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Averaging of Stochastic Equations for Flow and Transport in Porous Media

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Abstract

It is well known that at present exact averaging of the equations of flow and transport in random porous media have been realized for only a small number of special fields. Moreover, the approximate averaging methods are not yet fully understood. For example, the convergence behavior and the accuracy of truncated perturbation series are not well known; and in addition, the calculation of the high-order perturbations is very complicated. These problems for a long time have stimulated attempts to find the answer for the question: Are there in existence some *exact* general and sufficiently universal forms of averaged equations? If the answer is positive, there arises the problem of the construction of these equations and analyzing them. There are many publications on different applications of this problem to various fields, including: hydrodynamics, flow and transport in porous media, theory of elasticity, acoustic and electromagnetic waves in random fields, etc. Here, we present a method of finding some *general form of exactly averaged* equations for flow and transport in random fields by using (1) some general properties of the Green's functions for appropriate stochastic problems, and (2) some basic information about the random fields of the conductivity, porosity and flow velocity. We present general forms of *exactly* averaged non-local equations for the following cases:(1) steady-state flow with sources in porous media with random conductivity, (2) transient flow with sources in compressible media with random conductivity and porosity; and (3) Nonreactive solute transport in random porous media. We discuss the problem of uniqueness and the properties of the non-local averaged equations for cases with some type of symmetry (isotropic, transversal isotropic and orthotropic), and we analyze the structure of the nonlocal equations in the general case of stochastically homogeneous fields.

1.INTRODUCTION

Recently the methods of analyzing flow and transport in random media are finding ever-widening applications in fields involving various physical processes. An effective description of flow and transport in irregular porous media entails interpreting porosity and permeability fields as random functions of the spatial coordinates and flow velocity as a random function of the spatial coordinates and time. Such a description also involves averaging of the stochastic system of flow and transport equations containing these functions (conservation laws, Darcy's law and closing relations). The averaging problem consists of finding the relations between the non-random functionals of the unknown and the given fields – means, variations, distributions, densities, etc., or a closed set of equations that contain these functionals. The certain interest attaches to the equations for the averaged functionals that are the laws of conservation of mass, momentum, and energy that are invariant with respect to some set of conditions that uniquely determine the process (for example, the initial and boundary conditions). This is fundamentally possible, for example, in cases where the length scale of heterogeneity is extremely small and the fields of local space averaged pressure, flow velocity, etc., are weakly variable and measurable. In physics literature, this phenomenon some time is referred to as *self-averaging* (Lifshits et al.,1988).

It is apparent that (in general) this is impossible because in practical situations the process depends on a set of parameters that are usually not small and thus an averaged description is used for computing the nonrandom characteristics (functionals) of random flow and transport processes for estimating the uncertainty of the processes. Different variations of stochastic approach and many results have been widely developed. For example see books by Shvidler (1964, 1985), Matheron (1967), Bakhvalov and Panasenco (1984, 1989), Dagan (1989), Gelhar (1993) and Zhikov *et al.* (1993, 1994).

Many related studies have been published for a number of different applications. They include hydrodynamics, theory of elasticity, acoustic and electromagnetic waves in random fields (e.g.: Batchelor, 1953: Monin and Yaglom, 1965, 1967; Tatarsky, 1967; Saffman, 1971; Klyatskin, 1975, 1980; Shermergor, 1979).

It is possible to select an investigation strategy from the following three approaches. **Use of a perturbation technique**: Usually the methods of averaging are approximate. It should be noted that the approximate methods of averaging and derivation of averaged equations of flow and transport are related to the method of perturbation in one way or another, using either Lagrangian or Eulerian approach, in real or frequency domain.

Every so often one can use the series expansion of small parameters, which (for example) specifies the deviation of some given fields from their mean values. This approach usually utilizes analytical techniques. Although it is possible to achieve many results, it should be pointed out that it involves significant difficulties. Even in a problem of a comparatively simple structure, we can usually find only the first few terms of expansions, because the analytical difficulties grow very quickly with the number of the terms. Moreover, the convergence of the expansion is not studied. For this reason even if we can write full perturbation expansion (for example see Herring, 1960; Kudinov and Moizhes, 1972, 1979; Shermergor, 1979; Shvidler, 1985; King, 1987; Teodorovich, 1997) there still exist the open question: is the similar expansion the exact solution of an appropriate problem?

Recently, there have been some studies in which the perturbation technique in Fourier space was extensively used for analysis of flow in an unbounded domain (Indelman and Abramovich, 1994, Indelman, 1996; and others). The authors reduce the problem to a Fredholm second kind integral equation for the Fourier transformation of random hydraulic head. Note that the meaning of this equation is not clear because the classical Fourier transformation of head and fluctuation of conductivity does not exist. For example, the fluctuations of conductivity of stochastically homogeneous field do not decrease at infinity and the flow velocity in any space dimension decreases slowly. As is well known in similar cases we must use the generalized Fourier transformation (see Section 4 and appropriate references).

The aforementioned studies assume that the Fourier transformation for the fluctuation of conductivity is small and thus solve the resulting Fredholm integral equation by iteration (Liouville-Neuman`s method). It is well known -see for example Yosida (1978)- that the Neuman`s series converges if the norm of the kernel for the appropriate integral operator is sufficiently small. In these works, the kernel of the operator depends linearly on the

Fourier transformed fluctuation of conductivity that is a generalized function. There are reasons to suspect that in this case Neuman`s series does not always converge.

One approach is to utilize a distinctive space scale for fast oscillating fields as a small parameter. This approach, so-called homogenization, was largely developed for investigating processes with periodical structures. Many rigorous results were obtained that justify the method, although the computation of the results is highly laborious. For random structures, which is the focus of the present paper, some results have been obtained, but the constructive theory is still absent.

It should be pointed out that there exist many problems that do not contain parameters naturally considered as small. Later we will analyze some similar problems.

Numerical approach: This is the approach by which appropriate equations for representative sets of random fields realizations are numerically solved - sometimes called the Monte Carlo technique. Information obtained in this way makes it possible to find the highest moments together with the local and mean fields of pressure, velocity, etc. However, this approach is restricted by the exceptionally large volume of calculation, as well as the difficulty of generalizing the results and finding the relations between the known and the unknown functionals.

Exact analytical approach: In this context it is difficult to overestimate the value of the exact solutions that are derived using the averaging theory. Specifically, these include exactly averaged flow and transport equations and an exact asymptotic estimation of averaged concentration field for small and large times. In the latter case, it must be remembered that the notation of small and large times for the fields with time dependent sources of solute needs some refinement, because the solute particles that enter the field at different times have different “ages”, and hence they can have significantly different parameters of dispersion.

It has been established that the exact analytical averaging of the equations of flow and transport in random porous media that turns out well can be realized only for a small number of special fields. In the case of steady-state flow the effective conductivity for stratified systems and for two-dimensional systems with special symmetry (Keller, 1964, Matheron, 1967, Dychne, 1970) are well known. The exact effective conductivity for a three-dimensional case is not known. Some examples of exact averaging of one-

dimensional transport were described by Indelman and Shvidler (1985) , Indelman (1986), Cvetcovic *et al*., (1991), and Shvidler and Karasaki (2003a, 2003b). Quasi one-dimensional transport in stratified media was described by Matheron and de` Marsily (1980). In the multi-dimensional case the exact classical Einstein-Fokker-Plank diffusion-advection averaged equation is only valid if the flow-velocity is a delta -correlated Gaussian random field in time (for example see Klyatskin, 1980; Rytov et al., 1989).

Generally speaking, direct averaging as well as defining the functionals and the relations between them is exceptionally complicated. However, the fundamental information contained in the local equations and their structure has not been sufficiently utilized. Later in this paper, we will show that investigation in this direction can lead to finding the forms for the exact relations between average fields. We show that this is possible in some cases without actually solving the appropriate stochastic equations - only by using the existence of the solutions and some of their general properties.

The following question has for a long time stimulated attempts to find the answer: Are there in existence some exact general and sufficiently universal forms of averaged equations for transport of mass, momentum, energy, etc? If the answer is positive, then there begins the quest to construct the equations and to analyze them.

We present a method of finding the general form of exactly averaged equations by using (1) some general properties of the Green`s functions for appropriate stochastic problems, (2) some information about the random fields of the conductivity, porosity, and flow velocity. We present a general form of the exactly averaged non-local equations for the following cases: (1)steady-state flow with sources in porous media with random conductivity, (2) transient flow with sources in compressible media with random conductivity and porosity, and (3) non-reactive solute transport in random porous media. In this paper, we discuss the properties of the non-local averaged equations. Case (1) is presented in detail and for the other cases we present only basic results.

We discuss the problem of uniqueness and the properties of the non-local averaged equations for the cases with some type of symmetry (isotropic, transversally isotropic, and orthotropic). We also present and analyze nonlocal equations for the general case of stochastically homogeneous fields.

In the present paper we further develop the approach and some of the results that were presented briefly in the earlier works by Shvidler and Karasaki (1999, 2001).

2. STEADY-STATE FLOW WITH SOURCES IN AN UNBOUNDED DOMAIN

We consider the steady flow with sources and sinks that are local or continuous, distributed in a single-connected heterogeneous d-dimensional, porous, unbounded domain. It should be noted that studying random flow in bounded and much less multiply connected domains is quite difficult. If the scale of heterogeneity is finite, the averaged description depends on the kind of external and internal boundaries. Because this problem is important for such applications as flows to wells, we will discuss the subject later in Appendix to this paper. We assume that liquid and rock are incompressible.

The local condition of mass conservation is given by the Equation:

$$\frac{\partial \ v_l(\mathbf{x})}{\partial \ x_l} = f(\mathbf{x}) \tag{1}$$

Here $\mathbf{x} = (x_1, ..., x_d)$ is a d-dimensional vector with components $x_l \ (l = 1, ..., d)$, $\mathbf{v}(\mathbf{x})$ is the D'arcy's velocity vector, the function $f(\mathbf{x})$ is the given density of sources and sinks which we assume is an integrable and compactly supported function or distribution with bounded support. In this case $q = \left| \int_{-\infty}^{\infty} f(\mathbf{x}) dx^d \right| < \infty$ where $dx^d = dx_1 ... dx_d$.

