', \qquad n' = N - 1. \tag{24}$$

The following theorem proves that these numerical conjectures are correct.

Theorem 1 *Let N, N', ν and ν' be integers satisfying $N > N' > 0$ and $m = N\nu' - N'\nu > 0$. There is a four-parameter family of time-periodic solutions connecting the traveling wave bifurcations $(N',\nu',N-1,m)$ and $(N,\nu,N-N',m)$. These solutions are of the form*

$$u(x,t) = \alpha_0 + \sum_{l=1}^{N} u_{\beta_l(t-t_0)}(x-x_0), \tag{25}$$

where $\beta_1(t)$, …, $\beta_N(t)$ are the roots of the polynomial $P(\cdot, e^{-i\omega t})$ defined by

$$P(z,\lambda) = z^N + A\lambda^{\nu'} z^{N-N'} + B\lambda^{\nu-\nu'} z^{N'} + C\lambda^{\nu}, \tag{26}$$

with

$$A = \sqrt{\frac{N-N'+s+s'}{N+s+s'}}\sqrt{\frac{(N+s')s'}{N'(N-N')+(N+s')s'}}, \tag{27}$$

$$B = \sqrt{\frac{(N+s')s'}{N'(N-N')+(N+s')s'}}\sqrt{\frac{s}{N-N'+s}}, \qquad C = \sqrt{\frac{s}{N-N'+s}}\sqrt{\frac{N-N'+s+s'}{N+s+s'}},$$

$$\alpha_0 = \frac{N^2\nu' - (N')^2\nu}{m} - 2s - \frac{2N'(\nu'-\nu)}{m}s', \qquad \omega = \frac{2\pi}{T} = \frac{N'(N-N')(N+2s')}{m}.$$

The four parameters are $s \geq 0$, $s' \geq 0$, $x_0 \in \mathbb{R}$ and $t_0 \in \mathbb{R}$. The N- and N'-hump traveling waves occur when $s' = 0$ and $s = 0$, respectively. When both are zero, we obtain the constant solution $u(x,t) \equiv \frac{N^2\nu' - (N')^2\nu}{m}$.

Proof: Without loss of generality, we may assume $x_0 = 0$ and $t_0 = 0$. It was shown in [3] that $P(z, \lambda)$ of the form (26) satisfies (15) if

$$\gamma = (3N - \alpha_0)N - \nu\omega, \tag{28}$$

$$[(N')^2 - 2NN' + N'\alpha_0 - \nu'\omega]B + [(N')^2 + 2NN' - N'\alpha_0 + \nu'\omega]AC = 0, \tag{29}$$

$$\begin{aligned}&\big[3N^2 - 4NN' + (N')^2 - (N - N')\alpha_0 + (\nu - \nu')\omega\big]BC \\ &\qquad - \big[N^2 - (N')^2 - (N - N')\alpha_0 + (\nu - \nu')\omega\big]A = 0,\end{aligned} \tag{30}$$

$$\begin{aligned}&(N\alpha_0 - \nu\omega - N^2) + \big[(2N' - N)\alpha_0 + (\nu - 2\nu')\omega + 3N^2 - 8NN' + 4(N')^2\big]B^2 \\ &\quad + \big[(N - 2N')\alpha_0 + 4(N')^2 - N^2 + (2\nu' - \nu)\omega\big]A^2 + \big[(3N - \alpha_0)N + \nu\omega\big]C^2 = 0.\end{aligned} \tag{31}$$

Using a computer algebra system, it is easy to check that (29)–(31) hold when A, B, C, α_0 and ω are defined as in (27). When $s' = 0$, we have $A = B = 0$ and $C = \sqrt{\dfrac{s}{N+s}}$ so that

$$\beta_l(t) = \sqrt[N]{-C\lambda^\nu} = \sqrt[N]{-C}e^{-ict}, \qquad c = \frac{\omega\nu}{N} = \frac{N'(N - N')\nu}{m} = \alpha_0 - N\frac{1 - 3C^2}{1 - C^2},$$

where each β_l is assigned a distinct Nth root of $-C$. By (17), this is an N-hump traveling wave with speed index ν and period $T = \frac{2\pi}{\omega}$. Similarly, when $s = 0$, we have $B = C = 0$ and $A = \sqrt{\dfrac{s'}{N' + s'}}$ so that

$$\beta_l(t) = \left\{ \begin{array}{cc} \sqrt[N']{-A}e^{-ict} & l \le N' \\ 0 & l > N' \end{array} \right\}, \qquad c = \frac{\omega\nu'}{N'} = \frac{(N - N')(N + 2s')\nu'}{m} = \alpha_0 - N'\frac{1 - 3A^2}{1 - A^2},$$

which is an N'-hump traveling wave with speed index ν' and period $T = \frac{2\pi}{\omega}$.

Finally, we show that the roots of $P(\cdot, \lambda)$ are inside the unit disk for any λ on the unit circle, S^1. We will use Rouché's theorem [2]. Let

$$\begin{aligned} f_1(z) &= z^N + A\lambda^{\nu'}z^{N-N'} + B\lambda^{\nu-\nu'}z^{N'} + C\lambda^\nu, \\ f_2(z) &= z^N + A\lambda^{\nu'}z^{N-N'}, \\ f_3(z) &= z^N \qquad\qquad\quad + B\lambda^{\nu-\nu'}z^{N'}. \end{aligned}$$

From (27), we see that $\{A, B, C\} \subseteq [0, 1)$, $A \ge BC$, $B \ge CA$ and $C \ge AB$. Thus,

$$\begin{aligned} d_2(z) := |f_2(z)|^2 - |f_1(z) - f_2(z)|^2 &= |\lambda^{-\nu'}z^{N'} + A|^2 - |B\lambda^{-\nu'}z^{N'} + C|^2 \\ &= 1 + A^2 - B^2 - C^2 + 2(A - BC)\cos\theta \ge (1 - A)^2 - (B - C)^2, \end{aligned} \tag{32}$$

where $\lambda^{-\nu'}z^{N'} = e^{i\theta}$. Similarly,

$$d_3 := |f_3(z)|^2 - |f_1(z) - f_3(z)|^2 \ge (1 - B)^2 - (A - C)^2. \tag{33}$$

Note that

$$\begin{array}{llllll}
B \le A, & C \le B & \Rightarrow & B - C \le B - AB < 1 - A & \Rightarrow & d_2(z) > 0 \text{ for } z \in S^1, \\
B \le A, & C > B & \Rightarrow & |C - A| < 1 - B & \Rightarrow & d_3(z) > 0 \text{ for } z \in S^1, \\
A \le B, & C \le A & \Rightarrow & A - C \le A - AB < 1 - B & \Rightarrow & d_3(z) > 0 \text{ for } z \in S^1, \\
A \le B, & C > A & \Rightarrow & |C - B| < 1 - A & \Rightarrow & d_2(z) > 0 \text{ for } z \in S^1.
\end{array}$$

Thus, in all cases, $f_1(z) = P(z, \lambda)$ has the same number of zeros inside S^1 as $f_2(z)$ or $f_3(z)$, which each have N roots inside S^1. Since $f_1(z)$ is a polynomial of degree N, all the roots are inside S^1. $\square$

4 An infinite hierarchy of interior bifurcations

Next we wish to find all possible cascades of interior bifurcations from these already non-trivial solutions to more and more complicated time-periodic solutions. The most interesting consequence of the following theorem is that there are some traveling waves with more bifurcations to non-trivial time-periodic solutions than predicted by counting the dimension of the kernel of the linearization of the map measuring deviation from time-periodicity. This will be illustrated in various examples in Section 5.

