



## The decay constants $f_B$ and $f_{D^+}$ from three-flavor lattice QCD

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### Fermilab Lattice and MILC Collaborations

We present new preliminary results for the leptonic decay constants  $f_B$  and  $f_{D^+}$  determined in  $2+1$  flavor lattice QCD at lattice spacings  $a = 0.09, 0.12$  and  $0.15$  fm. Results are obtained using the MILC Collaboration gauge configuration ensembles, clover heavy quarks in the Fermilab interpretation and improved staggered light quarks. Decay constants, computed at partially quenched combinations of the valence and sea light quark masses, are used to determine the low-energy parameters of staggered chiral perturbation theory. The physical decay constants are found in an extrapolation using the parameterized chiral formula.

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## 1. Introduction

The  $D$  meson decay constants, when compared to precise experimental results, are a critical check of the lattice methods needed for  $f_B$ . In Ref. [1] we predicted  $f_{D^+} = 201 \pm 3 \pm 17$  MeV in good agreement with the CLEO-c measurement  $f_{D^+} = 223 \pm 17 \pm 3$  MeV revealed days later [2].

In this work we present new results for the  $D$  and  $B$  meson decay constants. Precise determinations of  $f_B$ ,  $f_{B_s}$  and the ratio  $f_{B_s}/f_B$  are needed to study the Standard Model picture of  $B$ - $\bar{B}$  and  $B_s$ - $\bar{B}_s$  mixing. A progress report for the mixing matrix element study is presented in Ref. [3].

## 2. Simulation details

We use the MILC Collaboration three-flavor asqtad ensembles [4]. Details are tabulated in Table 1. For these ensembles,  $m_l$  denotes the mass of the two degenerate lighter sea quarks. A single heavier sea quark has a mass  $m_h$  near the strange quark mass. Upsilon spectroscopy tells us the heavy quark potential scale  $r_1 = 0.318(7)$  fm [5]. The number of valence quark masses,  $\#m_q$ , used in this study is listed in the last column of the table.

The leptonic decay constant  $f_{H_q}$  for a meson  $H_q$  is defined by

$$\langle 0 | A_\mu | H_q(p) \rangle = i f_{H_q} p_\mu . \quad (2.1)$$

The combination  $\phi_{H_q} = f_{H_q} \sqrt{m_{H_q}}$  emerges from a combined fit to lattice 2-pt functions:

$$C_O(t) = \langle O_{H_q}^\dagger(t) O_{H_q}(0) \rangle \quad (2.2)$$

$$C_{A_4}(t) = \langle A_4(t) O_{H_q}(0) \rangle , \quad (2.3)$$

where  $O_{H_q}$  can be either a smeared or local operator.

The axial current renormalization is taken to be

$$Z_{A_4}^{Qq} = \rho_{A_4}^{Qq} \sqrt{Z_{V_4}^{QQ} Z_{V_4}^{qq}} . \quad (2.4)$$

$a$ [fm]	$am_h$	$am_l$	$\beta$	$r_1/a$	configs	$\# m_q$
0.09	0.031	0.0031	7.08	3.69	435	11
		0.0062	7.09	3.70	557	10
		0.0124	7.11	3.72	518	8
0.12	0.05	0.005	6.76	2.64	529	12
		0.007	6.76	2.63	833	12
		0.01	6.76	2.62	592	12
		0.02	6.79	2.65	460	12
		0.03	6.81	2.66	549	12
0.15	0.0484	0.0097	6.572	2.13	631	9
		0.0194	6.586	2.13	631	9
		0.029	6.600	2.13	440	9

**Table 1:** MILC three-flavor lattice parameters. The last column lists the number of valence light quarks used in this study.

Factors  $Z_{V_4}^{ff}$  are fixed nonperturbatively from scattering 3-pt functions and the known normalization of the vector current. Factors  $\rho_{A_4}^{Qq}$  are known to one-loop order and are close to unity [6].

### 3. Staggered Chiral Perturbation Theory (S $\chi$ PT)

With staggered quarks the (squared) taste-nonsinglet pseudoscalar meson masses are split:

$$M_{ab,\xi}^2 = (m_a + m_b)\mu + a^2\Delta_\xi, \quad (3.1)$$

where  $m_a, m_b$  are quark masses and the (sixteen) mesons are labeled by their taste representation  $\xi = P, A, T, V, I$  with  $\Delta_P = 0$ .