The Darcy`s velocity vector $\mathbf{v}(\mathbf{x})$, tensor of conductivity $\boldsymbol{\sigma}(\mathbf{x})$, hydraulic field intensity-vector $\mathbf{h}(\mathbf{x})$ and reduced pressure (or hydraulic head) $u(\mathbf{x})$ obey the Darcy`s Law

$$\mathbf{v}(\mathbf{x}) = \boldsymbol{\sigma}(\mathbf{x}) \mathbf{h}(\mathbf{x}) \ , \tag{2}$$

$$\mathbf{h}(\mathbf{x}) = -\nabla u(\mathbf{x}) \tag{3}$$

and because the $\mathbf{h}(\mathbf{x})$ is a potential vector, we can write relation

$$rot\, \mathbf{h}(\mathbf{x}) = 0 \tag{4}$$

When $u(\mathbf{x})$ is reduced pressure $p^*(\mathbf{x}) = p(\mathbf{x}) - \rho \mathbf{g}\mathbf{x}$, the reduced conductivity is $\boldsymbol{\sigma}(\mathbf{x}) = \mathbf{k}(\mathbf{x})/\mu$, and when $u(\mathbf{x})$ is the head $p^*(\mathbf{x})/\rho g$, the conductivity is $\boldsymbol{\sigma}(\mathbf{x}) = \mathbf{k}(\mathbf{x})\rho g/\mu$. Here $p(\mathbf{x})$ is pressure, $\mathbf{k}(\mathbf{x})$-tensor of permeability, $\mu = const$ and $\rho = const$ are liquid viscosity and density respectively, $\mathbf{g}$ is acceleration of gravity vector.

We assume that conductivity $\boldsymbol{\sigma}(\mathbf{x}) = \{\sigma_{lm}(\mathbf{x})\}$ is the second rank tensor symmetric by subscripts and is a positive definite and limited local tensor, i.e., for any vector $\boldsymbol{x}$ and $\boldsymbol{\xi}$, the conditions $\nu|\boldsymbol{\xi}|^2 \le \xi_m \sigma_{lm}(\mathbf{x})\xi_l \le \theta|\boldsymbol{\xi}|^2, \quad (\theta > 0, \nu > 0)$ are satisfied. The left inequality denotes that the tensor $\boldsymbol{\sigma}(\mathbf{x})$ is elliptic and the right denotes that it is bounded. In this case a unique and positive definite limited tensor $\mathbf{r}(\mathbf{x}) = \boldsymbol{\sigma}^{-1}(\mathbf{x})$ exists and we can write the conservative form of Darcy`s Law as a condition for momentum balance:

$$\mathbf{r}(\mathbf{x})\mathbf{v}(\mathbf{x}) = \mathbf{h}(\mathbf{x}) \tag{5}$$

It is evident from Equations (2), (3), (4) and (5) that the fields $\mathbf{v}(\mathbf{x})$ and $\mathbf{h}(\mathbf{x})$ are one-to-one vector-functions.

We can use the Equations (1)-(5) and the conditions for the function $f(\mathbf{x})$, to derive some estimations for the functions $\mathbf{v}(\mathbf{x}), \mathbf{h}(\mathbf{x})$ and $u(\mathbf{x})$ for large $|\mathbf{x}|$. Indeed, integrating Equations (1) over a spherical volume that includes the area where $f(\mathbf{x}) \ne 0$, assuming that the radius of the sphere $|\mathbf{x}|$ is significantly larger than the maximal scale of the area, and using the theorem of divergence, for $d = 2,3$, we can write $v_l \ \mathbf{x} \ \sim \ 2 \ \ ^{1\ d} qx_l |\mathbf{x}|^{\ d}$. It is clear from Equation (3) that $h(\mathbf{x})$ for large $|\mathbf{x}|$ has the same order. Now we estimate the function $u(\mathbf{x})$: If $d = 3$, we can write $|u \ \mathbf{x}| \sim c_3 qx^{\ 1} \ \ C$, for $d = 2$ we have $|u \ \mathbf{x}| \sim c_2 q \ln x \ \ C$, and for $d = 1$, if $u(x) \to C$ when $x \to -\infty$ for large positive x we have $u \ x \ \sim c_1 qx \ \ C$. Here c_1, c_2, and c_3 are certain constants and C is an arbitrary constant.

Contrary to the boundary-value problems in bounded domains the conditions for the solution at infinity cannot be arbitrarie assigned. These conditions must be consistent with the equations, that is, they must be consistent with present estimations. For example, if $d=3$ for the functions $\mathbf{v}(\mathbf{x})$ and $\mathbf{h}(\mathbf{x})$, we have the conditions:

$$\mathbf{v}(\mathbf{x})=0,\ \mathbf{h}(\mathbf{x})=0 \quad \text{for } |\mathbf{x}|\to\infty \tag{6}$$

The function $u(\mathbf{x})$ for $|\mathbf{x}|\to\infty$ can be an arbitrary constant. It is easy to see that without a loss of generality, this constant can be defined as zero. Thus, we have

$$u(\mathbf{x})=0 \ \text{ for } |\mathbf{x}|\to\infty \tag{7}$$

We can demonstrate that the conditions (6) and (7) ensure a unique solution for the system of Equations (1), (2) and (3). Indeed, let the problem have two different solutions: $\{u_1(\mathbf{x}),\mathbf{v}_1(\mathbf{x}),\mathbf{h}_1(\mathbf{x})\}$ and $\{u_2(\mathbf{x}),\mathbf{v}_2(\mathbf{x}),\mathbf{h}_2(\mathbf{x})\}$. Then, the functions $u_3(\mathbf{x})=u_1(\mathbf{x})-u_2(\mathbf{x}),\mathbf{v}_3(\mathbf{x})=\mathbf{v}_1(\mathbf{x})-\mathbf{v}_2(\mathbf{x})$ and $\mathbf{h}_3(\mathbf{x})=\mathbf{h}_1(\mathbf{x})-\mathbf{h}_2(\mathbf{x})$ are also a solution of the system of Equations (1) and (2) in which the function $f(\mathbf{x})\equiv 0$. In this case, since the tensor $\boldsymbol{\sigma}(\mathbf{x})$ is elliptic, the energy dissipation $e_3(\mathbf{x})=\mathbf{h}_3(\mathbf{x})\mathbf{v}_3(\mathbf{x})=\mathbf{h}_3(\mathbf{x})\boldsymbol{\sigma}(\mathbf{x})\mathbf{h}_3(\mathbf{x})>0$. But, this is impossible because if $f(\mathbf{x})\equiv 0$, the energy does not enter in the field, and, therefore, there cannot be the any dissipation of energy. This contradiction indicates that the first and second solutions are not different. So, $u_3(\mathbf{x})\equiv 0,\mathbf{v}_3(\mathbf{x})\equiv 0$ and $\mathbf{h}_3(\mathbf{x})\equiv 0$. The solution of system (1), (2) and (3), which satisfies the conditions (6) and (7), is unique.

In the two-dimensional case ($d=2$), the estimation showed that if $q\neq 0$ and $|\mathbf{x}|\to\infty$, then functions $\mathbf{v}(\mathbf{x})\to 0$ and $\mathbf{h}(\mathbf{x})\to 0$ but $|u(\mathbf{x})|\to\infty$. The solution of the system of equations (1), (2), and (3) is not unique. On the other hand, we can reformulate the problem and find the solution of the system (1), (2) and (4). This system is closed with respect to the vector-functions $\mathbf{v}(\mathbf{x})$ and $\mathbf{h}(\mathbf{x})$. As we showed earlier, these functions vanish at infinity, and the solution of Equations (1), (2) and (4) is unique.

In the one-dimensional case $(d=1)$ we have the relation $v(+\infty)-v(-\infty)=q$ and if at least one of the velocity at infinity is non- zero, the function h (x) is not limited at infinity and $|u(x)|\to\infty$.

It should be noted that if $f(\mathbf{x})\neq 0$ but $q=0$, the vector-functions $\mathbf{v}(\mathbf{x})$, $\mathbf{h}(\mathbf{x})$ will tend to zero at infinity for $d=2,3$. In these two cases without loss of generality we can write $u(\mathbf{x})=0$ for $x\to\infty$ and the system of Equations (1), (2) and (3) has a unique solution. If the flow is one-dimensional ($d=1$), we have $v(+\infty)=v(-\infty)$, and if flow at infinity is zero, again without loss of generality we can write $u(\pm\infty)=0$. In this case the solution is also unique.

Next we consider in detail a three-dimensional flow and we will return to the steady-state flow in two-dimensional and one-dimensional spaces in section **7.**

3. STOCHASTIC FORMULATION

We assume that the tensor $\boldsymbol{\sigma}(\mathbf{x})$ is a stochastically homogeneous random field in three-dimensional space. That is, for any vector **x** and for an arbitrary vector **a**, all the finite-dimensional probability distributions for the random field $\boldsymbol{\sigma}(\mathbf{x}+\mathbf{a})$ doesn't depend on the arbitrary vector **a**. Let $f(\mathbf{x})$ be a given, non-random density function. We introduce a random Green`s function $g(\mathbf{x},\mathbf{y})$ and a random vector- functions $\boldsymbol{\gamma}(\mathbf{x},\mathbf{y})$ and $\mathbf{s}(\mathbf{x},\mathbf{y})$, for the problem described in Equations (1), (2), (3), and (7), so that for almost all realizations of the differentiable field $\boldsymbol{\sigma}(\mathbf{x})$, the functions $g(\mathbf{x},\mathbf{y})$ and $\boldsymbol{\gamma}(\mathbf{x},\mathbf{y})$ satisfy the following Equations:

$$\frac{\partial\gamma_l(\mathbf{x},\mathbf{y})}{\partial x_l}=\delta(\mathbf{x}-\mathbf{y}),\ \gamma_l(\mathbf{x},\mathbf{y})=\sigma_{lm}(\mathbf{x})s_m(\mathbf{x},\mathbf{y}),\ s_m(\mathbf{x},\mathbf{y})=-\frac{\partial g(\mathbf{x},\mathbf{y})}{\partial x_m} \tag{8}$$

$$g(\mathbf{x},\mathbf{y})=0\ \text{ for } |\mathbf{x}|\to\infty \tag{9}$$

Below, we name $\boldsymbol{\gamma}(\mathbf{x},\mathbf{y})$ and $\mathbf{s}(\mathbf{x},\mathbf{y})$ as random Green`s velocity and hydraulic intensity vector-functions, respectively.

For the case in which the conductivity tensor components $\sigma_{lm}(\mathbf{x})$ are piecewise smooth functions, the Green`s function $g(\mathbf{x},\mathbf{y})$ satisfies the integral identity $\int \sigma_{lm}(\mathbf{x}) \frac{\partial g(\mathbf{x},\mathbf{y})}{\partial x_m} \frac{\partial \varphi(\mathbf{x})}{\partial x_l} dx^3 = \varphi(\mathbf{y})$, where $\varphi(\mathbf{x})$ is an arbitrary testing function that is infinitely differentiable and tends to zero at infinity. It is well known that in this case, the generalized solution $g(\mathbf{x},\mathbf{y})$ satisfies the Equations (8) and (9) at all points where $\boldsymbol{\sigma}(\mathbf{x})$ is smooth. On the surface where tensor $\boldsymbol{\sigma}(\mathbf{x})$ is disconnected, the function $g(\mathbf{x},\mathbf{y})$ and normal component of velocity-vector $\boldsymbol{\gamma}(\mathbf{x},\mathbf{y})$ are uninterrupted.