Theorem 2 *Let $M \ge 2$ be an integer and choose*

$$\begin{array}{ll}
k_1, \dots, k_M \in \mathbb{N}, & \textit{(positive integers, not necessarily distinct or monotonic),} \\
\nu_1, \dots, \nu_M \in \mathbb{Z}, & \textit{(arbitrary integers satisfying } \nu_{j-1} < \dfrac{k_{j-1}}{k_j}\nu_j \textit{ for } j \ge 2\textit{).}
\end{array}$$

Now define the positive quantities

$$m_j = k_{j-1}\nu_j - k_j\nu_{j-1}, \quad \tau_j = \frac{k_j(k_j + k_{j-1})k_{j-1}}{m_j}, \quad \gamma_j = \frac{2k_jk_{j-1}}{m_j}, \quad (2 \le j \le M). \tag{34}$$

Let $J = \operatorname{argmax}_{2 \le j \le M} \tau_j$. If there is a tie, J can be any of the candidates. Then there is an $M + 2$ parameter family of time-periodic solutions of the Benjamin-Ono equation parametrized by

$$s_1 \ge 0, \qquad s_J \ge 0, \qquad x_{j0} \in \mathbb{R}, \qquad (1 \le j \le M) \tag{35}$$

and constructed as follows. First, we define

$$s_j = \frac{\tau_J - \tau_j}{\gamma_j} + \frac{\gamma_J}{\gamma_j}s_J, \qquad (2 \le j \le M, \;\; j \ne J), \tag{36}$$

$$q_1 = s_1, \qquad p_1 = s_1 + k_1, \qquad q_j = p_{j-1} + s_j, \qquad p_j = q_j + k_j, \qquad (2 \le j \le M) \tag{37}$$

so that $s_j \geq 0$ *for* $1 \leq j \leq M$ *and*

$$0 \overset{s_1}{\leq} q_1 \overset{k_1}{<} p_1 \overset{s_2}{\leq} q_2 \overset{k_2}{<} \cdots \overset{s_{M-1}}{\leq} q_{M-1} \overset{k_{M-1}}{<} p_{M-1} \overset{s_M}{\leq} q_M \overset{k_M}{<} p_M. \tag{38}$$

For any subset S *of* $\mathcal{M} = \{1, \dots, M\}$, *we denote the complement by* $S' = \mathcal{M} \setminus S$ *and define*

$$k_S = \sum_{j \in S} k_j, \qquad \nu_{S'} = \sum_{m \in S'} \nu_m, \qquad C_S = \left(\prod_{(j,m) \in S \times S'} a_{jm} \right)\left(\prod_{m \in S'} b_m \right), \tag{39}$$

where

$$a_{jm} = \sqrt{\frac{(p_m - q_j)(q_m - p_j)}{(q_m - q_j)(p_m - p_j)}}, \qquad b_m = \sqrt{\frac{q_m}{p_m}} e^{-ik_m x_{m0}}. \tag{40}$$

Then

$$u(x,t) = \alpha_0 + \sum_{l=1}^{N} u_{\beta_l(t)}(x) \tag{41}$$

is a periodic solution of the Benjamin-Ono equation, where $N = k_{\mathcal{M}} = \sum_{j=1}^{M} k_j$,

$$\alpha_0 = \left(2N - k_1 + \frac{\nu_1}{k_1} \tau_J \right) - 2s_1 + \frac{\nu_1}{k_1} \gamma_J s_J, \qquad \omega = \frac{2\pi}{T} = \tau_J + \gamma_J s_J, \tag{42}$$

and $\beta_1(t), \dots, \beta_N(t)$ *are the roots of the polynomial* $z \mapsto P(z, e^{-i\omega t})$ *given by*

$$P(z,\lambda) = \sum_{S \in \mathcal{P}(\mathcal{M})} C_S \lambda^{\nu_{S'}} z^{k_S}. \tag{43}$$

When $M = 2$, *this representation coincides with that of Theorem 1 if we set*

$$\begin{aligned} &k_1 = N - N', \qquad &&\nu_1 = \nu - \nu', \qquad &&s_1 = s, \qquad &&x_{10} = x_0 - \frac{\nu_1}{k_1}\omega t_0, \\ &k_2 = N', &&\nu_2 = \nu', &&s_2 = s', &&x_{20} = x_0 - \frac{\nu_2}{k_2}\omega t_0. \end{aligned}$$

It reduces to a traveling wave when $s = 0$ *or* $s' = 0$ *and to a constant solution when both are zero. Similarly, when* $M \geq 3$, *the solution reduces to a simpler solution in this same hierarchy (with* M *replaced by* $\widetilde{M} = M - 1$*) when* s_1 *or* s_J *reaches zero. Specifically, if* $s_1 = 0$, *then* $P(z,\lambda) = z^{k_1} \widetilde{P}(z,\lambda)$, *where* $\widetilde{P}(z,\lambda)$ *corresponds to the parameters*

$$\tilde{k}_j = k_{j+1}, \quad \tilde{\nu}_j = \nu_{j+1}, \quad \tilde{s}_j = s_{j+1}, \quad \tilde{x}_{j0} = x_{j+1,0}, \quad (1 \leq j \leq \widetilde{M}). \tag{44}$$

We interpret this as an annihilation of k_1 *particles* β_l *at the origin. If* $s_J = 0$, *we have* $P(z,\lambda) = \widetilde{P}(z,\lambda)$, *where*

$$\begin{aligned} &\tilde{k}_j = k_j, &&\tilde{\nu}_j = \nu_j, &&\tilde{s}_j = s_j, &&\tilde{x}_{j0} = x_{j0}, &&(1 \leq j \leq J-2), \\ &\tilde{k}_j = k_j + k_J, &&\tilde{\nu}_j = \nu_j + \nu_J, &&\tilde{s}_j = s_j, &&\tilde{x}_{j0} = \tfrac{k_j x_{j0} + k_J x_{J0}}{k_j + k_J}, &&(j = J-1), \\ &\tilde{k}_j = k_{j+1}, &&\tilde{\nu}_j = \nu_{j+1}, &&\tilde{s}_j = s_{j+1}, &&\tilde{x}_{j0} = x_{j+1,0}, &&(J \leq j \leq \widetilde{M}). \end{aligned} \tag{45}$$

If several s_j are zero when $s_J = 0$ (i.e. if a tie occurs when choosing $J = \operatorname{argmax}_{2\le j\le M} \tau_j$), the bifurcation is degenerate and any subset of the indices for which $s_j = 0$ may be removed using the rule (45) repeatedly (once for each index removed, with J ranging over these indices in reverse order to avoid re-labeling), allowing for bifurcations that jump several levels in the hierarchy at once.

PROOF: Rather than show directly that $P(z,\lambda)$ in (43) satisfies (15), we show that each of our solutions differs from a multi-periodic solution [22, 16] by at most a transformation of the form (22). We give a direct proof that all the zeros of $P(\cdot,\lambda)$ lie inside the unit circle in Appendix A as concluding this from the combined results of [10] and [16] is complicated. If all the inequalities in (38) are strict, then it is known [16] that

$$U = 2i\frac{\partial}{\partial x}\log\frac{f'}{f} \tag{46}$$

satisfies the Benjamin-Ono equation (2) with

$$f' = \sum_{\mu=0,1}\exp\left[\sum_{j=1}^{M}\mu_j\left(i\theta_j - \phi_j - \frac{1}{2}\sum_{m\neq j}^{M}A_{jm}\right) + \sum_{j<m}^{(M)}\mu_j\mu_m A_{jm}\right], \tag{47}$$

$$f = \sum_{\mu=0,1}\exp\left[\sum_{j=1}^{M}\mu_j\left(i\theta_j + \phi_j - \frac{1}{2}\sum_{m\neq j}^{M}A_{jm}\right) + \sum_{j<m}^{(M)}\mu_j\mu_m A_{jm}\right], \tag{48}$$

$$\theta_j = k_j(x - c_jt - x_{j0}), \qquad e^{2\phi_j} = \frac{p_j}{q_j}, \qquad k_j = p_j - q_j, \qquad c_j = p_j + q_j, \tag{49}$$

$$\exp(A_{jm}) = \frac{(q_m - q_j)(p_m - p_j)}{(p_m - q_j)(q_m - p_j)} = \frac{(c_m - c_j)^2 - (k_m - k_j)^2}{(c_m - c_j)^2 - (k_m + k_j)^2}, \tag{50}$$

where $x_{j0} \in \mathbb{R}$ are M arbitrary phase parameters and the notation $\sum_{\mu=0,1}$ indicates a summation over all possible combinations of $\mu_1 = 0,1$; $\mu_2 = 0,1$; $\dots$; $\mu_M = 0,1$. (The notation $\sum_{j<m}^{(M)}$ indicates that j and m both vary between 1 and M such that $j<m$, while $\sum_{m\neq j}^{M}$ indicates that m varies from 1 to M omitting $m = j$).

