Parameterized Beyond-Einstein Growth

Eric V. Linder and Robert N. Cahn

Lawrence Berkeley National Laboratory, University of California, Berkeley, CA 94720

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A single parameter, the gravitational growth index γ, succeeds in characterizing the growth of density perturbations in the linear regime separately from the effects of the cosmic expansion. The parameter is restricted to a very narrow range for models of dark energy obeying the laws of general relativity but can take on distinctly different values in models of beyond-Einstein gravity. Motivated by the parameterized post-Newtonian (PPN) formalism for testing gravity, we analytically derive and extend the gravitational growth index, or Minimal Modified Gravity, approach to parameterizing beyond-Einstein cosmology. The analytic formalism demonstrates how to apply the growth index parameter to early dark energy, time-varying gravity, DGP braneworld gravity, and some scalar-tensor gravity.

I. INTRODUCTION

The acceleration of the cosmic expansion points to new physics beyond the standard models of particle physics or gravitation. But the nature of this physics is not clear. While current data constraints are consistent with Einstein's cosmological constant Λ, the uncertainties are still substantial. Even the simplest deviation from this picture, an assumed constant effective equation of state w, is determined to no better than 20% (at 95% confidence level) when combining the full set of current supernova distance, cosmic microwave background, and baryon acoustic oscillation measurements, *with* systematic uncertainties included [1]. Given that virtually no physical explanation beyond Λ predicts a simple, purely constant equation of state, it is clear that we cannot claim to have zeroed in on Einsteinian physics as the solution.

As we can look beyond Einstein's cosmological constant, so too we can look beyond the framework of Einstein gravity. To address the question of the nature of the new physics requires measuring both the expansion history of the universe and the history of structure growth. The growth history can provide independent information on the accelerating physics in the case when gravity deviates from general relativity (e.g. [2, 3, 4, 5]). As we describe in detail below, the structure growth history is conveniently described by a parameter γ, first introduced by [6] and expanded upon by [7], within Einstein gravity, to supplement the conventional cosmological parameters of the matter density and the equation of state of the dark energy.

The goal of this article is to motivate and explore further the idea of a model independent parametrization of the deviations of growth from Einstein gravity. Just as the model independent parametrization of the expansion history equation of state $w(z)$ proves valuable even without a particular Lagrangian in mind, we use γ without the intent to adopt a specific gravitational action, though we do show results for some specific (toy) models. We find that γ is confined to a narrow range of values near 6/11, independent of the nature of the dark energy if general relativity holds. In contrast, beyond-Einstein models of gravity can lead to rather different values for γ. This difference then establishes a standard for measurements of the growth of structure required to exclude alternate models of gravity.

Linder [8] proposed that this independent information, apart from the expansion physics, be parameterized by a single gravitational growth index γ, defined through

$$g(a) = e^{\int_0^a (da'/a') \, [\Omega_m(a')^\gamma - 1]}, \qquad (1)$$

where $\Omega_m(a)$ is the matter density fraction of the total energy density as a function of scale factor, $g = \delta/a$ and $\delta = \delta\rho_m/\rho_m$ is the linear density perturbation in the matter. (See [7] for an early approach to the growth index within general relativity and tied to the expansion history.) In [9], this was combined with the expansion history into a full cosmological fitting framework called Minimal Modified Gravity (MMG), using that the expansion history of the universe, $a(t)$, can be phrased in terms of an effective equation of state ratio $w(a) = w_0 + w_a(1-a)$ shown to be extremely successful for describing a wide variety of physics, including gravitational modifications [10, 11]. Comparison of the parameters $\{w_0, w_a, \gamma\}$ fitted to data (see examples in [9]) could establish statistically significant evidence for beyond-Einstein physics and provide information on its origin.

In §II we reexamine and derive analytically certain important properties of the parameter γ, which were originally obtained numerically. The physical motivation for the growth index approximation to the growth history is strengthened in §III, and evaluated for time varying and early dark energy. We derive a relationship between the growth index and the PPN formalism for gravity theories in §IV, and show the success of the growth index in characterizing DGP braneworld gravity and some scalar-tensor gravity theories. The analytic derivations serve an important role as a foundation for understanding both extensions and breakdowns of the growth index. Of course exact calculations of growth within specific gravitational models, not treated here, play an important role as well.