At next-to-leading order (NLO) in  $\chi$ PT the expression for the decay constants is

$$\phi_{H_q} = \Phi_H [1 + \Delta f_H(m_q, m_l, m_h) + p_H(m_q, m_l, m_h)] \quad (3.2)$$

where  $\Delta f_H$  denotes the ‘‘chiral logs’’ and  $p_H$  denotes terms analytic in the meson masses.

With staggered quarks

$$\Delta f_H = -\frac{1 + 3g_{H^*H\pi}^2}{2(4\pi f_\pi)^2} [\bar{h}_q + h_q^I + a^2 (\delta'_A h_q^A + \delta'_V h_q^V)]. \quad (3.3)$$

Taste-breaking effects arise at finite  $a$  from the meson mass splittings and the  $\delta'_A$  and  $\delta'_V$  hair-pin terms [7]. Finite  $a$  effects reduce the chiral logarithm curvature, however, the expected QCD chiral logarithm is recovered in the continuum limit.

The NLO analytic terms are

$$p_H = \frac{1}{2(4\pi f_\pi)^2} [p_1(m_l, m_h) + p_2(m_q)] \quad (3.4)$$

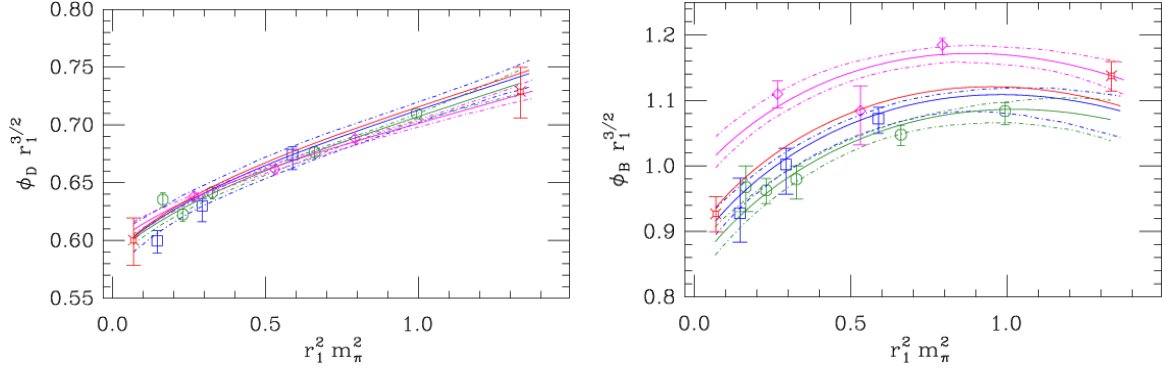
$$p_1 = f_1(\Lambda_\chi) \left[ \frac{11}{9}\mu(2m_l + m_h) + a^2 \left( \frac{3}{2}\bar{\Delta} + \frac{1}{3}\Delta_I \right) \right] \quad (3.5)$$

$$p_2 = f_2(\Lambda_\chi) \left[ \frac{5}{3}\mu m_q + a^2 \left( \frac{3}{2}\bar{\Delta} - \frac{2}{3}\Delta_I \right) \right], \quad (3.6)$$

where  $\bar{\Delta}$  is the weighted average of taste splittings. The  $O(a^2)$  terms ensure that dependence upon the chiral logarithm scale,  $\Lambda_\chi$ , in  $f_1$  and  $f_2$  cancels that of  $\Delta f_q$ .

Equation (3.2) with the addition of four NNLO analytic terms parameterizes our chiral extrapolations. We fit  $\phi_{H_q}$  to determine the parameters. Constraints (value and width) for  $\mu$ ,  $\Delta_\xi$ ,  $f_\pi$ ,  $\delta'_A$  and  $\delta'_V$  come from  $\chi$ PT for lattice pions and kaons [8]. The coupling  $g_{D^*D\pi}^2 = 0.35 \pm 0.14$  is likewise constrained by the CLEO measurement [9]. From heavy quark symmetry we expect  $g_{B^*B\pi}^2 \approx g_{D^*D\pi}^2$ . The remaining parameters  $\Phi_H$ ,  $f_1$  and  $f_2$  and the NNLO analytic parameters are determined in the fit.

In order to extrapolate to the physical results we set  $\Delta_\xi = \delta'_{A,V} = 0$ ,  $m_h \rightarrow m_s$  and  $m_l \rightarrow (m_u + m_d)/2$ . Then  $\phi_{H_d}$  ( $\phi_{H_s}$ ) is found in the limit  $m_q \rightarrow m_d$  ( $m_s$ ).