We write f' and f in (47) and (48) as sums over all subsets S of $\mathcal{M} = \{1,\dots,M\}$:

$$f', f \;=\; \sum_{S\in\mathcal{P}(\mathcal{M})}\left(\prod_{(j,m)\in S\times S'} e^{-\frac{1}{2}A_{jm}}\right)\prod_{j\in S} e^{ik_j(x - c_jt - x_{j0})\mp\phi_j}, \qquad (S' = \mathcal{M}\setminus S), \tag{51}$$

where $-\phi_j$ is used for f' and $+\phi_j$ is used for f. Next we observe that

$$u(x,t) = U(x - ct, t) + c \tag{52}$$

will be time-periodic with period $T = \frac{2\pi}{\omega}$ if there exist integers $\nu_j \in \mathbb{Z}$ such that

$$k_j(c + c_j) = \nu_j\omega, \qquad (1 \le j \le M). \tag{53}$$

Then we have

$$
\begin{aligned}
f'(x-ct,t) &= \Bigg(\prod_{j\in\mathcal{M}} e^{ik_jx}\Bigg) \sum_{S\in\mathcal{P}(\mathcal{M})} C_S \Bigg(\prod_{m\in S'} e^{-i\nu_m\omega t}\Bigg)\Bigg(\prod_{j\in S} e^{-ik_jx}\Bigg), \\
f(x-ct,t) &= \Bigg(\prod_{j\in\mathcal{M}} e^{-ik_jx_{j0}} e^{-i\nu_j\omega t} e^{\phi_j}\Bigg) \sum_{S\in\mathcal{P}(\mathcal{M})} \overline{C_S} \Bigg(\prod_{m\in S'} e^{i\nu_m\omega t}\Bigg)\Bigg(\prod_{j\in S} e^{ik_jx}\Bigg),
\end{aligned} \tag{54}
$$

with C_S as in (39) above. The complex conjugation in f comes from interchanging S and S' in the sum after factoring out $\big(\prod_{j\in\mathcal{M}}\cdots\big)$. It follows that u in (52) is given by

$$
u = c - 2N + 2i\partial_x \log\frac{g}{h} = \alpha_0 + 2\left(\frac{i\partial_x g}{g} - N\right) + 2\left(\frac{-i\partial_x h}{h} - N\right), \tag{55}
$$

where

$$
\begin{aligned}
&N = \sum_{j=1}^{M} k_j, \qquad \alpha_0 = c + 2N, \qquad g(x,t) = P(e^{-ix}, e^{-i\omega t}), \qquad h = \bar{g}, \\
&P(z,\lambda) = \sum_{S\in\mathcal{P}(\mathcal{M})} C_S \lambda^{\nu_{S'}} z^{k_S}, \qquad \nu_{S'} = \sum_{m\in S'} \nu_m, \qquad k_S = \sum_{j\in S} k_j.
\end{aligned} \tag{56}
$$

With λ fixed, P is a monic polynomial in z of degree N. If we complexify x and fix t, then

$$
P(e^{-ix}, e^{-i\omega t}) = 0 \quad\Leftrightarrow\quad f'(x-ct,t) = 0 \quad\Leftrightarrow\quad f(\bar{x}-ct,t) = 0, \tag{57}
$$

so all the zeros of P are inside the unit circle iff all the zeros of f are in the upper half-plane and all the zeros of f' are in the lower half-plane. These properties of f and f' were assumed to be true in [22], leaving a small gap in their proof (acknowledged in the paper); we give a proof in Appendix A. The right hand side of (55) is equal to the right hand side of (14), which establishes the representation (41) of $u(x,t)$ in terms of the trajectories $\beta_l(t)$ of the roots of $P(\cdot, e^{-i\omega t})$.

Eliminating c from (53) and using $(c_j - c_{j-1}) = (k_{j-1} + k_j + 2s_j)$, we find that

$$
k_j k_{j-1}(k_{j-1} + k_j + 2s_j) = (k_{j-1}\nu_j - k_j\nu_{j-1})\omega, \qquad (2 \le j \le M). \tag{58}
$$

This shows that m_j in (34) must be positive. Eliminating ω, we find that

$$
\tau_j + \gamma_j s_j = \tau_J + \gamma_J s_J, \qquad\qquad j, J \in \{2,\dots,M\}. \tag{59}
$$

Choosing $J = \operatorname{argmax}_{2\le j\le M} \tau_j$ and solving (59) for s_j in terms of s_J yields (36), which ensures that $s_j > 0$ whenever $s_J > 0$. From

$$
\alpha_0 = c + 2N, \qquad c = -c_1 + \frac{\nu_1}{k_1}\omega, \qquad c_1 = k_1 + 2s_1, \qquad \omega = \tau_J + \gamma_J s_J, \tag{60}
$$

we obtain the formulas in (42) for α_0 and ω.

Finally, we drop the assumption that the inequalities in (38) are strict and observe what happens to these solutions when $s_1 = 0$ or $s_J = 0$. If $s_1 = 0$, then $b_1 = 0$, so

$$1 \notin S \;\Rightarrow\; C_S = 0, \qquad 1 \in S \;\Rightarrow\; C_S = \left(\prod_{(j,m)\in(S\setminus\{1\})\times S'} a_{jm} \right) \left(\prod_{m\in S'} a_{1m} b_m \right).$$

But since $q_1 = 0$ and $p_1 = k_1$ when $s_1 = 0$, we have

$$a_{1m} b_m = \sqrt{\frac{(p_m - q_1)(q_m - p_1)}{(q_m - q_1)(p_m - p_1)}} \sqrt{\frac{q_m}{p_m}} e^{-ik_m x_{m0}} = \sqrt{\frac{q_m - k_1}{p_m - k_1}} e^{-ik_m x_{m0}}. \tag{61}$$

Thus, if we define $\widetilde{M} = M - 1$ and shift indices down as in (44), the parameters q_j and p_j will decrease by k_1 as illustrated here,

$$\begin{array}{c} 0 \overset{s_1=0}{=} q_1 \overset{k_1}{<} p_1 \overset{s_2}{<} q_2 \overset{k_2}{<} p_2 \overset{s_3}{\le} q_3 \overset{k_3}{<} \cdots \overset{s_M}{\le} q_M \overset{k_M}{<} p_M \\ 0 \underset{\tilde{s}_1=s_2}{\le} \tilde{q}_1 \underset{\tilde{k}_1=k_2}{<} \tilde{p}_1 \underset{\tilde{s}_2=s_3}{\le} \tilde{q}_2 \underset{\tilde{k}_2=k_3}{<} \cdots \underset{\tilde{s}_{M-1}=s_M}{\le} \tilde{q}_{M-1} \underset{\tilde{k}_{M-1}=k_M}{<} \tilde{p}_{M-1} \end{array} \tag{62}$$

and hence $\tilde{a}_{jm} = a_{j+1,m+1}$ and $\tilde{b}_m = a_{1,m+1} b_{m+1}$ for $1 \le j, m \le \widetilde{M}$, $j \neq m$. Therefore, with the notation $\widetilde{S}' = \widetilde{M} \setminus \widetilde{S}$, the only difference between

$$P(z,\lambda) = \sum_{S\in\mathcal{P}(\mathcal{M}),1\in S} C_S \lambda^{\nu_{S'}} z^{k_S}, \qquad \widetilde{P}(z,\lambda) = \sum_{\widetilde{S}\in\mathcal{P}(\widetilde{\mathcal{M}})} \widetilde{C}_{\widetilde{S}} \lambda^{\tilde{\nu}_{\widetilde{S}'}} z^{\tilde{k}_{\widetilde{S}}},$$

is that each term in the former sum carries an extra factor of z^{k_1} when the sets S and $\widetilde{S}$ are matched up in the natural way. Thus $P(z,\lambda) = z^{k_1} \widetilde{P}(z,\lambda)$.