II. PARAMETERIZING GROWTH

Considering linear density perturbations in the matter, $\delta = \delta\rho_m/\rho_m$, the equation for their growth within general relativity is given by

$$\frac{d^2\delta}{dt^2} + 2H(a)\frac{d\delta}{dt} - 4\pi\rho_m\delta = 0, \qquad (2)$$

where $H = \dot{a}/a$ is the Hubble parameter. This can be rewritten as

$$\frac{dG}{d\ln a} + \left(4 + \frac{1}{2}\frac{d\ln H^2}{d\ln a}\right)G + G^2$$
$$+ 3 + \frac{1}{2}\frac{d\ln H^2}{d\ln a} - \frac{3}{2}\Omega_m(a) = 0, \qquad (3)$$

where $G = d\ln(\delta/a)/d\ln a$. Note that by normalizing δ by a we remove the pure matter universe growth behavior, which would give $G = 0$. The dimensionless matter density is $\Omega_m(a) = \Omega_m a^{-3}/[H(a)/H_0]^2$, where Ω_m is its present value.

The growth history can be solved formally by quadrature, giving the solution

$$G(a) = -1 + [a^4 H(a)]^{-1}\int_0^a \frac{da'}{a'}\,a'^4 H(a')$$
$$\times \left[1 + \frac{3}{2}\Omega_m(a') - G^2(a')\right]. \qquad (4)$$

This can also be written in terms of the dark energy density $\Omega_w(a) = 1 - \Omega_m(a)$, for a flat universe, as

$$G(a) = -1 + [a^4 H(a)]^{-1}\int_0^a \frac{da'}{a'}\,a'^4 H(a')$$
$$\times \left[\frac{5}{2} - \frac{3}{2}\Omega_w(a') - G^2(a')\right]. \qquad (5)$$

For growth during the matter-dominated era, G will be small, and this holds reasonably well even as dark energy comes to dominate, since Ω_m is still non-negligible today: for the concordance cosmology, today G is of order $-1/2$. So let us neglect the G^2 term in Eqs. (4)-(5) – while this is motivated by the asymptotic growth behavior during matter domination, it should be a reasonable approximation throughout the growth history. In practice we find that $g(a) = \delta/a$ agrees with the exact solution to $\sim 0.2\%$. This now gives an explicit solution for the growth.

We can use

$$H^2/H_0^2 = \Omega_m a^{-3}[1 + \Omega_w(a)/\Omega_m(a)] \qquad (6)$$

to write at the same level of approximation (that the universe is not too far from matter-dominated),

$$G(a) = -\frac{1}{2}\Omega_w(a) - \frac{1}{4}a^{-5/2}\int_0^a \frac{da'}{a'}\,a'^{5/2}\Omega_w(a'). \qquad (7)$$

For any particular model of $H(a)$, or $\Omega_m(a)$ or $\Omega_w(a)$, we can then evaluate the growth history.

Connecting this new analytic approach to the growth index defined in Eq. (1), we find immediately

$$G(a) = \Omega_m(a)^\gamma - 1. \qquad (8)$$

In the same approximation used to derive the quadratures for G, we can write

$$\gamma \approx -G/\Omega_w(a), \qquad (9)$$

and then using Eq. (7) we have

$$\gamma \approx \frac{1}{2} + \frac{1}{4}a^{-5/2}\Omega_w(a)^{-1}\int_0^a \frac{da'}{a'}\,a'^{5/2}\Omega_w(a')$$
$$= \frac{1}{2} + \frac{1}{4}\int_0^1 \frac{du}{u}\,u^{5/2}\Omega_w(au)/\Omega_w(a). \qquad (10)$$

To first order in deviations from matter domination, e.g. at early times, $\Omega_w(a) \sim a^{-3w_\infty}$, where $w_\infty = w(a \ll 1)$, and we can evaluate Eq. (10) to obtain

$$\gamma_\infty = \frac{3(1 - w_\infty)}{5 - 6w_\infty}. \qquad (11)$$

Equivalently, we can substitute Eq. (9) into Eq. (3) and find

$$\gamma_\infty = \frac{3 + \bar{G}}{5 + 2\bar{G}}, \qquad (12)$$

where $\bar{G} \equiv d\ln G/d\ln a = -3w_\infty$ to this level of approximation.