Now we can convey the random solution for the problem of Equations (1), (2), (3), and (4) through density function $f(\mathbf{x})$ and the random solution of the system of Equations (8) and (9) and write:

$$u(\mathbf{x}) = \int g(\mathbf{x},\mathbf{y}) f(\mathbf{y}) dy^3,\ h_m(\mathbf{x}) = \int s_m(\mathbf{x},\mathbf{y}) f(\mathbf{y}) dy^3,\ v_l(\mathbf{x}) = \int \gamma_l(\mathbf{x},\mathbf{y}) f(\mathbf{y}) dy^3 \quad (10)$$

where the integration is over the entire unbounded 3-D space. Note that although the functions $g(\mathbf{x},\mathbf{y}), \mathbf{s}(\mathbf{x},\mathbf{y})$ and $\boldsymbol{\gamma}(\mathbf{x},\mathbf{y})$ are integrable in any bounded domain in R^3 and vanish at infinity, they are not integrable in a full three-dimensional space. But the integrals in Equations (10) make sense, because the function $f(\mathbf{x})$ is compactly supported.

We introduce the averaged fields over the ensemble of realizations of the random function $\boldsymbol{\sigma}(\mathbf{x})$:

$$U(\mathbf{x}) = \langle u(\mathbf{x}) \rangle,\ \mathbf{V}(\mathbf{x}) = \langle \mathbf{v}(\mathbf{x}) \rangle,\ \mathbf{H}(\mathbf{x}) = \langle \mathbf{h}(\mathbf{x}) \rangle, \quad (11)$$

$$G(\mathbf{x},\mathbf{y}) = \langle g(\mathbf{x},\mathbf{y}) \rangle,\ \mathbf{S}(\mathbf{x},\mathbf{y}) = \langle \mathbf{s}(\mathbf{x},\mathbf{y}) \rangle = -\nabla_x G(\mathbf{x},\mathbf{y}),\ \boldsymbol{\Gamma}(\mathbf{x},\mathbf{y}) = \langle \boldsymbol{\gamma}(\mathbf{x},\mathbf{y}) \rangle$$

As long as $\boldsymbol{\sigma}(\mathbf{x})$ is a stochastically homogeneous field, the mean Green's function $G(\mathbf{x},\mathbf{y})$, the mean Green`s velocity-vector $\boldsymbol{\Gamma}(\mathbf{x},\mathbf{y})$ and mean hydraulic intensity vector-functions $\mathbf{S}(\mathbf{x},\mathbf{y})$ are invariant over translation in space, and therefore, depend only on

the difference $\mathbf{x}-\mathbf{y}$. Hence, after averaging the Equation (10) over the ensemble, we have:

$$U(\mathbf{x})=\int G(\mathbf{x}-\mathbf{y})f(\mathbf{y})dy^3\ ,\ \mathbf{H}(\mathbf{x})=\int \mathbf{S}(\mathbf{x}-\mathbf{y})\,f(\mathbf{y})dy^3 \tag{12}$$

Then, we can write the equation for mean velocity as

$$V_l(\mathbf{x})=\int \Gamma_l(\mathbf{x}-\mathbf{y})\,f(\mathbf{y})dy^3, \tag{13}$$

$$\Gamma_l(\mathbf{x}-\mathbf{y})=\left\langle \sigma_{lm}(\mathbf{x})s_m(\mathbf{x},\mathbf{y})\right\rangle=-\left\langle \sigma_{lm}(\mathbf{x})\ \frac{\partial g(\mathbf{x},\mathbf{y})}{\partial x_m}\right\rangle \tag{14}$$

Thus the functions $U(\mathbf{x}), \mathbf{H}(\mathbf{x})$ and $\mathbf{V}(\mathbf{x})$ are presented as convolutions which have sense because the function $f(\mathbf{x})$ has bounded support.

After averaging the equations (1) and (3) we have:

$$\frac{\partial V_l(\mathbf{x})}{\partial x_l}=f(\mathbf{x}),\ H_l(\mathbf{x})=-\frac{\partial U(\mathbf{x})}{\partial x_l} \tag{15}$$

and after averaging the first equation from (8) we find the relation of compatibility for the components $\Gamma_l(\mathbf{x}-\mathbf{y})$:

$$\frac{\partial \Gamma_l(\mathbf{x}-\mathbf{y})}{\partial x_l}=\delta(\mathbf{x}-\mathbf{y}) \tag{16}$$

Therefore, the mean pressure $U(\mathbf{x})$, mean hydraulic intensity $\mathbf{H}(\mathbf{x})$ and the mean velocity $\mathbf{V}(\mathbf{x})$ are convolution integrals of the source density $f(\mathbf{x})$ and the mean Green's function $G(\mathbf{x})$, mean intensity $\mathbf{S}(\mathbf{x})$ and the Green`s vector-function $\mathbf{\Gamma}(\mathbf{x})$, respectively.

Equations (12), (13), and (15) make up a system of equations for the averaged fields $U(\mathbf{x})$, $\mathbf{H}(\mathbf{x})$ and $\mathbf{V}(\mathbf{x})$. This system contains the kernels, $G(\mathbf{x}-\mathbf{y})$, $\mathbf{S}(\mathbf{x}-\mathbf{y})$, and $\mathbf{\Gamma}(\mathbf{x}-\mathbf{y})$ which are non-random functional from the random conductivity field $\boldsymbol{\sigma}(\mathbf{x})$ and the random Green's function $g(\mathbf{x},\mathbf{y})$. Of course, explicit definitions of the functionals G, $\mathbf{S}$ and $\mathbf{\Gamma}$ are very difficult to obtain in the general case (for any random field $\boldsymbol{\sigma}(\mathbf{x})$).

For now the existence of these functionals in itself is sufficient. It is possible to determine some of their features that help find a general form of the

averaged equations, of which Equations (12), (13) and (15) are a part of. Later in this paper we will find them in different forms.

4. FOURIER ANALYSIS

For analyzing equations with convolutions in all space we consider the classical Fourier transform T_F of a $\varphi(\mathbf{x})$ that is absolute integrable in full space function and it`s inverse Fourier transform T_F^{-1} :

$$T_F\left[\varphi(\mathbf{x})\right]=\bar{\varphi}(\mathbf{k})=\int \exp\left[-2\pi j(\mathbf{x}\cdot\mathbf{k})\right]\varphi(\mathbf{x})\,dx^3 \tag{17}$$

$$T_F^{-1}\left[\bar{\varphi}(\mathbf{k})\right]=\varphi(\mathbf{x})=\int \exp\left[2\pi j(\mathbf{k}\cdot\mathbf{x})\right]\bar{\varphi}(\mathbf{k})\,dk^3 \tag{18}$$

where $j=\sqrt{-1}$.

Since the functions $G(\mathbf{x})$ and $U(\mathbf{x})$, $\mathbf{S}(\mathbf{x})$ and $\mathbf{H}(\mathbf{x})$, $\mathbf{\Gamma}(\mathbf{x})$ and $\mathbf{V}(\mathbf{x})$ slowly vanish at infinity, they are not integrable in full space. In this case the integrals in (17) for these functions differ, and so-called generalized Fourier transformation of similar functions or distributions should be used (for example, see L.Schwartz, 1962; R.Courant, 1962; I.Gelfand and G.Shilov, 1964; K. Yosida, 1978; and A.H.Zemanian, 1987).

The definition of this transformation is: The generalized Fourier transformation $\bar{T}_F$ of distribution (functional) q over A-space testing functions φ is a distribution (functional) $\bar{q}$ over B-space functions $\bar{\varphi}=\mathrm{T}_F\left[\varphi\right]$. So, for $\bar{q}=\bar{\mathrm{T}}_F[q]$ and $q=\bar{\mathrm{T}}_F^{-1}\left[\bar{q}\right]$ we have equality for scalar products $\left(\bar{T}_F\left[q\right],\bar{\varphi}\right)=\left(q,\varphi\right)$ that is

$$\int \bar{q}^*(\mathbf{k})\bar{\varphi}(\mathbf{k})\,dk^3=\int q^*(\mathbf{x})\varphi(\mathbf{x})\,dx^3 \tag{19}$$

Here, the star-superscript indicates the complex conjugate.

It is known that the relation (19) is the definition of generalized direct and inverse Fourier transformations, and if $g(\mathbf{x})$ is an absolute integrable function, the Equation (19) is equivalent to (17) and (18).

The commonly used examples of testing functions spaces are: (1) $\varphi(\mathbf{x})$ is the so-called testing function of functional space S_∞, which includes all infinitely differentiable functions that decrease more rapidly at infinity than any power of $1/|\mathbf{x}|$; (2)- $\varphi(\mathbf{x})$ belongs to K functional space of infinitely differentiable functions with finite support.

It is known (see, for example L.Schwartz (1962), A.H Zemanian (1987)) that a generalized Fourier transformation for arbitrary distribution does not exist. But this definition is valid for some special distributions : for example, so- called slow growth or tempered distributions. Any local integrable function that does grow more rapidly at infinity than any power function, is tempered. The generalized functions with bounded support are tempered. For example with (19) we can find:

$$\bar{T}_F\left[\delta(\mathbf{x})\right]=1,\ \bar{T}_F\left[\frac{\partial^m\delta(\mathbf{x})}{\partial x_l^m}\right]=\left(2\pi jk_l\right)^m \tag{20}$$

$$\bar{T}_F[1]=\delta(\mathbf{k}),\ \bar{T}_F\left[\exp\left(ja_lx_l\right)\right]=\prod_l\delta\left(k_l+a_l\right),\ \bar{T}_F\left[x_l^2\right]=-\frac{\partial^2\delta(\mathbf{k})}{\partial k_l^2} \tag{21}$$

Above, for scalar functions $G(\mathbf{x}), U(\mathbf{x})$ and vector-functions $\mathbf{S}(\mathbf{x}), \mathbf{H}(\mathbf{x}), \mathbf{\Gamma}(\mathbf{x}), \mathbf{V}(\mathbf{x})$ we write a system of equations, part of which contains convergent convolutions. Now we apply to all these equations the generalized Fourier transformation defined with equations (19). Taking into account that the functions $G(\mathbf{x}), U(\mathbf{x})$ and $\mathbf{\Gamma}(\mathbf{x})$ are tempered (they even decrease at infinity) and the function $f(\mathbf{x})$ has bounded support, we can present the generalized Fourier transformation of the convolutions as a product of generalized Fourier transformations of appropriate functions and $\bar{f}(\mathbf{k})\neq 0$. As a result, from Equations (12), (13), (15) and (16) we can write the following system of linear algebraic equations in **k**-space

$$\bar{U}(\mathbf{k})=\bar{G}(\mathbf{k})\,\bar{f}(\mathbf{k}) \tag{22}$$

$$\bar{\mathbf{H}}(\mathbf{k})=\bar{\mathbf{S}}(\mathbf{k})\,\bar{f}(\mathbf{k})=2\pi j\mathbf{k}\bar{U}(\mathbf{k}) \tag{23}$$

$$\bar{V}_l(\mathbf{k})=\bar{\Gamma}_l(\mathbf{k})\,\bar{f}(\mathbf{k}) \tag{24}$$

$$2\pi jk_l\bar{V}_l(\mathbf{k})=\bar{f}(\mathbf{k}) \tag{25}$$

$$2\pi jk_l \bar{\Gamma}_l(\mathbf{k}) = 1 \tag{26}$$

If we assume that the functions $\bar{G}(\mathbf{k})$, $\bar{\mathbf{\Gamma}}(\mathbf{k})$ and $\bar{f}(\mathbf{k})$ are known, the Equations (22)-(25) are a closed system with respect to functions $\bar{U}(\mathbf{k})$, $\bar{\mathbf{H}}(\mathbf{k})$, and $\bar{V}_l(\mathbf{k})$.