Now consider the case $s_J = 0$ with $J = \operatorname{argmax}_{2\le j\le M} \tau_j$. This time $a_{J-1,J} = 0$ due to $q_J = p_{J-1}$, so $C_S = 0$ unless $J-1$ and J are both in S or both in S'. Let us define

$$\mathcal{P}_J(\mathcal{M}) = \{ S \in \mathcal{P}(\mathcal{M}) \;:\; J-1, J \in S \;\text{ or }\; J-1, J \in S' \}. \tag{63}$$

When we perform the sum over $S \in \mathcal{P}_J(\mathcal{M})$ to construct $P(z,\lambda)$, we can consider $J-1$ and J as a single unit. The following factors always appear together in any C_S that contains one of them:

$$a_{J-1,m} a_{Jm} = \sqrt{\frac{(p_m - q_{J-1})(q_m - p_J)}{(q_m - q_{J-1})(p_m - p_J)}}, \qquad b_{J-1} b_J = \sqrt{\frac{q_{J-1}}{p_J}} e^{-ik_{J-1}x_{J-1,0}} e^{-ik_J x_{J0}}.$$

Thus we can remove p_{J-1} and q_J from the sequence if we short circuit the diagram

$$\begin{array}{c} 0 \overset{s_1}{\le} q_1 \overset{k_1}{<} p_1 \overset{s_2}{\le} \cdots \overset{s_{J-1}}{\le} q_{J-1} \overset{k_{J-1}}{<} p_{J-1} \overset{s_J=0}{=} q_J \overset{k_J}{<} p_J \overset{s_{J+1}}{\le} q_{J+1} \overset{k_{J+1}}{<} \cdots \overset{s_M}{\le} q_M \overset{k_M}{<} p_M \\ 0 \underset{s_1}{\le} \tilde{q}_1 \underset{k_1}{<} \tilde{p}_1 \underset{s_2}{\le} \cdots \underset{s_{J-1}}{\le} \tilde{q}_{J-1} \underset{\tilde{k}_{J-1}=k_{J-1}+k_J}{<} \tilde{p}_{J-1} \underset{\tilde{s}_J=s_{J+1}}{\le} \tilde{q}_J \underset{\tilde{k}_J}{<} \cdots \underset{\tilde{s}_{M-1}}{\le} \tilde{q}_{M-1} \underset{\tilde{k}_{M-1}}{<} \tilde{p}_{M-1} \end{array}$$

and set

$$\tilde{k}_{J-1} = k_{J-1} + k_J, \qquad \tilde{\nu}_{J-1} = \nu_{J-1} + \nu_J, \qquad \tilde{x}_{J-1,0} = \frac{k_{J-1}x_{J-1,0} + k_J x_{J0}}{k_{J-1} + k_J}. \tag{64}$$

The other parameters are simply copied from the original sequence as was indicated in (45). We then have

$$P(z,\lambda) = \sum_{S\in\mathcal{P}_J(\mathcal{M})} C_S \lambda^{\nu_{S'}} z^{k_S} = \sum_{\widetilde{S}\in\mathcal{P}(\widetilde{\mathcal{M}})} \widetilde{C}_{\widetilde{S}} \lambda^{\tilde{\nu}_{\widetilde{S}'}} z^{\tilde{k}_{\widetilde{S}}} = \widetilde{P}(z,\lambda),$$

where again the sets S and $\widetilde{S}$ in these sums are in natural 1-1 correspondence.

Finally, we verify that the parameters $\tilde{k}_j$, $\tilde{s}_j$ and $\tilde{\nu}_j$ of the reduced system are consistent with the construction, i.e. if $\widetilde{M} \geq 2$ and we define $\tilde{J} = \operatorname{argmax}_{2\leq j\leq \widetilde{M}} \tilde{\tau}_j$, then

$$\tilde{s}_j = \frac{\tilde{\tau}_{\tilde{J}} - \tilde{\tau}_j}{\tilde{\gamma}_j} + \frac{\tilde{\gamma}_{\tilde{J}}}{\tilde{\gamma}_j}\tilde{s}_{\tilde{J}}, \qquad (2 \leq j \leq \widetilde{M}, \;\; j \neq \tilde{J}). \tag{65}$$

Recall that these equations are obtained by eliminating $\tilde{c}$ and $\tilde{\omega}$ from

$$\tilde{k}_j(\tilde{c} + \tilde{c}_j) = \tilde{\nu}_j \tilde{\omega}, \qquad (1 \leq j \leq \widetilde{M}). \tag{66}$$

In the first case where $s_1 = 0$, (44) and (53) together with

$$\tilde{c}_j = \tilde{p}_j + \tilde{q}_j = c_{j+1} - 2k_1, \qquad (1 \leq j \leq \widetilde{M}) \tag{67}$$

imply that (66) is satisfied if we define

$$\tilde{c} = c + 2k_1, \qquad \tilde{\omega} = \omega. \tag{68}$$

Since we annihilate k_1 particles in this case, $\widetilde{N} = N - k_1$, and the mean

$$\tilde{\alpha}_0 = \tilde{c} + 2\widetilde{N} = c + 2N = \alpha_0 \tag{69}$$

remains unchanged in spite of the change in c and N (as it must for the solution to vary continuously through the bifurcation). In the remaining case where $s_J = 0$, we have

$$\tilde{c}_j = \begin{cases} c_j, & j < J-1, \\ c_j + k_J = c_J - k_j, & j = J-1, \\ c_{j+1}, & J \leq j \leq \widetilde{M}. \end{cases} \tag{70}$$

Thus, by (45) and (53), all the equations in (66), except possibly $j = J-1$, are trivially satisfied if we define

$$\tilde{c} = c, \qquad \tilde{\omega} = \omega. \tag{71}$$

The remaining equation is

$$(k_{J-1} + k_J)(c + c_{J-1} + k_J) = (\nu_{J-1} + \nu_J)\omega. \tag{72}$$

To see that this is true, note that by (53),

$$k_{J-1}(c + c_{J-1}) = \nu_{J-1}\omega, \qquad k_J(c + c_J) = \nu_J\omega. \tag{73}$$

Adding these equations and using $k_J c_J = k_J(c_{J-1} + k_{J-1} + k_J)$ gives (72), as required. Since $\tilde{c} = c$ and $\widetilde{N} = N$, the mean $\alpha_0 = c + 2N$ does not change as a result of the bifurcation.

Thus, we have shown that when $s_1 = 0$ or $s_J = 0$, the function $u(x,t)$ in (41) agrees with another function in the hierarchy with M reduced by one and the parameter s_1 or s_J removed. Continuing in this fashion, we can remove all the zero indices, eventually yielding the case where the p_j and q_j are distinct from one another, or the case that $M \in \{0, 1\}$. This shows that $u(x,t)$ is a solution of (2), where we rely on Theorem 1 to handle the reduction from $M = 2$ to the traveling wave case $M = 1$, or the constant solution case $M = 0$. □