These asymptotic values accord exceedingly well with the fitting formula of [8]. We can write

$$\gamma_\infty = \frac{6 - 3(1 + w_\infty)}{11 - 6(1 + w_\infty)} \approx \frac{6}{11} + \frac{3}{121}(1 + w_\infty). \qquad (13)$$

We can compare this to the fitting formula from [8] that gives

$$\gamma = 0.55 + 0.02\,[1 + w(z=1)], \qquad w < -1 \quad (14)$$
$$= 0.55 + 0.05\,[1 + w(z=1)], \qquad w > -1 \quad (15)$$

over the whole range $0 < a < 1$. For the cosmological constant case, $w = -1$, the asymptotic value is $\gamma_\infty = 6/11 = 0.545$ compared to the numerically obtained fit $\gamma = 0.55$ for the whole growth history.

The first order correction to the cosmological constant case for a different value of constant w is quite small for the asymptotic formula (13), with a coefficient of 0.025 times $1 + w$. This agrees with the fitting correction from Eqs. (14)-(15), which have a coefficient of 0.02 (0.05) times $1 + w$ for the case when $1 + w < (>) 0$. The asymptotic value is closer to the fit for the $w < -1$ case because then matter domination lasts longer. The asymptotic form is also in agreement with the pioneering work of [7], who expanded the growth equation about $\Omega_m(a) = 1$ for constant w. Their asymptotic term agrees with Eq. (11) and they find the next order is given by

$(15/1331)[1 - \Omega_m(a)]$ for the $w = -1$ case; even when $1 - \Omega_m(a)$ is much larger than the $\sim 10^{-9}$ for ΛCDM at CMB last scattering, this correction is negligible.

The new elements in Eq. (10) include the integral nature of the relation, allowing for treatment of the whole growth history to some redshift, rather than an instantaneous measure of growth (thus allowing a prediction of G_{today} unlike previous work), the ability to treat time varying dark energy, and the clear identification of the key assumptions such as early matter domination that allow the single parameter to succeed.

We emphasize that the numerical fitting form of [8] covers the entire growth history, not just the asymptotic high redshift behavior, and does not assume $G \ll 1$ but rather fits to the exact numerical solution. Nevertheless, it is instructive to pursue the analytic arguments further to understand the motivation for phrasing the growth history in terms of a gravitational growth index, and why a single parameter proves so successful, even in the case of time varying equations of state.

III. GROWTH AND DARK ENERGY

One question to ask is why the fitting form (1) is physically appropriate. In the asymptotic limit, G will be linearly proportional to $\Omega_w(a) = 1 - \Omega_m(a)$ since Eq. (3) is linear (for small G; cf. Eq. 7), but one can imagine other forms besides $\Omega_m(a)^\gamma - 1$ that have this property. We consider three possibilities for fitting forms of G: the standard one of $\Omega_m(a)^\gamma - 1$, one directly proportional to $\Omega_w(a)$, and $\ln \Omega_m(a)^\gamma$. Each of these has the appropriate limit that G vanishes linearly in $1 - \Omega_m(a)$ as $a \to 0$. Defining γ for the entire growth history in terms of these forms gives the options

$$\gamma \equiv \frac{\ln(1 + G)}{\ln \Omega_m(a)}, \quad -\frac{G}{\Omega_w(a)}, \quad \frac{G}{\ln \Omega_m(a)}. \quad (16)$$

As a guide to defining γ as a useful parameter (e.g. nearly constant) over the entire growth history from $a = 0$ to the present, we can examine its late time behavior, when $a \gg 1$. For dark energy domination in the future, the matter density $\Omega_m(a) \to 0$ and matter density perturbations δ stop growing, so $G \to -1$, in agreement with Eq. (4). The second definition of γ in Eq. (16) leads to $\gamma \to 1$ in the future; the third definition has $\gamma \to 0$. Only the first, original definition of the growth index preserves its stability over the entire growth history from asymptotic past to asymptotic future. Indeed, γ varies from its present value by less than 2% back to $a \ll 1$ and 6% forward to $a = 5$, for the cosmological constant case. Since γ enters into an integral relation, then even this mild variation is further smoothed over, giving a constant γ as an excellent approximation. [8] found it to reproduce the exact growth for a wide variety of models to better than 0.2%. Overall, the definition of a gravitational growth index through Eq. (1) is therefore physically well motivated.