**Figure 1:** Chiral fits for the  $D$  (left) and  $B$  (right) mesons. Each fit is viewed along the direction in which  $m_q = m_l$ . Each fit is shown as a set of solid curves with the 68% confidence limits denoted by broken curves. Only statistical errors are shown. The  $a = 0.09$  fm curve and data points are shown in blue, 0.12 fm in green and 0.15 fm in magenta. The  $a^2 \rightarrow 0$  extrapolation curve and the  $\phi_{H_{d,s}}$  points at physical values of  $m_u$ ,  $m_d$  and  $m_s$  are shown in red. The statistical errors on the  $D$  and  $B$  physical points are of comparable size.

#### 4. The Fit and Extrapolation for $D$ and $B$

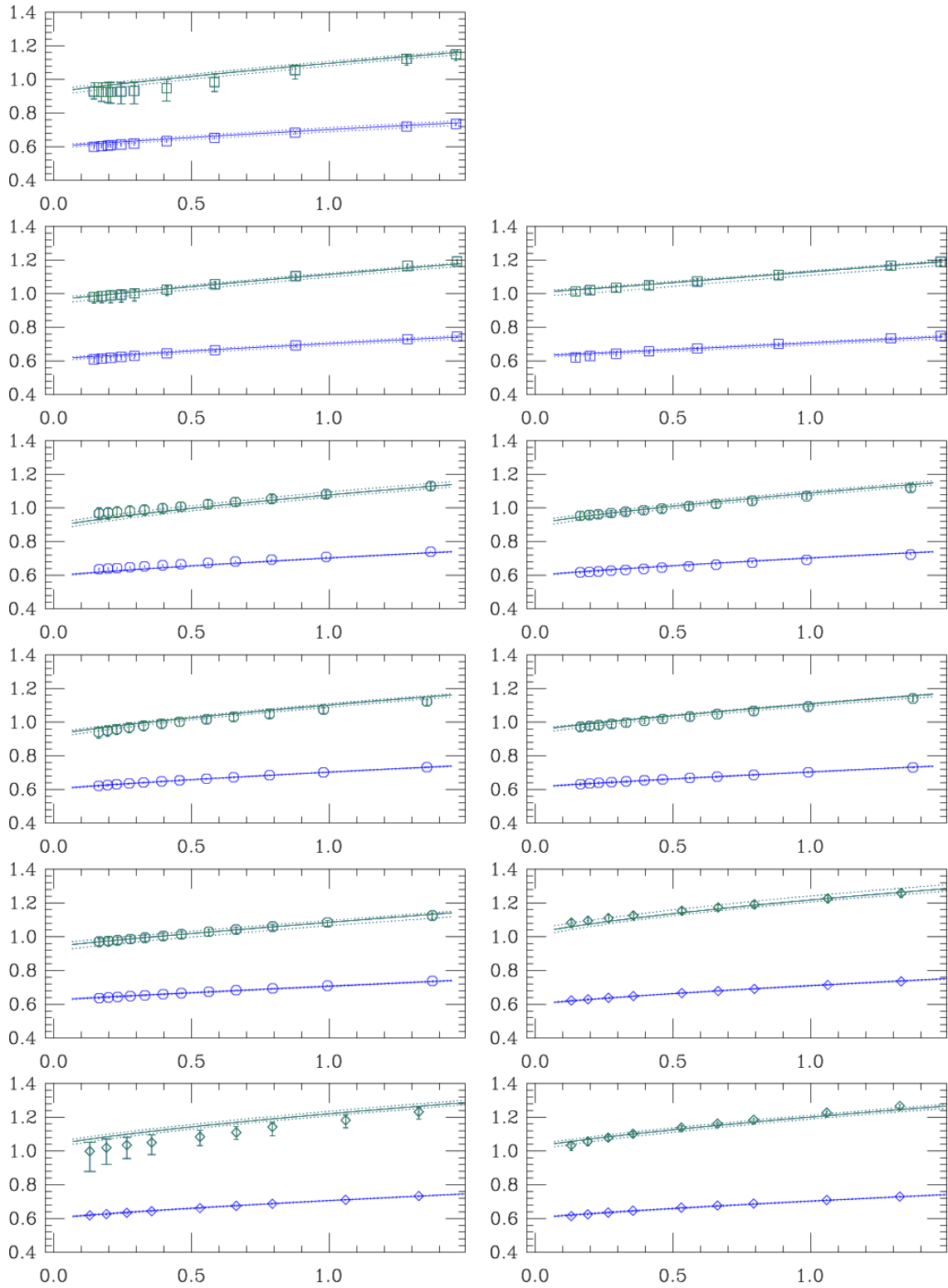
We determine both  $\phi_{D^+}$  and  $\phi_{D_s}$  from a single fit of  $\phi_{D_q}$  simulation results using the expression in Eqn. (3.2), adding the four NNLO analytic terms and allowing for an explicit  $O(a^2)$  term. We combine simulation results from 11 gauge ensembles at lattice spacings of  $a = 0.09$ , 0.12 and 0.15 fm in the fit. A total of 116 points are included in the fit. A bootstrap procedure propagates errors and correlations among the simulated results through to the statistical errors on our results. An analogous fit procedure for the  $B$  meson simulation results yields  $\phi_{B_d}$  and  $\phi_{B_s}$ .