After eliminating $\bar{f}(\mathbf{k}) \neq 0$ from the Equations (22) and (24) we find equations that bind the scalar field $\bar{U}(\mathbf{k})$ and the vector field $\bar{\mathbf{V}}(\mathbf{k})$:

$$\bar{V}_l(\mathbf{k}) = \bar{\Pi}_l(\mathbf{k})\bar{U}(\mathbf{k}) \tag{27}$$

$$\bar{\Pi}_l(\mathbf{k}) = \bar{\Gamma}_l(\mathbf{k})\left[\bar{G}(\mathbf{k})\right]^{-1} \tag{28}$$

From (26) and (28) we obtain the condition of compatibility for the components of vector $\bar{\mathbf{\Pi}}(\mathbf{k})$:

$$2\pi jk_l \bar{\Pi}_l(\mathbf{k}) = \left[\bar{G}(\mathbf{k})\right]^{-1} \tag{29}$$

The scalar function $G(\mathbf{x}-\mathbf{y}) = G(\mathbf{y}-\mathbf{x})$ and the vector-function $\mathbf{\Gamma}(\mathbf{x}-\mathbf{y}) = -\mathbf{\Gamma}(\mathbf{y}-\mathbf{x})$ and their generalized Fourier transformations $\bar{G}(\mathbf{k})$ and $\bar{\Gamma}(\mathbf{k})$ are real-even and imaginary-odd functions of $\mathbf{k}$, respectively. The vector-function $\bar{\mathbf{\Pi}}(\mathbf{k})$ is also imaginary-odd. The Equations (22) and (27) are also a closed system with respect to functions $\bar{U}(\mathbf{k})$ and $\bar{V}_l(\mathbf{k})$.

It easy to see that the exact averaged Equation (27) is reversible. If we know the scalar field $\bar{U}(\mathbf{k})$, from (27) we can directly define the vector field $\bar{\mathbf{V}}(\mathbf{k})$ and vice versa. If we know the field $\bar{V}_l(\mathbf{k})$, using (27) we can write $\bar{U}(\mathbf{k}) = \bar{V}_l(\mathbf{k}) / \bar{\Pi}_l(\mathbf{k})$ for any *l*.

It is interesting to note in a simple case when the porous media is nonrandom and homogeneous, the correct result is evident after the entire stage of mathematical operations: the averaged equations will be the same as the local equations. Of course, we miss the averaging stage, but we can in detail see some difficulties associated with the Fourier transformation of distributions (functionals).

So, let $\sigma_{lm}(\mathbf{x}) = \sigma_{lm} = const$. In this case the function $G(\mathbf{k})$, the vectors $\bar{\Gamma}_l(\mathbf{k})$, and $\bar{\Pi}_l(\mathbf{k})$ and the kernel-vector $\Pi_l(\mathbf{x}-\mathbf{y})$ are:

$$\bar{G}(\mathbf{k}) = \left(4\pi^2 k_l \sigma_{lm} k_m\right)^{-1},\ \bar{\Gamma}_l(\mathbf{k}) = -j\sigma_{lm}k_m / 2\pi k_l \sigma_{lm} k_m \tag{30}$$

$$\bar{\Pi}_l(\mathbf{k}) = -2\pi j \sigma_{lm} k_m\ ,\ \Pi_l(\mathbf{x}) = -\sigma_{lm}\frac{\partial\delta(\mathbf{x})}{\partial x_m} \tag{31}$$

$$\bar{V}_l(\mathbf{k}) = -2\pi j \sigma_{lm} k_m U(\mathbf{k})\ \ or\ \ r_{ml}\bar{V}_l(\mathbf{k}) = -2\pi j k_m U(\mathbf{k}) \tag{32}$$

As we can see in this example, the function $\bar{G}(\mathbf{k})$ diminishes more rapidly than $\bar{\mathbf{\Gamma}}(\mathbf{k})$ and therefore the vector-function $\bar{\mathbf{\Pi}}(\mathbf{k})$ is a tempered function (in this case-linear function).

Since the function $f(\mathbf{x})$ has bounded support and Green`s function $G(\mathbf{x})$ is a tempered distribution, we can use the theorem of convolution with equation (22) and write

$$U(\mathbf{x}) = \int G(\mathbf{x}-\mathbf{y}) f(\mathbf{y})\, dy^3 \tag{33}$$

Similarly, because $U(\mathbf{x})$ is a tempered function and $\mathbf{\Pi}(\mathbf{x})$ is a distribution with local support, we have from Equation (27)

$$\mathbf{V}(\mathbf{x}) = \int \mathbf{\Pi}(\mathbf{x}-\mathbf{y}) U(\mathbf{y})\, dy^3 \tag{34}$$

We then have

$$V_l(\mathbf{x}) = -\sigma_{lm}\int \frac{\partial\delta(\mathbf{x}-\mathbf{y})}{\partial x_m} U(\mathbf{y})\, dy^3 = -\sigma_{lm}\int \delta(\mathbf{y}-\mathbf{x})\frac{\partial U(\mathbf{y})}{\partial y_m} dy^3 = -\sigma_{lm}\frac{\partial U(\mathbf{x})}{\partial x_m} \tag{35}$$

and similarly we have the conservative form of Darcy`s equation

$$r_{ml}\bar{V}_l(\mathbf{x}) = -\frac{\partial U(\mathbf{x})}{\partial x_m} \tag{36}$$

Now, let us compare the approaches for solving the direct and inverse problems when we use the local description of flow introduced with the system of Equations (1)-(4) and the

averaged description presented above, with the system of Equations (22)-(26) or (27)-(29).

It is evident that when we know the non-random density-function $f(\mathbf{x})$ and a second rank random tensor-function $\boldsymbol{\sigma}(\mathbf{x})$, we can find a unique scalar field $u(\mathbf{x})$ and vector-field $\mathbf{v}(\mathbf{x})$ for almost all realizations. The inverse problem of finding the density-function $f(\mathbf{x})$ and the tensor $\boldsymbol{\sigma}(\mathbf{x})$ is more complicated and requires a special approach. If we know some velocity-field $\mathbf{v}_1(\mathbf{x})$, we can compute the function $f_1(\mathbf{x})$ from the Equation (1). The tensor $\boldsymbol{\sigma}(\mathbf{x})$ is symmetric, and we have nine unknown components in a three-dimensional space and three conditions of symmetry $\sigma_{lm}(\mathbf{x}) = \sigma_{ml}(\mathbf{x})$. Both fields $u_1(\mathbf{x})$ and $\mathbf{v}_1(\mathbf{x})$ depend on the same density- function $f_1(\mathbf{x})$, which makes it possible to use the *Darcy`s* Law in (2) to obtain a system of three scalar linear equations. Each of them contains three unknown components. Thus, we have an underdetermined system of 6 linear algebraic equations with nine unknown components. It is clear that if we use two linearly independent pairs, $\{u_1(\mathbf{x}), \mathbf{v}_1(\mathbf{x})\}$ and $\{u_2(\mathbf{x}), \mathbf{v}_2(\mathbf{x})\}$, we can add to the system three independent equations, that is *Darcy`s* Law for the second pair of fields. In this case, we have a closed system of nine equations for nine components. Since the local fields $u_i(\mathbf{x})$ and $\mathbf{v}_i(\mathbf{x})$ depend on the density-function $f_i(\mathbf{x})$ in all x-space, in order for the pairs $\{u_i(\mathbf{x}), \mathbf{v}_i(\mathbf{x})\}$ to be linearly independent, the functions $f_i(\mathbf{x})$ must be linearly independent.

When analyzing the averaged description, the direct problem is to define the fields $\bar{U}(\mathbf{k})$ and $\bar{\mathbf{V}}(\mathbf{k})$ under a known scalar function $\bar{f}(\mathbf{k})$ and a **vector**-function $\bar{\boldsymbol{\Pi}}(\mathbf{k})$. It is evident that we can find from Equation (29) the function $\bar{G}(\mathbf{k})$ and then find the field $\bar{U}(\mathbf{k})$ from Equation (22). Finally, we find the field $\bar{\mathbf{V}}(\mathbf{k})$ from Equation (27), and after using the generalized Fourier transformation definition (19) we can find the fields $U(\mathbf{x})$ and $\mathbf{V}(\mathbf{x})$. Thus, the direct problem is fully defined. Remember that to fully define the

direct local problem we need to use the scalar function $f(\mathbf{x})$ and the **tensor**-function $\sigma_{lm}(\mathbf{x})$.

The inverse problem under the averaged descriptions is to define the scalar function $\bar{f}(\mathbf{k})$ and vector-function $\bar{\mathbf{\Pi}}(\mathbf{k})$. It is evident that if we know the scalar-field $\bar{U}(\mathbf{k})$ and the vector-field $\bar{\mathbf{V}}(\mathbf{k})$, the appropriate function $\bar{f}(\mathbf{k})$ can be found from the Equation (24) and vector $\bar{\mathbf{\Pi}}(\mathbf{k})$ from Equation (27). Note that if we only know the vector-field $\bar{\mathbf{V}}(\mathbf{k})$, we can only find the function $\bar{f}(\mathbf{k})$. If in addition we know the scalar-function $\bar{G}(\mathbf{k})$, we can find the field $\bar{U}(\mathbf{k})$ from the Equation (22) after computing $\bar{f}(\mathbf{k})$.

It is clear that if we know only the fields $\bar{U}(\mathbf{k})$ and $\bar{G}(\mathbf{k})$, we can only find the function $\bar{f}(\mathbf{k})$. Thus, contrary to a local model, the inverse problem is fully defined if we know one pair of fields $\{\bar{U}(\mathbf{k}), \bar{\mathbf{V}}(\mathbf{k})\}$.