5 Examples

In this section we present several examples to illustrate the types of bifurcation that occur in the hierarchy of time-periodic solutions described in Theorem 2. We begin with the simplest example that leads to a degenerate bifurcation, namely

$$M = 3, \qquad \vec{k} = (1, 1, 1), \qquad \vec{\nu} = (-2, -1, 0). \tag{74}$$

This solution and the three $M = 2$ solutions connected to it have parameters shown in Figure 1. We hold the mean $\alpha_0 = 0.544375$ fixed, which is the value used in several of the numerical simulations in [4, 3], and construct a single bifurcation diagram showing all four solutions; see Figure 2. Note that paths B,C,D are actually parametrized by

$$\tilde{s}_1^{(B)} = s_2, \quad \tilde{s}_2^{(B)} = s_3, \qquad \tilde{s}_1^{(C)} = s_1, \quad \tilde{s}_2^{(C)} = s_3, \qquad \tilde{s}_1^{(D)} = s_1, \quad \tilde{s}_2^{(D)} = s_2 \tag{75}$$

in formulas (34)–(43), but we use the original variables s_j here for the convenience of making a single bifurcation diagram. The one confusing aspect of doing this is that we obtain different traveling waves depending on the order in which we set the s_j to zero. This is why we drew two axes for s_2 and s_3 in Figure 2. For example, if we start on path A and decrease s_1 to zero (moving to path B) and then decrease s_3 to zero, we obtain the traveling wave bifurcation $(2, -1, 1, 1)$; however, if we first set $s_3 = 0$ (moving to path D) and then set $s_1 = 0$, we obtain the bifurcation $(2, -1, 2, 3)$. Both traveling waves have $N = 2$ humps and speed index $\nu = -1$, but the amplitude and period of the two solutions are different as

path A		path B		path C		path D	
$k =$	$1,1,1,$	$k =$	$1,1,$	$k =$	$2,1,$	$k =$	$1,2,$
$\nu =$	$-2,-1,0,$	$\nu =$	$-1,0,$	$\nu =$	$-3,0,$	$\nu =$	$-2,-1,$
$m =$	$1,1,$	$m =$	$1,$	$m =$	$3,$	$m =$	$3,$
$\tau =$	$2,2,$	$\tau =$	$2,$	$\tau =$	$2,$	$\tau =$	$2,$
$\gamma =$	$2,2,$	$\gamma =$	$2,$	$\gamma =$	$4/3,$	$\gamma =$	$4/3,$
$s_3 =$	$s_2,$	$s_1 =$	$0,$	$s_2 =$	$0,$	$s_3 =$	$0,$
$\alpha_0 =$	$1-2s_1-4s_2,$	$\alpha_0 =$	$1-2s_2-2s_3,$	$\alpha_0 =$	$1-2s_1-2s_3,$	$\alpha_0 =$	$1-2s_1-\frac{8}{3}s_2,$
$\omega =$	$2+2s_2,$	$\omega =$	$2+2s_3,$	$\omega =$	$2+\frac{4}{3}s_3,$	$\omega =$	$2+\frac{4}{3}s_2.$

Figure 1: Parameters of four paths of time-periodic solutions connected by bifurcations.

they have different bifurcation indices. Similarly, although starting on path A and setting s_2 and s_3 to zero in either order leads to the same stationary solution, the period T is different depending on whether we follow path B to $(1,0,1,1)$ or path C to $(1,0,2,3)$. This only happens at the bottom ($\widetilde{M}=1$) level of the hierarchy, where $\tilde{k}_1$, $\tilde{\nu}_1$ and $\tilde{s}_1$ are not sufficient to uniquely determine the traveling wave; if we start with $M \geq 4$ and follow two paths down several levels to $\widetilde{M} \geq 2$ with the same parameters $\tilde{k}_j$, $\tilde{\nu}_j$ and $\tilde{s}_j$, the resulting solution is independent of the path.

In Figure 3, we plot the particle trajectories of several solutions on paths C, D and A in the bifurcation diagram of Figure 2. We parametrize each path linearly by a variable $\theta \in [0,1]$. For example, on path A,

$$s_1 = \frac{1-\alpha_0}{2}\theta, \qquad s_2 = s_3 = \frac{1-\alpha_0}{4}(1-\theta), \qquad 0 \leq \theta \leq 1. \tag{76}$$

Path C connects the one-hump stationary solution to the three-hump traveling wave. When $\theta = 4.0 \times 10^{-6}$, two particles have nucleated at the origin and execute small, nearly circular orbits around each other while the third particle travels around its original resting position. As θ increases, the orbits deform and coalesce into a single path, as shown in the middle two panels of this row. At the critical value $\theta = 0.002649485$, the particles collide at $t = T/6$, $t = 3T/6$ and $t = 5T/6$, so the solution of the ODE (8) ceases to exist for all time; nevertheless, $u(x,t)$ in (41) remains smooth and satisfies (2) for all t. As θ increases to 1, the common trajectory of the three particles becomes more and more circular until the traveling wave is reached, where it is exactly circular. The solutions on this path are

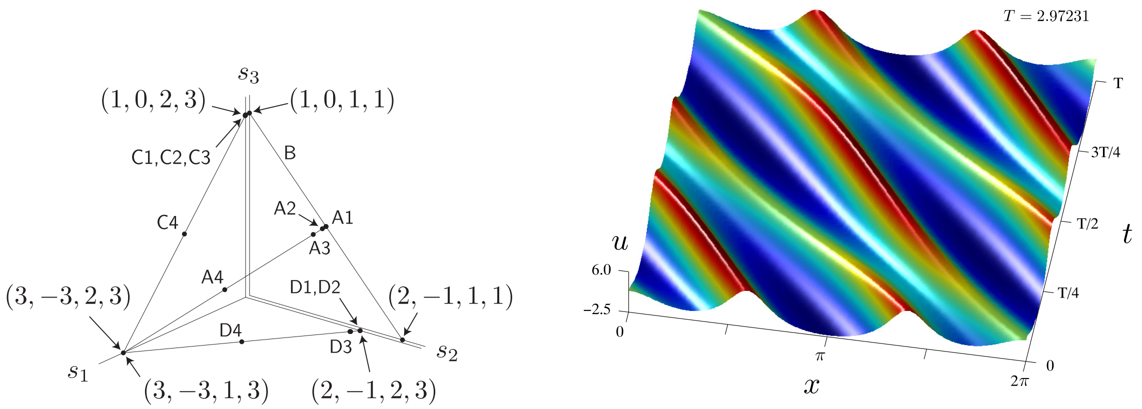

Figure 2: *Left:* degenerate bifurcation from $M = 1$ to $M = 2$ (paths C and D) and $M = 3$ (path A) with α_0 held fixed. *Right:* three dimensional plot of the solution labeled D4.