We can carry this further, showing that a single growth parameter suffices rather than needing a function (of redshift). Considering dynamical dark energy, rather than a constant w model, expand

$$\Omega_w(a) \sim e^{3 \int_a^1 (da'/a') w(a')} \sim a^{-3w_\infty} (1 + Ba^x), \quad (17)$$

in Eq. (10), where we approximate the time variation of the equation of state at high redshift as a power law correction to the asymptotic w_∞ value. The solution for the growth index is

$$\gamma \approx \frac{3(1 - w_\infty)}{5 - 6w_\infty} - \frac{Bx\, a^x}{(5 - 6w_\infty)(5 - 6w_\infty + 2x)}. \quad (18)$$

For the standard equation of state parameterization $w(a) = w_0 + w_a(1 - a) = w_\infty - w_a a$, we have $B = 3w_a$, $x = 1$. The correction to γ is exceedingly small, at early times proportional to a, with furthermore a small coefficient $-(3/143)w_a$, generally $\lesssim 0.01$. So even in the dynamical dark energy case, γ is found to be nearly constant. (The growth index fit was examined numerically in [9].)

Suppose we consider early dark energy, where its energy density is not completely negligible at high redshifts, but possibly contributing up to a couple percent of the total energy density at CMB last scattering [12, 13, 14]. We might expect this to upset the gravitational growth index since early dark energy directly affects the matter domination used to derive the analytic asymptotic behavior. In order for dark energy to contribute non-negligibly to the early energy density, its equation of state must approach $w = 0$. Substituting $w_\infty = 0$ in Eq. (11), we see $\gamma_\infty = 3/5$. However, we can actually obtain this solution without approximating $G \ll 1$; the exact solution for growth in a universe with two components, each evolving (at early times) with density proportional to a^{-3}, as ordinary matter, but with a fraction Ω_e not clustering, is [14, 15, 16]

$$g = \delta/a \sim a^{(-5 + \sqrt{25 - 24\Omega_e})/4} \approx a^{-(3/5)\Omega_e}, \quad (19)$$

or $\gamma = 3/5$. (To next order, $\gamma \approx 3/5 + (3/125)\Omega_e$, in agreement with [7].)

The redshift distortion factor β [17] used in galaxy redshift surveys can be written as $\beta \sim 1 + G = \Omega_m^\gamma$, and one often sees $\Omega_m^{0.6}$ used. For the concordance model we have seen that $\gamma = 0.55$ is a much better fit. Using $\Omega_m^{0.6}$ introduces a needless 6% error in β, or a systematic bias of 0.03 (11%) in the value of Ω_m derived, so more accurate results will be achieved if $\Omega_m^{0.55}$ is used in place of $\Omega_m^{0.6}$.

IV. GROWTH AND GRAVITY

To this point, the growth behavior has been completely specified by the expansion history. That is, in Eq. (3) the terms only involve $H(a)$ or $\Omega_m(a)$, and so the gravitational growth index γ can be defined in terms of the

effective equation of state $w(a)$. Furthermore, we have seen that γ has quite a small range over a reasonably large variety of dark energy models, i.e. for $w = -1$ to $-1/3$ (including effective curvature energy), γ only varies over 0.55-0.57.

Now we consider what happens when we allow alterations to the gravity theory. There is no unique prescription for how modified gravity theories affect the growth equation, even in the linear regime, though see [8] for an attempt to provide a somewhat general, if formal, treatment. Effects on the growth equation include the introduction of scale dependence, anisotropic stress, and variation of the gravitational coupling (i.e. Newton's constant). We briefly discuss the first two of these in §§IV D and V, and concentrate here on the last one. Again we emphasize that our goal is a model independent approach, rather than adopting a specific theory.

A. Varying gravity

The last term of Eq. (3) contains a hidden dependence on the gravitational coupling, multiplying the source term $\Omega_m(a)$. Here we rewrite the growth equation, explicitly showing the effect:

$$\frac{dG}{d\ln a} + \left(4 + \frac{1}{2}\frac{d\ln H^2}{d\ln a}\right) G + G^2 + 3 + \frac{1}{2}\frac{d\ln H^2}{d\ln a}$$
$$- \frac{3}{2}[1 + (Q(a) - 1)]\,\Omega_m(a) = 0, \quad (20)$$

where $Q - 1$ gives the fractional deviation of the coupling from the general relativistic case (i.e. $Q = 1$ means the coupling is given by Newton's constant).