Notice that the equations for an averaged steady-state flow are associated with non-random functionals of random fields and thus are not as detailed as the local models. Similar to any variant of upscaling, we lose some information about flow, but in return, we have simpler tools to study the important property of the process. As we show here, instead of the second rank tensor $\boldsymbol{\sigma}(\mathbf{x})$, the random media characteristic in local model, it is sufficient to use the first rank tensor, i.e., the vector - $\bar{\mathbf{\Pi}}(\mathbf{k})$ for the description of the averaged model.

5. GLOBAL SYMMETRY

We continue the analysis of the averaged equations and assume that the random field $\boldsymbol{\sigma}(\mathbf{x})$ satisfies some symmetry conditions that are related to the structural properties of the field as a whole. We shall call this type of symmetry global.

***ISOTROPY*:** Let the random conductivity tensor $\boldsymbol{\sigma}(\mathbf{x})$ be an isotropic field. In this case the imaginary vector $\bar{\mathbf{\Pi}}(\mathbf{k})$ in any orthogonal coordinate system is proportional to the uniquely defined vector $2\pi j\mathbf{k}$ in frequency space. It is invariant for any rotation and

reflection on the coordinate planes $k_l = 0$ and the proportionality coefficient depends entirely on k = $|\mathbf{k}|$ only. We can write

$$\bar{\Pi}_l^{(i)}(\mathbf{k}) = -\bar{\Pi}_{*l}^{(i)}(k)\, 2\pi j k_l \tag{37}$$

Where $\Pi_{*l}^{(i)}(|\mathbf{k}|)$ is a scalar and positive even function, such that $\bar{\Pi}_{*1}^{(i)} = \bar{\Pi}_{*2}^{(i)} = \bar{\Pi}_{*3}^{(i)} = \bar{\Pi}_{*}^{(i)}$.

Then

$$\bar{V}_l(\mathbf{k}) = \mathrm{B}_{lm}^{(i)}(k)\bar{H}_m(\mathbf{k}) \;\; , \; \bar{\mathrm{B}}_{lm}^{(i)}(k) = \Pi_*^{(i)}(k)\delta_{lm}, \; \bar{H}_m(\mathbf{k}) = -2\pi j k_m \bar{U}(\mathbf{k}) \tag{38}$$

and therefore in *x*-space, if appropriate convolutions converge, we have the relations:

$$V_l(\mathbf{x}) = -\int \mathrm{B}_{lm}^{(i)}(|\mathbf{x}-\mathbf{y}|)\frac{\partial U(\mathbf{y})}{\partial y_m} dy^3 \;\; , \text{ or } \; V_l(\mathbf{x}) = -\int \Pi_*^{(i)}(|\mathbf{x}-\mathbf{y}|)\frac{\partial U(\mathbf{y})}{\partial y_l} dy^3 \tag{39}$$

Here $\mathbf{B}^{(i)}(|\mathbf{x}|)$ is a unique spherical tensor and $\Pi_*^{(i)}(|\mathbf{x}|)$ is a unique scalar function.

It is evident that the Equation (38) is reversible and we can write:

$$\bar{R}_{ml}^{(i)}(k)\bar{V}_l(\mathbf{k}) = \bar{H}_m(\mathbf{k}) \, , \; \bar{R}_{ml}^{(i)}(k) = \left[\Pi_*^{(i)}(k)\right]^{-1}\delta_{ml} \tag{40}$$

In x- space, if an appropriate convolution converges; we have the non-local averaged condition of the momentum balance with unique kernel:

$$\int R_{ml}^{(i)}(|\mathbf{x}-\mathbf{y}|)V_l(\mathbf{y})\, dy^3 = H_m(\mathbf{x}) \tag{41}$$

Here the isotropic resistance tensor $\mathbf{R}(|\mathbf{x}|) = T_F^{-1}\left[\bar{\mathbf{R}}(|\mathbf{k}|)\right]$.

Note again that non-local equations (39) and (41) are meaningful only when appropriate integrals (convolutions) converge. Of course, in the framework of the approach used here the proof of convergence or divergence in a general case is hardly possible because we utilize only the existence of Green`s function and Green`s velocity and some simple properties of them. Then we again describe a partial case of global isotropic porous media and, in addition, use the information on the asymptotic behavior of mean Green`s function $G(\mathbf{x})$ for large and small $|\mathbf{x}|$.

Inserting into Equation (29) the expression $\bar{\mathbf{\Pi}}(\mathbf{k}) = -2\pi \mathbf{k}\bar{\mathbf{R}}^{-1}(k)$, we find $\bar{R}_{ml}(k) = 4\pi^2 k^2 \bar{G}(k)\delta_{ml}$ or in x-space we have $R_{ml}(x) = -\nabla^2 G(x)\delta_{ml}$. Installing the last formula in Equation (41) yields

$$-\int \nabla^2 G(\mathbf{y}) V_m(\mathbf{x}-\mathbf{y}) dy^3 = H_m(\mathbf{x}) \tag{42}$$

If the field $\mathbf{V}(\mathbf{x}-\mathbf{y})$ is differentiable, the convergence of the convolution in (42) depends on the behavior of the integrand at $\mathbf{y}=0$ and $|\mathbf{y}| \to \infty$. An investigation of the asymptotic behaviors of function $G(\mathbf{y})$ (see Shvidler, 1966, 1985) showed that if scale heterogeneity is finite and when y is very small the principal part of Green`s function is $G(\mathbf{y}) \sim 1/4\pi\sigma_* y$. Here $\sigma_* = \langle \sigma^{-1} \rangle^{-1}$ is the mean harmonic conductivity. For very large y, it was also shown that $G(\mathbf{y}) \sim 1/4\pi\sigma^* y$, where σ^* is an effective conductivity. In the Section 2 for very large y, we present the estimates of velocity $V_m(\mathbf{y}) \sim q y_m / 4\pi^2 y^3$ that vanish but are not integrable. It is convenient to transform the tensor $\bar{R}_{ml}(k)$ in the form $\bar{R}_{ml}(k) = 4\pi^2 k^2 \bar{G}(k)\delta_{ml} = 4\pi^2 k_r \bar{G}(k)\delta_{rp} k_p \delta_{ml}$. Using equation (41), we can write $\int \frac{\partial G(\mathbf{y})}{\partial y_l} \frac{\partial V_m(\mathbf{x}-\mathbf{y})}{\partial y_l} dy^3 = H_m(\mathbf{x})$. Now we can see that for small y, we have estimation $\frac{G \quad y}{y_l} \sim \frac{y_l}{4 \quad y^3}$, and because $\frac{\partial V_m(\mathbf{x}-\mathbf{y})}{\partial y_l}$ is bounded, the integral converges at **y=0.** For very large *y,* we use the following estmations

$$\frac{G \quad \mathbf{y}}{y_l} \sim \frac{y_l}{4 \quad y^3}, \frac{V_m \quad \mathbf{y}}{y_l} \sim \frac{q}{4} \frac{{}_{lm} y^2 \quad 3 y_l y_m}{y^5} \quad \text{and} \quad \frac{G \quad \mathbf{y}}{y_l} \frac{V_m \quad \mathbf{y}}{y_l} \sim \frac{q y_m}{16 \quad {}^2 \quad y^6} .$$

Now we see that boht singularities in the integrand are integrable. A consideration convolution like the one in equation (42) does exist.

***ORTHOTROPY*:** If the field $\boldsymbol{\sigma}(\mathbf{x})$ is globally orthotropic, then there exists some orthogonal coordinate system such that all the stochastic multipoint moments of the random field are invariant to the reflection on the planes $k_l = 0$. In this case, the components $\bar{\Pi}_l(\mathbf{k})$ can be written as

$$\bar{\Pi}_l^{(o)}(\mathbf{k}) = -\bar{\Pi}_{*l}^{(o)}(\mathbf{k}) 2\pi j k_l \tag{43}$$

(summation over l is not implied!)

The functions $\bar{\Pi}_{*l}^{(o)}(\mathbf{k})$ are positive and even of **k,** therefore depends on $|k_1|, |k_2|, |k_3|$. In the global orthotropic system, the averaged equations are in the forms:

$$\bar{V}_l(\mathbf{k}) = \bar{\Pi}_{*l}^{(o)}(\mathbf{k}) \bar{H}_l(\mathbf{k}) \quad , \quad \bar{H}_l(\mathbf{k}) = -2\pi j k_l \bar{U}(\mathbf{k}) \tag{44}$$

(no summation over *l!*)

$$\bar{V}_l(\mathbf{k}) = \bar{\mathrm{B}}_{ml}^{(o)}(\mathbf{k}) \bar{H}_m(\mathbf{k}) \tag{45}$$

where the components tensor $\bar{\mathbf{B}}^{(o)}(\mathbf{k})$ takes the form:

$$\bar{\mathrm{B}}_{ml}^{(o)}(\mathbf{k}) = \delta_{ml} \bar{\Pi}_{*m}^{(o)}(\mathbf{k}) \tag{46}$$

(no summation over m!)

which means that the tensor $\bar{\mathbf{B}}^{(o)}$ is diagonal.

In x-space, if appropriate convolutions converge, in the corresponding coordinate system we have non-local equations with unique kernels

$$V_l(\mathbf{x}) = -\int \Pi_{*l}^{(o)}(\mathbf{x}-\mathbf{y}) \frac{\partial U(\mathbf{y})}{\partial y_l} dy^3 \ , \ V_l(\mathbf{x}) = -\int B_{lm}^{(o)}(\mathbf{x}-\mathbf{y}) \frac{\partial U(\mathbf{y})}{\partial y_m} dy^3 \tag{47}$$

Evidently, Equation (45) is reversible, and the averaged equation has the form:

$$\bar{R}_{ml}^{(o)}(\mathbf{k}) \bar{V}_l(\mathbf{k}) = \bar{H}_m(\mathbf{k}) \tag{48}$$

where $\bar{\mathbf{R}}^{(o)}(\mathbf{k}) = \left[\bar{\mathbf{B}}^{(o)}(\mathbf{k})\right]^{-1}$ is the diagonal orthotropic tensor of resistance.