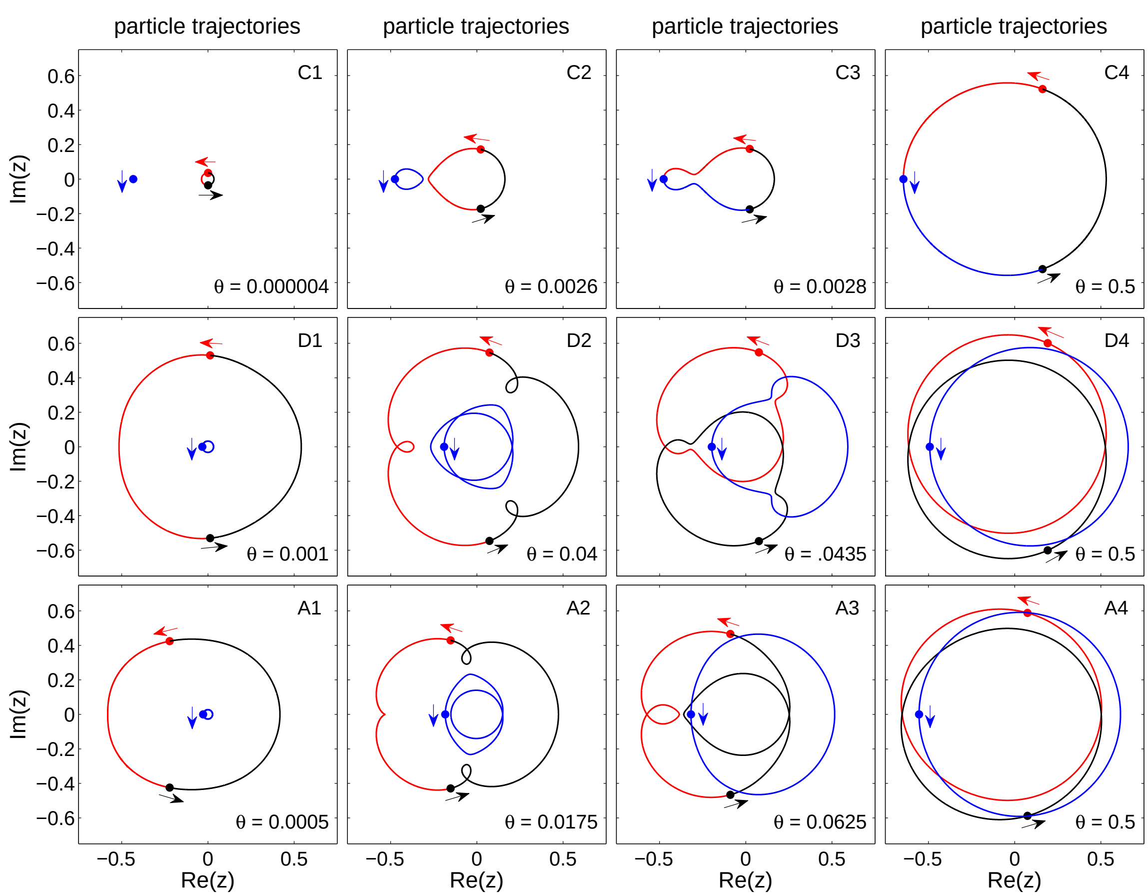

Figure 3: Particle trajectories along paths C,D,A in the bifurcation diagram of Figure 2.

reducible in the sense that their natural period is $1/3$ of the period T used here. However, in order to bifurcate to paths A and D, we have to use this solution rather than the reduced solution.

Path D connects the two-hump traveling wave with speed index $\nu = -1$ to the three-hump traveling wave with speed index $\nu = -3$. When we bifurcate from the two-hump traveling wave, a new particle nucleates at the origin and grows in amplitude until its trajectory joins up with the orbits of the outer particles. As θ increases further, the three orbits become nearly circular and eventually coalesce into a single circular orbit at the three-hump traveling wave. Note that the particles on path D follow different trajectories, which leads to a "braided" effect in the peaks and troughs of the solution $u(x,t)$ shown in Figure 2; unlike path C, these solutions are not reducible to a shorter period.

Path A connects an interior bifurcation on path B to this same three-hump traveling wave. When $\theta = 0.0005$, a particle has nucleated at the origin without destroying the periodicity of the orbit of the other two particles. This path involves two topological changes in the particle trajectories, as shown in the middle two panels of the bottom row of Figure 3. As $\theta \to 1$, this solution also approaches the three-hump traveling wave. This is interesting because, up to a phase shift in space and time, the linearized Benjamin-Ono equation about this traveling wave [3] has only two linearly independent, time-periodic solutions corresponding to the bifurcations $(3,-3,1,3)$ and $(3,-3,2,3)$; this degenerate bifurcation is not predicted by linear theory.

In a similar way, we can construct a bifurcation from a traveling wave to an arbitrary level of the hierarchy by taking

$$M \text{ arbitrary}, \qquad \vec{k} = (1,1,\dots,1), \qquad \nu = (-M, -M+1, \dots, -2, -1, 0). \tag{77}$$

We find that $m_j = 1$, $\tau_j = 2$ and $\gamma_j = 2$ for $2 \le j \le M$, so any subset of the indices $J = 2, \dots, M$ can be removed to obtain a solution at a lower level of the hierarchy. If all the indices are removed, we obtain a traveling wave with $2^{M-1} - 1$ bifurcations to higher levels of the hierarchy, but only $M-1$ of them (to the second level) are predicted by linear theory. The case $M = 4$ is depicted in the bifurcation diagram of Figure 4, where each tetrahedron contains paths with one of the s_j set to zero, and the outermost point of the outer three tetrahedra corresponds to one and the same traveling wave. Linear theory predicts the bifurcations $(4,-6,1,6)$, $(4,-6,2,8)$, $(4,-6,3,6)$ from this traveling wave to the $M = 2$ level of the hierarchy, but does not predict the three $M = 3$ families of solutions that connect this traveling wave to interior bifurcations on paths B, C and D, nor the $M = 4$ solution connecting this traveling wave to path A from the previous example. In Figure 5, we show six solutions on this $M = 4$ path labeled E in the bifurcation diagram.

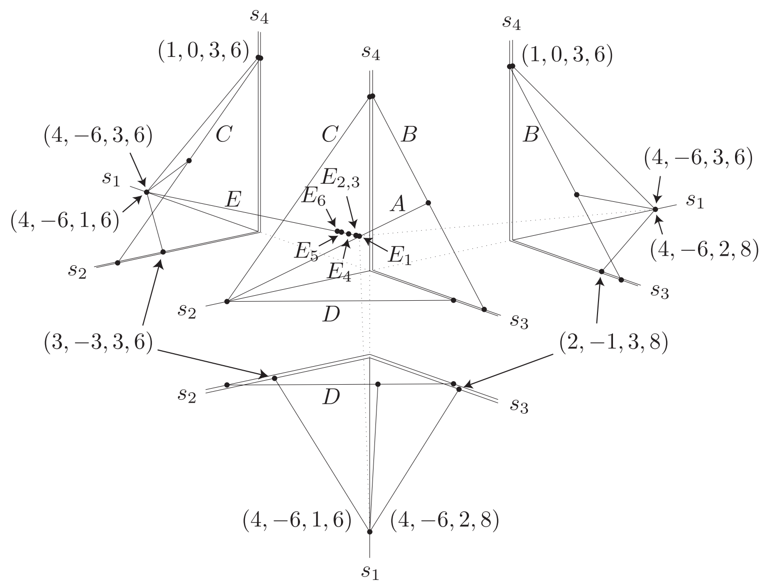

Figure 4: Degenerate bifurcation from $M = 1$ to $M = 2, 3, 4$ with α_0 held fixed.

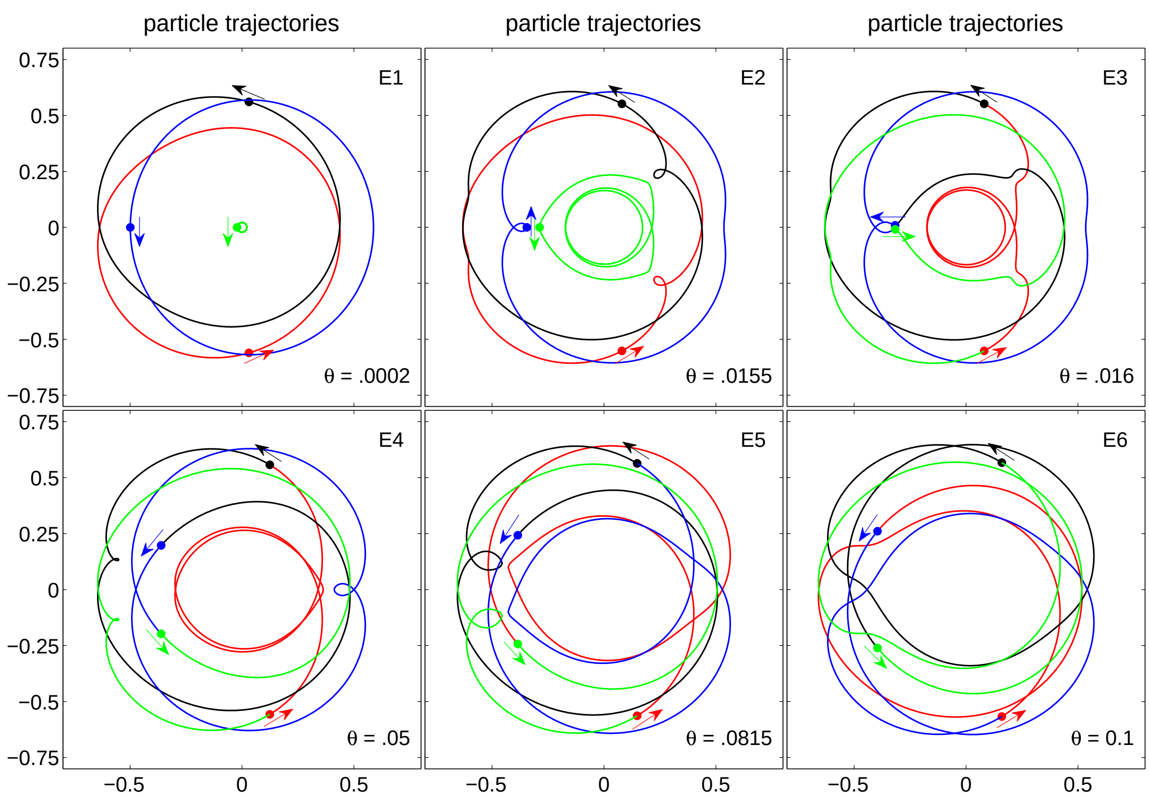

Figure 5: Particle trajectories along path E in the bifurcation diagram of Figure 4.