We can now repeat the analysis of §II, finding

$$G(a) = -1 + (a^4 H)^{-1} \int_0^a \frac{da'}{a'}\,a'^4 H\left[1 + \frac{3}{2}Q(a')\,\Omega_m(a')\right]. \quad (21)$$

This is turn leads to a revised gravitational growth index

$$\gamma \approx \frac{1}{2} + \frac{1}{4}\int_0^1 \frac{du}{u}\,u^{5/2}\Omega_w(au)/\Omega_w(a)$$
$$- \frac{3}{2}\int_0^1 \frac{du}{u}\,u^{5/2}\,[Q(au) - 1]/\Omega_w(a). \quad (22)$$

There are three cases to consider for the gravitational deviation. If $(Q-1) \sim a^q$ at early times, and $q > -3w_\infty$, then the modification to γ from the altered gravitational coupling will be negligible. However, if $q < -3w_\infty$ then there will be strong modification. In fact, in this case the usual matter-dominated growth behavior at high redshift is broken ($G = 0$, or $\delta \sim a$, is no longer a solution at early times), and large scale structure would not accord with observations. This leaves the main, and physically best motivated, case of the scaling behavior $q = -3w_\infty$ where the same physics responsible for the variation in the force of gravity also affects the expansion rate, giving rise to an effective energy density $\Omega_w(a)$. We discuss this further in the specific DGP braneworld and scalar-tensor gravity examples below.

In the scaling case, we can write $Q - 1 = A\Omega_w(a)$ and find the asymptotic growth index to be

$$\gamma_\infty = \frac{3(1 - w_\infty - A)}{5 - 6w_\infty}. \quad (23)$$

We have verified that, as in §III, it is best to define γ in the usual way through $G = \Omega_m(a)^\gamma - 1$, not $G = [Q\Omega_m(a)]^\gamma - 1$, even in the presence of beyond-Einstein gravity leading to $Q \neq 1$. That is, the gravitational deviation $Q - 1$ enters strictly through the growth index γ, while the expansion history determines the growth history G apart from the value of γ. This separation of physical effects from the expansion rate and from the gravity theory through distinct beyond-Einstein parameterizations w and γ is an important point that clarifies the nature of the beyond-Einstein physics.

B. Braneworld gravity

To test our new expression for the gravitational growth index we consider several examples of gravity beyond general relativity. First, we examine a theory altering the Einstein-Hilbert action by a term arising from large extra dimensions, the DGP braneworld theory [18, 19]. On scales smaller than the Hubble scale but still within the linear density perturbation regime, the effect on the growth behavior is that of a variation in gravitational coupling, with [2]

$$(Q - 1)_{\mathrm{DGP}} = -\frac{1}{3}\left(\frac{1 - \Omega_m^2(a)}{1 + \Omega_m^2(a)}\right). \quad (24)$$

Note that, as predicted, one has the scaling behavior $(Q-1) \sim a^{-3w_\infty}$ asymptotically and

$$A \equiv \frac{Q - 1}{1 - \Omega_m(a)} = -\frac{1}{3}\frac{1 + \Omega_m(a)}{1 + \Omega_m^2(a)} \to -\frac{1}{3}, \quad (25)$$

is of order unity. This ensures that the gravitational deviation neither violates matter domination nor has negligible effect on growth.

The effective equation of state from DGP gravity is $w(a) = -1/[1 + \Omega_m(a)]$ [2], approaching $w_\infty = -1/2$ at high redshift. Substituting the DGP values for A and w_∞ into Eq. (23), we find the growth index

$$\gamma_{\infty,\mathrm{DGP}} = \frac{11}{16} = 0.6875. \quad (26)$$

This accords exactly with the asymptotic numerical solution and extremely well with the gravitational growth index fit $\gamma = 0.68$ over the whole history, given by [8].

If for any point in the growth history we naively substitute into Eq. (23) the values of A from Eq. (25) and

$w(a) = -1/[1 + \Omega_m(a)]$, we obtain

$$\gamma_{\mathrm{DGP}} \approx \frac{7 + 5\Omega_m(a) + 7\Omega_m^2(a) + 3\Omega_m^3(a)}{[1 + \Omega_m^2(a)][11 + 5\Omega_m(a)]}, \qquad (27)$$

which agrees well with the exact numerical solution. The growth index takes on the value $11/16 = 0.6875$ in the asymptotic past and 0.634 in the asymptotic future (vs. $7/11 = 0.636$ from the formula, which was derived under matter domination). At the present the numerical value is 0.665 (for $\Omega_m = 0.28$, vs. 0.674 from the formula), so the single parameter $\gamma = 0.68$ is an excellent approximation for growth at any time through the present, holding constant to 2%. In addition to the remarkable constancy of the growth index, γ stays well distinct of the pure expansion history prediction within general relativity of $\gamma = 0.55 - 0.56$ as the DGP equation of state evolves from $w = -1/2$ in the past to -1 in the future.