In x-space, in the corresponding coordinate system, we have a nonlocal averaged condition of the momentum balance with a unique kernel:

$$\int R_{ml}^{(o)}(\mathbf{x}-\mathbf{y}) V_l(\mathbf{y}) dy^3 = -\frac{\partial U(\mathbf{x})}{\partial x_m} \tag{49}$$

***TRANSVERSAL ISOTROPY*:** In the case of global transversal isotropy, the equations are invariant relative to the rotation around only one axis of the coordinate system, for example, for k_3, and reflection on any planes $k_l = 0$. Then we have $\overline{\Pi}_{*1}^{(t)} = \overline{\Pi}_{*2}^{(t)} \neq \overline{\Pi}_{*3}^{(t)}$, where the scalar functions $\overline{\Pi}_{*l}^{(t)}(\mathbf{k})$ stays invariant over rotation around axis k_3 and reflections on any planes that are perpendicular to axis k_1, k_2 *and* k_3. In this case of symmetry,

$$\overline{\Pi}_{l}^{(t)}(\mathbf{k}) = -\overline{\Pi}_{*l}^{(t)}(\mathbf{k}) 2\pi j k_l \tag{50}$$

(no summation over l!)

The function $\overline{\Pi}_{*l}^{(t)}(\mathbf{k})$ is positive and even, and depends on $\left(k_1^2 + k_2^2\right)^{1/2}$, $|k_1|$, $|k_2|$, and $|k_3|$. In the transversal isotropic system, the averaged equations are:

$$\overline{V}_l(\mathbf{k}) = \overline{\Pi}_{*l}^{(t)}(\mathbf{k}) \overline{H}_l(\mathbf{k}), \overline{H}_l(\mathbf{k}) = -2\pi k_l \overline{U}(\mathbf{k}) \tag{51}$$

(no summation over l!)

$$\overline{V}_l(\mathbf{k}) = -\overline{\mathrm{B}}_{lm}^{(t)}(\mathbf{k}) 2\pi j k_m \overline{U}(\mathbf{k}) \tag{52}$$

where the components of the tensor $\overline{\mathbf{B}}^{(t)}$ are:

$$\overline{\mathrm{B}}_{lm}^{(t)}(\mathbf{k}) = \delta_{lm} \overline{\Pi}_{*l}^{(t)}(\mathbf{k}) \tag{53}$$

It is evident that the tensor $\overline{\mathbf{B}}^{(t)}$ is diagonal and $\overline{\mathrm{B}}_{11}^{(t)}(\mathbf{k}) = \overline{\mathrm{B}}_{22}^{(t)}(\mathbf{k}) \neq \overline{\mathrm{B}}_{33}^{(t)}(\mathbf{k})$. In x-space, if appropriate convolution converge, in the corresponding coordinate system we have non-local averaged equations with unique kernels

$$V_l(\mathbf{x}) = -\int \Pi_{*l}^{(t)}(\mathbf{x}-\mathbf{y}) \frac{\partial U(\mathbf{y})}{\partial y_l} dy^3 \quad , V_l(\mathbf{x}) = -\int B_{lm}^{(t)}(\mathbf{x}-\mathbf{y}) \frac{\partial U(\mathbf{y})}{\partial y_m} dy^3 \tag{54}$$

The averaged equation (52) is reversible and has the form

$$R_{ml}^{(t)}(\mathbf{k}) V_l(\mathbf{k}) = -2\pi k_m \overline{U}(\mathbf{k}) \tag{55}$$

where $\mathbf{R}^{(t)}(\mathbf{k}) = \left[\mathbf{B}^{(t)}(\mathbf{k})\right]^{-1}$ is the diagonal transversal tensor of resistance.

In x-space coordinate system we have the non-local condition of the momentum balance with a unique kernel in corresponding coordinate system:

$$\int R_{ml}^{(t)}\left(\mathbf{x}-\mathbf{y}\right)V_l\left(\mathbf{y}\right)dy^3=-\frac{\partial U\left(\mathbf{x}\right)}{\partial y_m} \tag{56}$$

In summary, for any orthogonal coordinate systems in the case of isotropy the averaged equation is reversible and the tensors $\overline{\mathbf{B}}^{(i)}\left(k\right)$ and $\mathbf{R}^{(i)}\left(\mathbf{k}\right)$ are spherical. In the case of transversal isotropy if the orthogonal coordinate system is oriented so that one of the axes, for example, k_3, coincides with the axis of rotation, and the other two are oriented arbitrarily, the averaged equation is reversible, the tensors $\overline{\mathbf{B}}^{(t)}\left(\mathbf{k}\right)$ and $\mathbf{R}^{(t)}\left(\mathbf{k}\right)$ are diagonal, and $\overline{\mathrm{B}}_{11}^{(t)}\left(\mathbf{k}\right)=\overline{\mathrm{B}}_{22}^{(t)}\left(\mathbf{k}\right)\neq\overline{\mathrm{B}}_{33}^{(t)}\left(\mathbf{k}\right)$. Finally, in the case of orthotropy, if the axes of the orthogonal coordinate system are the orthotropy axes, the averaged equation is also reversible and the tensors $\overline{\mathbf{B}}^{(o)}\left(\mathbf{k}\right)$ and $\mathbf{R}^{(o)}\left(\mathbf{k}\right)$ are diagonal. However, we shall bear in mind that in each of the studied cases of symmetry, the components of the tensors $\overline{\mathrm{B}}^{(\alpha)}\left(k\right)$, where $\alpha=i,t,o$, remain invariant in relation to the superscript α. Thus, in all three basic cases of symmetry ($\alpha=i,t,o$) with a suitable orientation of the coordinate axes, the averaged equation is

$$\overline{V}_l^{(\alpha)}\left(\mathbf{k}\right)=\overline{\mathrm{B}}_{lm}^{(\alpha)}\left(\mathbf{k}\right)\overline{H}_m\left(\mathbf{k}\right) \tag{57}$$

where

$$\overline{\mathrm{B}}_{lm}^{(\alpha)}\left(\mathbf{k}\right)=\delta_{lm}\overline{\Pi}_{*m}^{(\alpha)}\left(\mathbf{k}\right)\ ,\ \overline{R}_{ml}^{(\alpha)}\left(\mathbf{k}\right)=\delta_{ml}\left[\overline{\Pi}_{*m}^{(\alpha)}\left(\mathbf{k}\right)\right]^{-1} \tag{58}$$

(no summations assumed in (58) over subscript m!)

Equation (57) is reversible and for any α we have

$$\overline{R}_{ml}^{(\alpha)}\left(\mathbf{k}\right)\overline{V}_l\left(\mathbf{k}\right)=\overline{H}_m\left(\mathbf{k}\right) \tag{59}$$

In x-space in corresponding coordinate system, if appropriate convolutions converge, we have the non-local equations with unique kernels

$$V_l^{(\alpha)}(\mathbf{x}) = -\int B_{lm}^{(\alpha)}(\mathbf{x}-\mathbf{y})\frac{\partial U(\mathbf{y})}{\partial y_m}dy^3,\ \int R_{ml}^{(\alpha)}(\mathbf{x}-\mathbf{y})V_l(\mathbf{y})dy^3 = -\frac{\partial U(\mathbf{x})}{\partial x_m} \tag{60}$$

Now we will consider in detail the tensors $\overline{\mathbf{B}}^{(\alpha)}(\mathbf{k})$ and $\mathbf{B}^{(\alpha)}(\mathbf{x})$. Because $\overline{\Pi}_l^{(\alpha)}(\mathbf{k})$ are imaginary and odd functions of vector $\mathbf{k}$, the components of vector $\overline{\mathbf{\Pi}}^{(\alpha)}(\mathbf{k})$ and diagonal tensor $\overline{\mathbf{B}}^{(\alpha)}(\mathbf{k})$ are even and real functions. Now we write the component $\overline{B}_{ll}^{(\alpha)}(\mathbf{k})$ in the following form:

$$\overline{\mathrm{B}}_{ll}^{(\alpha)}(\mathbf{k}) = \hat{\overline{\mathrm{B}}}_{ll}^{(\alpha)}\overline{F}_{ll}^{(\alpha)}(\tilde{\mathbf{k}}) \tag{61}$$

Here $\hat{\overline{B}}_{ll}^{(\alpha)} = \lim \overline{B}_{ll}^{(\alpha)}(\varepsilon\tilde{\mathbf{k}})$ where a positive dimensionless number $\varepsilon \to 0$, the variables $\tilde{k}_l = \Delta_l k_l$ and functions $\overline{F}_{ll}^{(\alpha)}(\tilde{\mathbf{k}})$ are dimensionless as well (Δ_l are linear scales of the random field $\boldsymbol{\sigma}(\mathbf{x})$, for example, the correlation scales). It is evident that the ratios between the scales of heterogeneity remain in the limit, which is the condition of special similarity. Assuming the existence of a *Taylor`s* expansion of the function $\overline{F}_{ll}^{(\alpha)}(\tilde{\mathbf{k}})$, we can write:

$$\overline{\mathrm{B}}_{ll}^{(\alpha)}(\mathbf{k}) = \hat{\overline{\mathrm{B}}}_{ll}^{(\alpha)}\sum_{n=n_1+n_2+n_3=0}^{\infty}\frac{1}{n!}\frac{\partial^n \overline{F}_{ll}^{(\alpha)}(0)}{\partial\tilde{k}_1\partial\tilde{k}_2\partial\tilde{k}_3}k_1^{n_1}k_2^{n_2}k_3^{n_3}\Delta_1^{n_1}\Delta_2^{n_2}\Delta_3^{n_3} \tag{62}$$

Substituting (62) into (57) and taking into account that all the odd derivatives of $\overline{F}_{ll}^{(\alpha)}(\tilde{\mathbf{k}})$ at $\tilde{\mathbf{k}} = 0$ are zero, we can write the expansions for the mean velocity $V_l^{(\alpha)}(\mathbf{x})$ in *x*-space:

$$V_l^{(\alpha)}(\mathbf{x}) = -\hat{\overline{B}}_{ll}^{(\alpha)}\sum_{n=n_1+n_2+n_3=0}^{\infty}\frac{(-1)^n\Delta_1^{2n_1}\Delta_2^{2n_2}\Delta_3^{2n_3}}{(2n)!(2\pi)^{2n}}\frac{\partial^{2n}\overline{F}_{ll}^{(\alpha)}(0)}{\partial\tilde{k}_1^{2n_1}\partial\tilde{k}_2^{2n_2}\partial\tilde{k}_3^{2n_3}}\frac{\partial^{2n+1}U(\mathbf{x})}{\partial x_l\partial x_1^{2n_1}\partial x_2^{2n_2}\partial x_3^{2n_3}} \tag{63}$$

$$V_l^{(\alpha)}(\mathbf{x}) = -\hat{\overline{\mathrm{B}}}_{ll}^{(\alpha)}\sum_{n=n_1+n_2+n_3=0}^{\infty}\frac{\Delta_1^{2n_1}\Delta_2^{2n_2}\Delta_3^{2n_3}\mathrm{I}_{ll,n}^{(\alpha)}(n_1,n_2,n_3)}{(2n)!}\frac{\partial^{2n+1}U(\mathbf{x})}{\partial x_l\partial x_1^{2n_1}\partial x_2^{2n_2}\partial x_3^{2n_3}} \tag{64}$$

Here, the power moment of the dimensionless function $F_{ll}^{(\alpha)}(\tilde{\mathbf{y}}) = T_F^{-1}\left[\overline{F}_{ll}^{(\alpha)}(\tilde{\mathbf{k}})\right]$ of the dimensionless variables $\tilde{y}_l = x_l/\Delta_l$ is $\mathrm{I}_{ll,n}^{(\alpha)}(n_1,n_2,n_3) = \int \tilde{y}_1^{2n_1}\tilde{y}_2^{2n_2}\tilde{y}_3^{2n_3}F_{ll}^{(\alpha)}(\tilde{\mathbf{y}})d\tilde{y}^3$.