The  $D$  and  $B$  meson fits combining the three lattice spacings are shown in Figs. 1 and 2. Figure 1 shows each fit and the data points along the  $m_q = m_l$  direction while Fig. 2 shows the valence mass dependence of the fit at fixed values of the sea quark mass. All of the fit points are visible in Fig. 2 while only the subset of points with  $m_q = m_l$  is visible in Fig. 1.

Figure 1 shows the  $D$  system on the left and the  $B$  system on the right. In each plot, the solid blue, green and magenta curves are the fit to the lattice data for lattice spacings 0.09, 0.12 and 0.15 respectively. These curves include the  $a^2$  effects described by the chiral fit function. The 68% confidence limits for each curve are indicated by dotted contours of the corresponding color.

In Fig. 2 we show the valence mass dependence the  $D$  and  $B$  systems. In each plot the  $D$  system (blue points and curves) is shown together with  $B$  system (green points and curves). Each plot in the figure corresponds to a single combination of lattice spacing and  $m_l$  from Table 1. Together, the points represent all of the simulation results for  $\phi_{H_q}$  used in this study. The fit curves include the  $a^2$  effects described by the chiral fit function. Each curve is shown with its 68% confidence contours. The expression in Eqn. (3.2) predicts a divergent logarithmic rise as  $m_q \rightarrow 0$  with  $m_l$  fixed. Our fits detect these logarithms even though taste breaking effects obscure them, so they are not immediately obvious in the plots.

The extrapolation,  $a \rightarrow 0$ ,  $m_h \rightarrow m_s$  and  $m_l = m_q$  is shown in Fig. 1 as a solid red curve for each of the  $D$  and  $B$  systems. Our result for  $\phi_{D^+}$  ( $\phi_{B_d}$ ) is found from the extrapolation by setting  $m_l = \hat{m} = (m_u + m_d)/2$  and  $m_q = m_d$ . Likewise,  $\phi_{D_d}$  ( $\phi_{B_s}$ ) is found by setting  $m_q = m_s$ . The physical



**Figure 2:** The  $m_q$  dependence of the single  $D$  fit (in blue) and the single  $B$  fit (in green) at fixed  $m_l$ . The figures ordered from left-to-right and top-to-bottom, have  $am_l$  equal to 0.0031, 0.0062, 0.0124 ( $a = 0.09$  fm), 0.005, 0.007, 0.01, 0.02, 0.03 (0.12 fm), 0.0097, 0.0194 and 0.029 (0.15 fm) respectively. On the y-axis is  $r_1^{3/2} \phi_{Hq}$  and on the x-axis is  $r_1^2 m_\pi^2$ . All 116 points in each fit are shown. The  $\chi^2 = 98.6$  for the  $D$  fit and 48.5 for the  $B$  fit.

quantity	value	
$\phi_{D_s}$	0.356(11)	GeV <sup>3/2</sup>
$\phi_{D_d}$	0.293(11)	GeV <sup>3/2</sup>
$R_{D_{d/s}}$	0.824(8)	
$\phi_{B_s}$	0.556(12)	GeV <sup>3/2</sup>
$\phi_{B_d}$	0.453(13)	GeV <sup>3/2</sup>
$R_{B_{d/s}}$	0.815(15)	

**Table 2:** The main results of this preliminary study.

results are indicated with the red burst symbols. Since these points are projected into the  $m_q = m_l$  plane of each figure, the central values do not lie on the red curve. The statistical errors for the physical  $\phi_{H_q}$  values are shown in each figure. We find statistical errors comparable in magnitude for the physical  $\phi_{D_q}$  and  $\phi_{B_q}$  results.