We illustrate this important property of separation of growth history parametrization from expansion history in Figure 1. Here the growth index is shown as a ratio to the (exact) general relativity (GR) value for the cases of quintessence and braneworld models. For the braneworld case, being a single parameter model, $w(z = 1)$ determines Ω_m (note Ω_m will be far from 0.28 when $w(z = 1)$ is far from -0.62). We see that the curve of γ as we vary the expansion history parameter $w(z = 1)$ is quite flat, showing success in obtaining a growth parameter distinct from expansion effects.

For the quintessence case (taking $\Omega_m = 0.28$, $w_0 = -1$, and then $w(z = 1)$ serves as a proxy for w_a) the curve representing the fitting form Eq. (15) (with a coefficient 0.04 rather than 0.05 as a better fit over the restricted range $w(z = 1) \in [-1, -0.5]$) is within 0.2% of unity, showing the success of this fitting form.

The growth index formalism thus possesses these important properties: 1) the constancy of γ in redshift, allowing a single parameter description of the gravity deviations beyond-Einstein, 2) the independence of γ from the expansion history, i.e. separating out the expansion influence on the growth so as to give a distinct window on the gravitational physics, and 3) clear signal of the origin of the beyond-Einstein physics, achieving in the braneworld case over 20% deviation from the general relativity prediction – while the "noise" from the expansion history w within general relativity affects γ at the 0.2% level, a factor of 100 less. Thus, the gravitational growth index provides a clear and effective parameterization of beyond-Einstein gravity. (While the theory signal-to-noise is high, achieving observational constraints is more challenging, with [9] estimating that next generation experiments will determine γ to within 8%.)

C. Scalar-tensor gravity

Next we consider scalar-tensor theories of gravity, involving coupling of a scalar field ϕ to the Ricci curvature R of the form of $F(\phi) R$ in the action. Because of
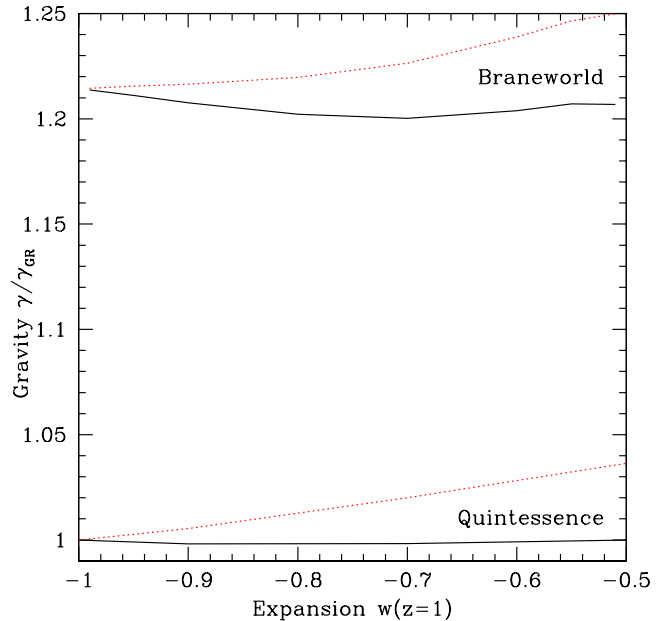

FIG. 1: In the parameterized beyond-Einstein approach, the growth index γ tests gravity separately from the expansion and clearly distinguishes beyond-Einstein gravity from physical dark energy. The curves show the values of γ change little as the expansion history, here represented by the equation of state $w(z = 1)$, varies. Solid curves show the ratio of the numerical solution for γ in the braneworld case to that in general relativity (GR), and in the quintessence case the ratio of the fitting form $\gamma = 0.55 + 0.04[1 + w(z = 1)]$ to the numerical GR solution. (Dotted curves assume $\gamma = 0.55$, the GR value in a cosmological constant model.)

the physical coupling between the modification of the expansion and the modification of the growth equation, we again might expect the scaling relation to hold, where the gravitational deviation $Q - 1$ is of order the effective energy density $\Omega_w(a)$ in the expansion, at least for theories consistent with observations of large scale structure. This would give an appreciable, but not pathological, influence on growth. Scalar-tensor theories where the scalar field is responsible for the current acceleration are often called extended quintessence theories [20].