It is evident that from the equation (61) we have $\overline{F}_{ll}^{(\alpha)}(0)=1$ and then $\mathrm{I}_{ll,0}^{(\alpha)}(0,0,0)=1$. The important question is: What is the behavior of the expansions (63) or (64) in the limiting case when $\Delta_l \to 0$, and for any l and m the ratio $\Delta_l / \Delta_m = const$, that corresponds to the theory of homogenization and the concept of effective conductivity (see Bakhvalov and Panasenko, 1984; Zhikov *et al.*, 1993)? We should note that in the first terms of both expansions (n=0), the coefficients of the derivatives do not contain Δ_l explicitly. By setting some restrictions to the density $f(\mathbf{x})$, the behavior of the leading derivatives of $U(\mathbf{x})$ can be sufficiently limited. All the other terms of these expansions tend to zero for $\Delta_l \to 0$. In this limiting case we have the averaged equation:

$$V_l^{(\alpha)}(\mathbf{x}) = -\hat{\overline{\mathrm{B}}}_{ll}^{(\alpha)} H_l(\mathbf{x}) \ , \ H_l(\mathbf{x}) = -\frac{\partial U(\mathbf{x})}{\partial x_l} \tag{65}$$

where $\hat{\overline{\mathrm{B}}}_{ll}^{(\alpha)} = const$ are the diagonal components of the effective conductivity tensor. Notice that according to the theory of homogenization, the tensor of the effective conductivity exists and is constant in all Euclidian space R^3. This is true if, for any limited domain $Q \subset R^3$, the source density function $f(\mathbf{x})$ belongs to Sobolev functional space $H^{-1}(Q)$ or to the square integrable functional space $L^2(Q)$, that is embedded in $H^{-1}(Q)$ space. Furthermore, if in any orthogonal coordinate system, the tensor of the local random conductivity is symmetric and positive definite, the tensor of the effective conductivity is also symmetric and positive definite, the so-called elliptic (see Zhikov *et al.*, 1993). Thus, the principal part of the expansions (63) and (64) corresponds with the theory of homogenization limit and can be used for computing the effective conductivity. In the cases of symmetry: isotropic $(\alpha = i)$, transversal isotropy $(\alpha = t)$ and orthotropy $(\alpha = o)$ for appropriate coordinate systems, the averaged equation has the form:

$$\mathbf{V}^{(\alpha)}(\mathbf{x}) = \overline{\mathbf{B}}_*^{(\alpha)} \mathbf{H}(\mathbf{x}) \ , \ \overline{\mathbf{R}}_*^{(\alpha)} \mathbf{V}^{\alpha}(\mathbf{x}) = \mathbf{H}(\mathbf{x}) \tag{66}$$

where the diagonal tensors of the effective conductivity $\overline{\mathrm{B}}_{*ml}^{(\alpha)} = \delta_{ml}\hat{\overline{\mathrm{B}}}_{ll}^{(\alpha)}$ and effective resistance $\overline{R}_{*lm}^{(\alpha)} = \delta_{lm}\left[\hat{\overline{B}}_{ll}^{(\alpha)}\right]^{-1}$ are

$$\overline{\mathbf{B}}_{*}^{(\alpha)} = \begin{pmatrix} \hat{\overline{\mathrm{B}}}_{11}^{(\alpha)} & 0 & 0 \\ 0 & \hat{\overline{\mathrm{B}}}_{22}^{(\alpha)} & 0 \\ 0 & 0 & \hat{\overline{\mathrm{B}}}_{33}^{(\alpha)} \end{pmatrix}, \quad \overline{\mathbf{R}}_{*}^{(\alpha)} = \begin{pmatrix} \left[\hat{\overline{B}}_{11}^{(\alpha)}\right]^{-1} & 0 & 0 \\ 0 & \left[\hat{\overline{B}}_{22}^{(\alpha)}\right]^{-1} & 0 \\ 0 & 0 & \left[\hat{\overline{B}}_{33}^{(\alpha)}\right]^{-1} \end{pmatrix} \tag{67}$$

Clearly the principal axes for all tensors $\overline{\mathbf{B}}^{(\alpha)}(\mathbf{k})$, $\overline{\mathbf{B}}_{*}^{(\alpha)}$ and $\overline{\mathbf{R}}^{(\alpha)}(\mathbf{k}), \overline{\mathbf{R}}_{*}^{(\alpha)}$ for any $\mathbf{k}$, for each α are identical to the respective coordinate axes.

Here we underline that Abramovich and Indelman (1994) presented equations (15) and (16) that resemble the first equations in (60) and (66) in the present paper. However, they are not equivalent but are dissimilar. In the first place, as we discussed in Section 1, the variant of perturbation method developed by Abramovich and Indelman (1994) is ineligible for finding the exact solution. Secondly, the definition of effective conductivity used by them as the conductivity of stochastically homogeneous media for mean uniform flow is senseless for any dimensional unbounded media, because the uniform flow in this media does not exist.

Up to this point, we have studied the fields with some symmetry in special orthogonal coordinate systems. If the orthogonal coordinate axes x_l' and k_l' are oriented arbitrarily and β_{lm} is the cosine of the angle between the axes x_l' and x_m, the effective conductivity tensor in the new coordinate system is $\overline{\mathbf{B}}_{*}'^{(\alpha)} = \boldsymbol{\beta}\,\overline{\mathbf{B}}_{*}^{(\alpha)}\boldsymbol{\beta}^{-1}$. This tensor is symmetric and positive definite (elliptic). The averaged equations in the arbitrary coordinate system x_l' have the forms $\mathbf{V}'^{(\alpha)}(\mathbf{x}') = -\overline{\mathbf{B}}_{*}'^{(\alpha)}\nabla U'(\mathbf{x}')$, $\overline{\mathbf{R}}_{*}'^{(\alpha)}\mathbf{V}'^{(\alpha)}(\mathbf{x}) = -\nabla U'(\mathbf{x}')$ and in the $\mathbf{k}'$-space we have $\overline{V}_l'^{(\alpha)}(\mathbf{k}') = -\hat{\overline{B}}_{lm}'^{(\alpha)} 2\pi\, jk_m'\overline{U}'(\mathbf{k}')$ and $\hat{\overline{R}}_{*ml}'^{(\alpha)}\overline{V}_l'^{(\alpha)}(\mathbf{k}') = -2\pi\, jk_m'\overline{U}'(\mathbf{k}')$. Evidently for $\overline{\Pi}_l'^{(\alpha)}(\mathbf{k}')$ we have a linear expression $\overline{\Pi}_l'^{(\alpha)}(\mathbf{k}') = -\hat{\overline{B}}_{*lm}'^{(\alpha)} 2\pi\, jk_m'$.

But what if the diagonal tensor $\overline{\mathbf{B}}_{*}^{(\alpha)}$ is unknown? Or to put it more precisely, what if we know that there exists some symmetry but the orientation of the principal axes is

unknown and the parameter α is unknown? In this case, we return to the Equation (27), which is valid for any stochastically homogeneous positive definite random fields $\boldsymbol{\sigma}(\mathbf{x})$, and study the vector $\bar{\Pi}'_l(\mathbf{k}')$ again and its formal Taylor expansion about $\mathbf{k}'=0$:

$$\bar{\Pi}'_l(\mathbf{k}') = \sum_{n=n_1+n_2+n_3=0}^{\infty} \frac{1}{n!} \frac{\partial^n \bar{\Pi}'_l(0)}{\partial k_1'^{n_1} \partial k_2'^{n_2} \partial k_3'^{n_3}} k_1'^{n_1} k_2'^{n_2} k_3'^{n_3} \tag{68}$$

The component $\bar{\Pi}'_l(\mathbf{k}')$ is an odd function of $\mathbf{k}'$ and therefore at point $\mathbf{k}'=0$ all even derivatives are zero. Thus,

$$\bar{\Pi}'_l(\mathbf{k}') = \sum_{2n-1=n_1+n_2+n_3=1}^{\infty} \frac{1}{(2n-1)!} \frac{\partial^{2n-1} \bar{\Pi}'_l(0)}{\partial k_1'^{n_1} \partial k_2'^{n_2} \partial k_3'^{n_3}} k_1'^{n_1} k_2'^{n_2} k_3'^{n_3} \tag{69}$$

The linear part of this expansion on variable $\mathbf{k}'$ is:

$$\bar{\Pi}'^1_l(\mathbf{k}') = \frac{\partial \bar{\Pi}'_l(0)}{\partial k'_m} k'_m \tag{70}$$

Inserting (70) in (27), we can write the linear approximation on $\mathbf{k}'$ for $\bar{V}'_l(\mathbf{k}')$:

$$\bar{V}'_l(\mathbf{k}') = -\left[-\frac{1}{2\pi j} \frac{\partial \bar{\Pi}'_l(0)}{\partial k'_m} \right] 2\pi j k'_m \bar{U}'(\mathbf{k}') \tag{71}$$

By imposing some restrictions on the density $f(\mathbf{x})$ as discussed earlier, the leading terms of the expansions of $\bar{V}_l(\mathbf{k}')$ vanish in the homogenization limit. In this case, we have the averaged equation:

$$V'_l(\mathbf{x}') = -\bar{\mathrm{B}}'_{*lm} \frac{\partial U'(\mathbf{x}')}{\partial x'_m} \tag{72}$$

and in the general case from (71), we find the real tensor of the effective conductivity, that is symmetric and positive definite

$$\bar{\mathrm{B}}'_{*lm} = -\frac{1}{2\pi j} \frac{\partial \bar{\Pi}'_l(0)}{\partial k'_m} \tag{73}$$

Thus, if we know the components $\bar{\Pi}'_l(\mathbf{k}')$, we can from Equation (73) find the effective conductivity tensor; and by using the standard method, we can find its real eigenvalues

and the orthogonal eigenvectors. Transition to a new orthogonal system associated with the eigenvectors and transformation of the tensor $\overline{\mathrm{B}}'_{*lm}$ to the new coordinates lead to a diagonal tensor $\overline{\mathrm{B}}_{*ml}$, whose components are the eigenvalues for tensor $\overline{\mathrm{B}}'_{*lm}$. As we mentioned earlier, for each α in the new coordinate system the tensor $\overline{\mathrm{B}}_{lm}(\mathbf{k})$ is diagonal with the following components:

$$\overline{\mathrm{B}}_{ll}(\mathbf{k}) = -\overline{\Pi}_l(\mathbf{k}) / 2\pi j k_l \ , \ \overline{B}_{lm}(\mathbf{k}) = 0 \text{, if } \ l \neq m \tag{74}$$

(no summation over l!)