## 5. Results and Outlook

Our preliminary results for the physical  $\phi_{H_q}$  values and ratios with their statistical errors are in Table 2. We tabulate the major sources of uncertainty in Table 3. We omit listing uncertainties arising from terms of order  $1/m_H$  in the chiral extrapolations since adding such terms changes the final results by less than the statistical errors. Such effects are still under investigation. Uncertainties from the input parameters  $r_1$  and the light quark masses are found by propagating the uncertainties found in the MILC  $f_\pi$  and  $f_K$  determinations [5]. We estimate a 3.8% uncertainty in the bare charm mass and a 6.8% uncertainty in the bare bottom mass from variations in tuning procedures for the 0.09 fm lattice. Using simulation results for two heavy-quark masses near both charm and bottom, we estimate the uncertainties in  $\phi_{H_q}$ . The uncertainties in  $Z_V^{ff}$  are statistical. Errors from unknown higher orders in  $\rho_{A_4}$  are estimated by considering higher orders effects to be as large as the 1-loop terms. Heavy quark discretization effects are estimated by power counting arguments. The dominant uncertainty in  $\phi_{H_q}$  comes from effects of order  $\alpha_s \Lambda a \times h(am)$  and  $a^2 \Lambda^2$ , where  $h(am)$  is some mild function of the heavy quark mass. The uncertainties in the ratios are smaller by a factor of  $m_s/\Lambda$ . Light quark discretization effects are estimated by varying the extrapolation procedure. Finite volume effects are estimated by comparing theories at finite volume to the continuum.

From Table 2 and the experimental  $D^+$ ,  $D_s$ ,  $B^0$  and  $B_s$  masses we compute the decay constants:

$$f_{D_s} = 254 \pm 8 \pm 11 \text{ MeV} \quad (5.1)$$

$$f_{D^+} = 215 \pm 8 \pm 11 \text{ MeV} \quad (5.2)$$

$$f_{B_s} = 240 \pm 5 \pm 11 \text{ MeV} \quad (5.3)$$

$$f_{B_d} = 197 \pm 6 \pm 12 \text{ MeV} \quad (5.4)$$

where each of the first errors is statistical. The second error is the systematic error combined in quadrature from Table 3.

We also consider ratios of  $B$  to  $D$  decay constants where statistical and systematic errors are expected to be reduced due to cancellations. Statistical errors in the ratios are from a bootstrap

source	$\phi_{D_s}$	$\phi_{D_d}$	$R_{d/s}$	$\phi_{B_s}$	$\phi_{B_d}$	$R_{d/s}$
statistics	3.1	3.8	1.0	2.1	3.1	1.8
inputs $r_1, m_s, m_d$ and $m_u$	1.4	2.0	0.5	3.1	3.8	0.6
input $m_c$ or $m_b$	2.7	2.7	<0.1	1.1	1.1	<0.1
$Z_V^{QQ}$ and $Z_V^{qq}$	1.4	1.4	0	1.4	1.4	0
higher-order $\rho_{A_4}$	0.3	0.3	<0.2	1.3	1.1	<0.2
heavy quark discretization	2.7	2.7	0.3	1.9	1.9	0.2
light quark discretization	1.0	2.7	1.8	2.0	3.8	1.8
finite volume	0.2	0.6	0.6	0.2	0.6	0.6
total systematic	4.4	5.3	2.0	4.7	6.1	2.0

**Table 3:** The error budget for the decay constants and their ratios. Uncertainties are quoted as a percentage. The total combines systematic errors in quadrature.

procedure in order to preserve statistical correlations.

$$f_{D^+}/f_{D_s} = 0.845 \pm 0.008 \pm 0.017 \quad (5.5)$$

$$f_{B_d}/f_{B_s} = 0.821 \pm 0.015 \pm 0.017 \quad (5.6)$$

$$f_{B_d}/f_{D^+} = 0.919 \pm 0.051 \pm 0.056 \quad (5.7)$$

$$f_{B_s}/f_{D_s} = 0.945 \pm 0.043 \pm 0.043 \quad (5.8)$$

The overall systematic errors for the first two ratios come from Table 3. Systematic errors for the last two ratios also come from combining errors in quadrature. These errors may be overestimates since we have not studied possible correlations at present.

We will extend this study to include a lattice spacing of  $a = 0.06$  fm. We will improve statistics at  $a = 0.09$  and  $0.12$  fm and add another ensemble (sea quark mass combination) at  $0.09$  fm. A finer lattice spacing, more sea quark combinations and better statistics will help control light- and heavy-quark discretization effects and improve statistical errors. The new gauge configurations will be used by MILC to refine  $r_1$  and the light quark masses inputs used in this study.

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