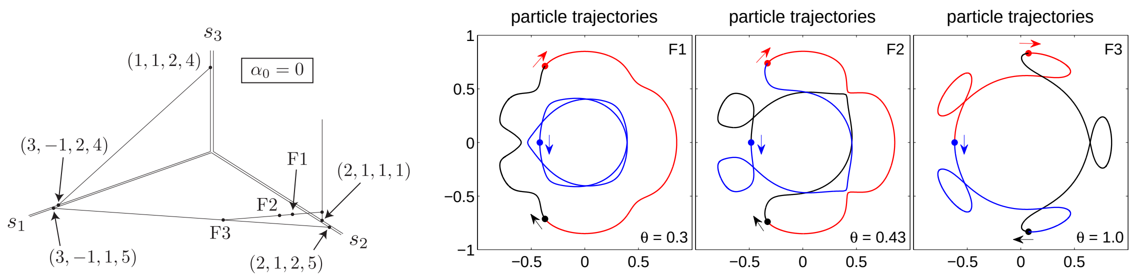

Figure 6: *Left:* Bifurcation diagram showing a path of time-periodic solutions at level $M = 3$ connecting two $M = 2$ solutions. *Right:* Three solutions on this path.

When $\theta = 0^+$, (where path E meets path A), a fourth particle nucleates at the origin without destroying periodicity of the other three. As θ increases to 1, the trajectories of the particles $\beta_j(t)$ undergo several topological changes (that determine which particles exchange positions over one period) until the trajectories coalesce into a single circular orbit at the degenerate traveling wave (with each particle moving counter-clockwise one and a half times per period).

Finally, in Figure 6, we show a path of solutions at level $M = 3$ that connects two $M = 2$ solutions by interior (non-degenerate) bifurcations. The parameters of this path are

$$M = 3, \qquad \vec{k} = (1,1,1), \qquad \vec{\nu} = (-2,0,1). \tag{78}$$

We parametrize this path by

$$s_1 = \frac{\theta}{2}, \qquad s_2 = \frac{3-\theta}{2}, \qquad s_3 = \frac{1-\theta}{4}, \qquad (0 \le \theta \le 1). \tag{79}$$

When $\theta = 0$, we obtain the $M = 2$ family of solutions that bifurcates from a two-hump, right traveling wave with indices $(2,1,1,1)$. For this family of solutions, the mean is related to the parameters $\tilde{s}_1 = s_2$ and $\tilde{s}_2 = s_3$ via $\alpha_0 = 3 - 2s_2$; hence, holding the mean fixed requires that s_2 remains constant. We will not reach the one-hump, right traveling wave at the bifurcation $(1,1,1,1)$ unless we increase α_0 to 3. As we increase θ in (79), the trajectory of the particle that nucleates at the origin at $\theta = 0^+$ grows and merges with the trajectories of the original two particles through three topological changes: one at F1, one not shown, and one at F3. Eventually, when $\theta = 1$, path F joins path G connecting $(2,1,2,5)$ to $(3,-1,1,5)$. Since the two-hump traveling wave moves to the right while the three-hump traveling wave moves to the left, the solution at F3 involves particles moving clockwise for part of their orbit and counter-clockwise at other times, leading to an interesting three-particle trajectory with 5-fold symmetry.

A Zeros of the polynomial $P(\cdot,\lambda)$

As mentioned in the proof of Theorem 2, there is a small gap in the paper [22] showing that the multiperiodic solutions (46) satisfy the Benjamin-Ono equation, for the bilinear formalism used to derive these solutions requires that the zeros of f (or f') in (47) and (48) lie in the upper (or lower) half of the complex plane. In this appendix, we prove the equivalent assertion (in the case that all the k_i are integers) that the zeros of the polynomial $P(\cdot,\lambda)$ in (43) lie inside the unit disk. The two key ideas of this proof, namely showing that P (or f) has a representation as a determinant, and that the matrix is non-singular for $|z|\geq 1$, are essentially due to Matsuno [16] and Dobrokhotov/Krichever [10], respectively.

THEOREM 3 *Suppose $M\geq 2$, $k_1,\dots,k_M\in\mathbb{N}$, $\nu_1,\dots,\nu_M\in\mathbb{R}$, $x_{10},\dots,x_{M0}\in\mathbb{R}$, and*

$$0<q_1<p_1<q_2<p_2<\cdots<q_M<p_M. \tag{80}$$

Let $\mathcal{M}=\{1,\dots,M\}$ and define the polynomial

$$P(z,\lambda)=\sum_{S\in\mathcal{P}(\mathcal{M})}C_S\lambda^{\nu_{S'}}z^{k_S},\qquad C_S=\left(\prod_{(i,j)\in S\times S'}a_{ij}\right)\left(\prod_{j\in S'}b_j\right), \tag{81}$$

where $S'=\mathcal{M}\setminus S$, $k_S=\sum_{i\in S}k_i$, $\nu_{S'}=\sum_{j\in S'}\nu_j$, and

$$a_{ij}=\sqrt{\frac{(p_j-q_i)(q_j-p_i)}{(q_j-q_i)(p_j-p_i)}},\qquad b_j=\sqrt{\frac{q_j}{p_j}}e^{-ik_jx_{j0}}. \tag{82}$$

Then all the zeros β_l of $P(\cdot,\lambda)$ lie inside the unit disk $\Delta\in\mathbb{C}$ provided $|\lambda|=1$.

PROOF: First we show that

$$P(z,\lambda)=\left[\prod_{j=1}^{M}(p_j-q_j)\right]\left[\prod_{j=1}^{M}b_j\lambda^{\nu_j}\right]\left[\prod_{i<j}^{(\mathcal{M})}a_{ij}^2\right]\det R(z,\lambda), \tag{83}$$

where the $M\times M$ matrix $R(z,\lambda)$ has entries

$$R_{ij}(z,\lambda)=r_i(z,\lambda)\delta_{ij}+\frac{1}{p_i-q_j},\qquad r_i(z,\lambda)=\frac{b_i^{-1}\lambda^{-\nu_i}z^{k_i}}{p_i-q_i}\prod_{j\neq i}^{M}a_{ij}^{-1}. \tag{84}$$

The symbol $\prod_{j\neq i}^{M}$ indicates a product over $j\in\mathcal{M}$ omitting $j=i$, while $\prod_{i<j}^{(\mathcal{M})}$ is a product over all pairs $(i,j)\in\mathcal{M}^2$ such that $i<j$. By expanding $\det R=\sum_\sigma \operatorname{sgn}(\sigma)R_{i,\sigma(i)}$ and collecting like products of the $r_i(z,\lambda)$, we find that

$$\det R=\sum_{S\in\mathcal{P}(\mathcal{M})}\left[\prod_{i\in S}r_i\right]\det R_{S'},\qquad \left(R_{S'}\right)_{ij}=\frac{1}{p_{S'_i}-q_{S'_j}}. \tag{85}$$

Here $\{S'_1,\dots,S'_n\}$ is an enumeration of S', i.e. $R_{S'}$ is the $n\times n$ (Cauchy) matrix obtained by removing the rows and columns with indices in S from the Cauchy matrix $\big\{(p_i-q_j)^{-1}\big\}_{i,j=1}^{M}$, and $\det R_\varnothing$ is taken to be 1. The determinant of a Cauchy matrix is well-known, giving

$$\det R_{S'} = \frac{\prod_{i<j}^{(S')}(p_i-p_j)(q_j-q_i)}{\prod_{i,j\in S'}(p_i-q_j)} = \Bigg[\prod_{j\in S'}\frac{1}{p_j-q_j}\Bigg]\prod_{i<j}^{(S')} a_{ij}^{-2}. \tag{86}$$