In extended quintessence (EQ), one can show that the main effects on expansion history come from the scalar potential (self-interaction) and the change in the gravitational coupling $(8\pi G_N)^{-1} \to F$, where G_N is Newton's constant. For the growth equation, deviations arise primarily from the variation of the coupling, with additional terms being suppressed by factors of $(k/H)^{-2}$, where k is the wavemode of the density perturbation. For calculations of $w(a)$ and the modified Poisson equation, see [21].

The effective modification to the growth history can be treated through a deviation in the source term, with $Q - 1 = (8\pi G_N F)^{-1} - 1$. Given a form for F, this can be

used in Eq. (23) to obtain the gravitational growth index for the scalar-tensor theory. In EQ, the R-boost mechanism [22] operating during matter domination drives the theory toward general relativity, so we have a consistent picture of the usual matter-dominated growth around the time of CMB last scattering. As the scalar field comes to dominate in the late universe, the theory may diverge from Einstein gravity. (Because we are exploring the growth index formalism, we do not here worry about constraints from solar system tests of gravity or about specific scalar-tensor theories.)

To give some flavor of calculating γ, we adopt a toy model with coupling

$$F = \frac{1}{8\pi G_N}\left[1 + \frac{B}{1 + (a/a_*)^{-q}}\right], \qquad (28)$$

where B is the amplitude of the coupling variation and a_* is a transition scale factor. At early times, $(Q-1) \sim (a/a_*)^q$. The dark energy density (the quintessence part of EQ) varies as $\Omega_w(a) \sim a^{-3w_\infty}$. As before, if $q > -3w_\infty$, the growth deviation arising from $(Q-1)/\Omega_w(a)$ in Eq. (23) will be negligible; if $q < -3w_\infty$ then the growth source term will be drastically affected at early times. As mentioned before, the coupling between the scalar field evolution, and hence gravity deviation, and the expansion provides a motivation for the scaling behavior $q = -3w_\infty$. In this regime,

$$A \equiv \frac{Q-1}{\Omega_w(a)} = -B\,\frac{\Omega_m}{1 - \Omega_m}\,a_*^{3w_\infty}. \qquad (29)$$

Substituting this into Eq. (23) predicts $\gamma = 0.571$ for $a_* = 0.5$, $B = 0.03$, and $w_\infty = -1$. Numerical solution of the growth equation gives $\gamma = 0.571$ at high redshift, $\gamma = 0.564$ today. This is close to the Einstein range since B is small, but the main point is that γ is quite constant over the growth history. However, for parameter values such that the gravity deviation causes A to approach unity, γ does start to vary (but of course there would be severe departures from general relativity today). As mentioned, this was a toy model giving but a brief taste of the rich phenomenology of scalar-tensor gravity and a full analysis should take into account all the observational constraints. (Also see [23] for analysis of linear perturbations within $f(R)$ gravity.)

D. Relation to PPN

As we survey a larger range of gravitational theories, the situation becomes more complex. The matter source term in the growth equation does not arise purely from the Newton-Poisson equation $\nabla^2\Phi_N = 4\pi\delta\rho$, or $\Phi_N = -4\pi k^{-2}\rho a^3(\delta/a)$ in Fourier space, where Φ_N is the Newtonian potential. The equations of motion actually depend on two potentials, Φ and Ψ, appearing in the metric as

$$ds^2 = -(1 + 2\Psi)dt^2 + a^2(t)\,(1 + 2\Phi)d\vec{x}^2, \qquad (30)$$

(written for a flat universe and in longitudinal gauge for simplicity).

The second order differential equation for the density contrast, à la Eq. (2), comes from both the density and velocity perturbation evolution, and involves both Ψ and Φ. The combination

$$\Psi + \Phi = -k^{-2}\Pi, \qquad (31)$$

is not necessarily zero, as in general relativity, in the presence of anisotropic stress Π. While saying anything general about growth is difficult, let us motivate one approach.