Thus, the right hand side of (83) is equal to

$$\sum_{S\in\mathcal{P}(\mathcal{M})}\Bigg[\prod_{j\in S'} b_j\lambda^{\nu_j}\Bigg]\Bigg[\prod_{i\in S} z^{k_i}\Bigg]\Bigg[\prod_{i\in S}\prod_{j\neq i}^{M} a_{ij}^{-1}\Bigg]\Bigg[\prod_{i<j}^{(\mathcal{S}')} a_{ij}^{-2}\Bigg]\Bigg[\prod_{i<j}^{(\mathcal{M})} a_{ij}^{2}\Bigg]. \tag{87}$$

If i and j are both in S or both in S', the terms a_{ij}^2 in the final product cancel with corresponding terms in one of the previous two products. If $i\in S$ and $j\in S'$, one of the factors of a_{ij} (or a_{ji} if $i>j$) in the final product cancels with a_{ij}^{-1} in the middle product, leaving behind $\prod_{(i,j)\in S\times S'} a_{ij}$, as required.

Next we show that $R(z,\lambda)$ is invertible for $|z|\geq 1$ and $|\lambda|=1$. Fix such a z and λ. Define $d_i = |b_i^{-1}\lambda^{-\nu_i}z^{k_i}|$ so that

$$|r_i|^2 = \frac{d_i^2}{(p_i-q_i)^2}\prod_{j\neq i}^{M}\frac{(q_j-q_i)(p_j-p_i)}{(p_j-q_i)(q_j-p_i)}, \qquad d_i > 1. \tag{88}$$

Suppose for the sake of contradiction that there is a non-zero vector $\gamma\in\mathbb{C}^M$ such that $R\gamma = 0$. This means that

$$\sum_{j=1}^{M}\left(r_i\delta_{ij}+\frac{1}{p_i-q_j}\right)\gamma_j = r_i\gamma_i+\psi(p_i) = 0, \quad (1\leq i\leq M), \qquad \psi(k) := \sum_{j=1}^{M}\frac{\gamma_j}{k-q_j}.$$

Then we define

$$\phi(k) = \left(\sum_i \frac{\gamma_i}{k-q_i}\right)\left(\sum_m \frac{\bar{\gamma}_m}{k-q_m}\right)\left(\prod_j \frac{k-q_j}{k-p_j}\right) \tag{89}$$

and observe that

$$\operatorname{res}_{k=p_i}\phi + \operatorname{res}_{k=q_i}\phi = \frac{|\gamma_i|^2}{p_i-q_i}\Bigg[\prod_{j\neq i}\frac{q_i-q_j}{q_i-p_j}\Bigg](d_i^2-1)\geq 0, \qquad (1\leq i\leq M) \tag{90}$$

where the inequality is strict if $\gamma_i\neq 0$ and we used $|\psi(p_i)|^2 = |r_i|^2|\gamma_i|^2$. Thus, the sum of all the residues of $\phi(k)$ is strictly positive, contradicting $\phi(k) = O(k^{-2})$ as $k\to\infty$. □

B Uniqueness of Periodic Traveling Waves

In this section we elaborate on the paper [5] showing that the only traveling wave solutions of the Benjamin-Ono equation are the solitary and periodic wave solutions found by Benjamin in [7]. The purpose of this section is to modify their argument to prove that we have found all 2π-periodic traveling solutions, and to simplify part of their analysis.

Consider any non-constant, 2π-periodic traveling solution of (2). After a transformation of the form (22), we may assume $u(x)$ is a stationary solution satisfying

$$uu_x = Hu_{xx}, \qquad (x \in \mathbb{R}/2\pi\mathbb{Z}). \tag{91}$$

After a translation, we may assume $u_x(0) = 0$. Integrating once, there is a constant c such that

$$\frac{1}{2}u^2 = Hu_x + \frac{c^2}{2}, \qquad (c > 0). \tag{92}$$

The integration constant must be positive since Hu_x is the derivative of a periodic function while the left hand side is positive. Now define the holomorphic function

$$f_1(z) = \frac{1}{\pi}\int_0^{2\pi} \frac{u(\theta)e^{i\theta}}{e^{i\theta} - z}\, d\theta, \qquad (z \in \Delta). \tag{93}$$

A direct calculation using Fourier series shows that

$$f_1(z^+) = \alpha_0 + u(\theta) + iHu(\theta), \qquad z = e^{i\theta}, \qquad \alpha_0 = \frac{1}{2\pi}\int_0^{2\pi} u(\theta)\, d\theta, \tag{94}$$

where $f_1(z^+)$ is the limit of $f_1(\zeta)$ as ζ approaches z from the inside. Now define

$$f_2(z) = f_1(e^{iz}) - \alpha_0, \qquad \operatorname{Im} z \geq 0. \tag{95}$$

Then $f_2(z)$ is analytic and bounded in the upper half-plane and satisfies

$$f_2(x) = u(x) + iHu(x), \qquad x \in \mathbb{R}. \tag{96}$$

Next we extend $u(x)$ to the upper half-plane via $u(x, y) = \operatorname{Re}\{f_2(x + iy)\}$ and define

$$U(x, y) = \frac{u(x/c, y/c) + c}{2c}, \tag{97}$$

where c was determined by $u(x)$ in (92). We then have

$$u_y(x, 0) = \operatorname{Re}\{if_2'(x)\} = -Hu_x(x) = \frac{c^2 - u(x)^2}{2}, \qquad U_y(x, 0) = U(x, 0) - U(x, 0)^2.$$

Amick and Toland [5] showed that any non-constant, bounded, harmonic function $U(x, y)$ defined in the upper half-plane and satisfying the nonlinear Neumann boundary condition $U_y = U - U^2$ on the real axis as well as $U_x(0, 0) = 0$, is given by

$$U(x, y) = \operatorname{Re}\{f(x + iy)\}, \qquad (y \geq 0), \tag{98}$$

where $f(z)$ is the (unique) solution of the complex ordinary differential equation

$$\frac{df}{dz}(z) = \frac{i}{2}\big[f(z)^2 - a^2\big], \qquad f(0) = U(0,0), \qquad a = \sqrt{2U(0,0) - U(0,0)^2} \tag{99}$$

over the upper half-plane. Moreover, such a solution $U(x,y)$ will satisfy $U(0,0) \in (0,1) \cup (1,2]$, and the case $U(0,0) = 2$ corresponds to the solitary wave solution $U(x,0) = 2/(1+x^2)$, which is ruled out by the assumption that u in (91) is periodic. Rather than treat the cases $U(0,0) \in (0,1)$ and $U(0,0) \in (1,2)$ separately as was done in [5], we choose the unique $\beta \in (-1,1)$ such that

$$U(0,0) = \frac{(1+\beta)^2}{1+\beta^2}, \qquad a = \frac{1-\beta^2}{1+\beta^2} \tag{100}$$

and check directly that

$$f(z) = a\left[1 + \frac{2\beta}{e^{-iaz} - \beta}\right] \tag{101}$$

satisfies (99). Since $U(x,y) = \mathrm{Re}\{f(x+iy)\}$ is x-periodic with (smallest) period $\frac{2\pi}{a}$ while $u(x,y) = [2cU(cx,cy) - c]$ is x-periodic with period 2π, it must be the case that $c = N/a$ for some positive integer N. But then

$$u(x) = \mathrm{Re}\{Nf_3(Nx)\}, \qquad f_3(z) = \frac{2}{a}f\left(\frac{z}{a}\right) - \frac{1}{a} = \frac{1-3\beta^2}{1-\beta^2} + \frac{4\beta e^{iz}}{1-\beta e^{iz}}, \tag{102}$$

i.e. $u(x)$ is one of the N-hump stationary solutions discussed in Section 3.

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