The fractional correction to the source term was defined as $Q - 1$ in §IV A. For wavemodes on sub-Hubble scales, the dominant correction to the growth equation takes the form (following the derivations of [24, 25])

$$-k^2\Psi \to -k^2\Psi + (2/3)\,k^4(\Psi + \Phi)/\rho_m. \qquad (32)$$

We note that we expect the scaling behavior to hold because the anisotropic stress is generated by the dark energy fluid. The deviation $Q - 1$, and the factor A in Eq. (23), is proportional to $1 + \Phi/\Psi$. We can now consider a direct connection to the parameterized post-Newtonian (PPN) formalism [26]. The first PPN parameter is just the ratio of the first order corrections of the g_{ii} and g_{00} parts of the metric, hence equivalent to $-\Phi/\Psi$. This PPN parameter is unfortunately also given the symbol γ, so we will denote it as $\gamma_{\rm PPN}$. Following §IV A, we see a correction to the growth index,

$$\gamma = \frac{3(1 - w_\infty + (2/9)(k/H)^2[1 - \gamma_{\rm PPN}])}{5 - 6w_\infty}. \qquad (33)$$

Note that in this case γ has a scale dependence. (For large wavemodes k, or small wavelengths, the k^2 behavior is cut off by other terms we neglected, such as the usual Jeans term of the matter pressure.) As the beyond-Einstein deviation $1 - \gamma_{\rm PPN}$ vanishes, the growth index approaches the general relativity expression. This illustrates but the barest outline of the complexity of beyond-Einstein gravity.

V. CONCLUSION

Observations, while broadly consistent with the concordance cosmology of Einstein's cosmological constant within Einstein's general relativity, still offer significant leeway for beyond-Einstein physics. Whether there is such a deviation and how to distinguish its physical origin – from a new field or an extended theory of gravity – are major questions to address with the next generation of experiments. However on the theory side, there is no universally accepted model beyond-Einstein. This drives us to develop a usable, model independent formalism, along the lines of the parameterized post-Newtonian approach, in which to evaluate future data and guide us to a theory.

Here we have demonstrated that the Minimal Modified Gravity (MMG) approach of combining a gravitational growth index γ to describe the mass perturbation growth history with equation of state $w(a)$ parameterization to describe the expansion history is in many cases robust, accurate, and broadly applicable. The growth index γ clarifies the nature of beyond-Einstein physics by separating the expansion effects on the growth from the gravitational theory effects.

Rather than needing a full function, the single growth index parameter γ (à la PPN) is extraordinarily successful: 1) it reproduces the exact growth history in a highly accurate manner ($< 0.2\%$ deviation), 2) stays constant to high accuracy, 3) describes both physical dark energy (§III) and beyond-Einstein gravity models consistent with observations (§IV), and 4) exhibits clear distinction between these different physical origins by deviating as the gravity theory does (up to $\sim 20\%$ from general relativity) even when the expansion histories are identical.

We emphasize, however, that while MMG is surprisingly broad in its physical realism and parameterized beyond-Einstein growth is an intriguing formalism for exploring new physics, there is considerable work still to do. This article has dealt only with the linear density perturbation regime of growth for modes below the Hubble scale; other cases, in particular the nonlinear regime, may not be susceptible to a model independent approach. However, [9] suggests first steps for addressing the translinear regime, $\delta \sim 1$, within MMG; this region will be important for weak gravitational lensing probes of cosmology. Modes approaching the Hubble scale can be probed to some extent by the integrated Sachs-Wolfe effect, and these observations may offer some clue to beyond-Einstein physics (see, e.g., [27]). The Hubble scale, and other features like a Vainshtein or Yukawa scale, can introduce scale dependence in the growth as well [28, 29]. Microphysics such as a dark energy sound speed or anisotropic stress (or coupling matter to dark energy, or non-minimal coupling to gravity), can confuse the interpretation of the growth history, as outlined by [30]. A specific formal model for this was proposed by [31], though the dark energy perturbations become large, possibly giving observational difficulties.

Our understanding of the full range of models beyond-Einstein has a long way to go, but Minimal Modified Gravity may provide a simple, robust, and broadly valid benchmark for testing future observations against new physics (cf. the role of minimal supergravity for dark matter physics). The beyond-Einstein parameterization $\{w_0, w_a, \gamma\}$ provides an accessible and reasonable model independent approach to studying the physics of the accelerating universe.

Acknowledgments

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