It is clear that the diagonal tensor $\overline{\mathbf{B}}(\mathbf{k})$ is unique and reversible.

6. DIFFERENT APPROACH

Majority of the works related to the present subject used a different approach. From the outset, many attempted to find the connections (algebraic or, more general, operator related) between the averaged flow velocity field and the gradient of mean pressure (head). To examine the validity of this approach, we return to the Equation (27). To recast it to the form like the Darcy`s Law, we introduce some tensor $\overline{B}_{lm}(\mathbf{k})$ that satisfies the equation:

$$\overline{\Pi}_l(\mathbf{k}) = -\overline{B}_{lm}(\mathbf{k}) 2\pi j k_m \tag{75}$$

and after inserting Equation (75) into (27) we have the following equation:

$$\overline{V}_l(\mathbf{k}) = -\overline{B}_{lm}(\mathbf{k}) 2\pi j k_m \overline{U}(k) \tag{76}$$

In $\mathbf{x}$-space, if the convolution converges, we have

$$V_l(\mathbf{x}) = -\int B_{lm}(\mathbf{x}-\mathbf{y}) \frac{\partial U(\mathbf{y})}{\partial y_m} dy^3 \tag{77}$$

If we insert Equation (76) into (29), the condition of compatibility for components vector $\overline{\mathbf{\Pi}}(\mathbf{k})$, we obtain the condition of compatibility for the components of tensor $\overline{B}_{lm}(\mathbf{k})$

$$4\pi^2 \overline{G}(\mathbf{k}) k_l \overline{B}_{lm}(\mathbf{k}) k_m = 1 \tag{78}$$

Assuming that tensor $\bar{\mathbf{B}}(\mathbf{k})$ is nonsingular, we can rewrite the equation (75) in the form

$$\bar{\mathbf{R}}(\mathbf{k})\bar{\mathbf{\Pi}}(\mathbf{k}) = -2\pi j\mathbf{k} \quad , \ \bar{\mathbf{R}}(\mathbf{k}) = \left[\bar{\mathbf{B}}(\mathbf{k})\right]^{-1} \tag{79}$$

Multiply the first equation from (79) by $\bar{U}(\mathbf{k})$, and taking into account the equation (27), we can write the averaged equation

$$\bar{\mathbf{R}}(\mathbf{k})\bar{\mathbf{V}}(\mathbf{k}) = -2\pi j\mathbf{k}\bar{U}(\mathbf{k}) \tag{80}$$

In $\mathbf{x}$-space, if the convolution converges, we have the conservative averaged equation of momentum balance

$$\int \mathbf{R}(\mathbf{x}-\mathbf{y})\mathbf{V}(\mathbf{x}-\mathbf{y})dy^3 = -\nabla U(\mathbf{x}) \tag{81}$$

The definition of the Fourier transformation $\bar{\mathbf{B}}(\mathbf{k})$ with the system (75) or $\bar{\mathbf{R}}(\mathbf{k})$ with system (79) leads to three linear algebraic equations for each $\mathbf{k}$ and each equation contains three from nine unknown components $\bar{B}_{ml}(\mathbf{k})$ or $\bar{R}_{lm}(\mathbf{k})$. In the x-space, this problem amounts to three differential equations with nine unknown function-components $B_{lm}(\mathbf{x})$ or three operator equations for the unknown nine functions $R_{lm}(\mathbf{x})$.

$$\Pi_l(\mathbf{x}) = -\frac{\partial B_{lm}(\mathbf{x})}{\partial x_m} \quad , \int R_{lm}(\mathbf{x}-\mathbf{y})\Pi_l(\mathbf{y})dy^3 = -\frac{\partial}{\partial x_m} \tag{82}$$

Both systems of Equation (75) and (79) and systems of Equations (82) are underdetermined and in general have unlimited sets of solutions.

It is well known that $\bar{\mathbf{B}}_{\otimes}(\mathbf{k})$ -the general solution (all infinite set of solutions) for a singular non-uniform system of linear algebraic equations can be presented as a sum of any particular solution of system $\bar{\mathbf{B}}_0(\mathbf{k})$ and $\bar{\mathbf{B}}_*(\mathbf{k})$, which is any solution of the uniform system $\bar{\mathbf{B}}_*(\mathbf{k})\mathbf{k} = 0$ (The geometric sense of the uniform system is that all three vector-rows for the tensor $\bar{\mathbf{B}}_*(\mathbf{k})$ are orthogonal to the vector $\mathbf{k}$). For this reason as indicated by Indelman and Abramovich, *1994*, any of the solutions $\bar{\mathbf{B}}_*(\mathbf{k})$ in Equation (76) with known vector $2\pi j\mathbf{k}\bar{U}(\mathbf{k})$, (i.e., *Fourier*-transformation of $\nabla U(\mathbf{x})$) do not affect the computation of $\bar{\mathbf{V}}(\mathbf{k})$.

When tensors $\bar{\mathbf{B}}_{\otimes}(\mathbf{k})$ are not singular, we can obtain the tensors $\bar{\mathbf{R}}_{\otimes}(\mathbf{k})=\left[\bar{\mathbf{B}}_{\otimes}(\mathbf{k})\right]^{-1}$, the general solutions of Equation (79), which are also not unique. In this case $\mathbf{R}_{\otimes}(\mathbf{k})$ can be written as a sum of any particular solution $\bar{\mathbf{R}}_0(\mathbf{k})$ and $\bar{\mathbf{R}}_*(\mathbf{k})$ - any solution of uniform system $\bar{\mathbf{R}}_*(\mathbf{k})\bar{\mathbf{\Pi}}(\mathbf{k})=0$. (The geometric sense of the uniform system is that all vector-rows of the tensor $\bar{\mathbf{R}}_*(\mathbf{k})$ are orthogonal to the vector $\bar{\mathbf{\Pi}}(\mathbf{k})$). If we select any particular solution $\bar{\mathbf{B}}_0(\mathbf{k})$, we can find $\bar{\mathbf{R}}_0(\mathbf{k})=\left[\bar{\mathbf{B}}_0(\mathbf{k})\right]^{-1}$. The corresponding tensor $\bar{\mathbf{R}}_*(\mathbf{k})=-\bar{\mathbf{R}}_0(\mathbf{k})\bar{\mathbf{B}}_*(\mathbf{k})\left[\bar{\mathbf{B}}_0(\mathbf{k})+\bar{\mathbf{B}}_*(\mathbf{k})\right]^{-1}$ does not affect the computation of $2\pi j\mathbf{k}U(\mathbf{k})$ from Equation (80) or $\nabla U(\mathbf{x})$ from Equation (81).

On the other hand, in the case of global symmetry as shown above, in corresponding coordinate system, the number of unknown functions reduces to three or less, and it is possible to find a unique solution, the diagonal tensor $\bar{\mathbf{B}}_0(\mathbf{k})$. In fact, if symmetry is global, the tensor $\bar{\mathbf{B}}_*(\mathbf{k})$ is diagonal also, and therefore $\bar{\mathbf{B}}_*(\mathbf{k})=0$. Even if there is no reason to believe that the types of the global symmetry discussed above exist, the fact remains that if the stochastically homogeneous field of the local random conductivity tensor $\sigma_{ml}(\mathbf{x})$ is symmetric and elliptic, the tensor of the effective conductivity $\hat{\bar{\mathrm{B}}}_{ml}$ is symmetric and elliptic as well. In this general case the "eigen" orthogonal coordinate system exists in which the tensor $\hat{\bar{\mathbf{B}}}$ is diagonal.

In a general case it would seem reasonable to find (for this coordinate system) special solutions to the system of Equation (75) and the uniform system as diagonal tensors for all $\mathbf{k}$, that are simple and convenient for matching and identification. Then, we select solutions $\bar{\tilde{\mathbf{B}}}(\mathbf{k})=0$ for the uniform system. In this case, we find diagonal tensors $\bar{\mathbf{B}}(\mathbf{k})$ and $\bar{\mathbf{R}}(\mathbf{k})=\left[\bar{\mathbf{B}}(\mathbf{k})\right]^{-1}$ to be:

$$\bar{B}_{ll}(\mathbf{k})=-\bar{\Pi}_l(\mathbf{k})/2\pi jk_l\ ,\ \bar{B}_{lm}(\mathbf{k})=0\ \text{ if }\ l\neq m$$

$$\bar{R}_{ll}(\mathbf{k})=-2\pi jk_l/\bar{\Pi}_l(\mathbf{k})\ ,\bar{R}_{lm}(\mathbf{k})=0\ \text{ if }\ l\neq m \tag{83}$$

(no summation over l!)

Note that if we use any orthogonal coordinate system k_l'' that is different from the "eigen" system, and write:

$$\bar{\mathrm{B}}_{ll}''\left(\mathbf{k}''\right) = -\bar{\Pi}_l''\left(\mathbf{k}''\right)/2\pi j k_l'' \quad , \quad \bar{\mathrm{B}}_{lm}''\left(\mathbf{k}''\right) = 0 \;\; if \;\; l \neq m \tag{84}$$

(no summation over l!)

which is also an exact solution of the system defined by Equation (75), we can see that the limit of $\bar{\mathrm{B}}_{ll}\left(\mathbf{k}''\right)$ does not exist when $\mathbf{k}'' \to 0$. In fact, inserting the linear part of the expansion $\bar{\Pi}_l\left(\mathbf{k}''\right)$ in the form of (70) into (72), we have:

$$\bar{\mathrm{B}}_{ll}''\left(\mathbf{k}''\right) = \frac{1}{2\pi j}\frac{\partial \bar{\Pi}_l''(0)}{\partial k_m''}\frac{k_m''}{k_l''} \tag{85}$$

(Here, the summation over m only is implied!)

Because $\dfrac{\partial \bar{\Pi}_l''(0)}{\partial k_m''} \neq 0$ for $l \neq m$ and because the $\mathbf{k}_l''$ coordinate system is not eigen, as $\mathbf{k}'' \to 0$ the $\lim \bar{\mathrm{B}}_{ll}''\left(\mathbf{k}''\right)$ does not exist.

Therefore in Equation (83) we have solutions that are exact, continuous, and reversible with the formulated constrains.

To find the continuous, exact, and reversible tensor- solution of system (75) in any orthogonal coordinate system, it should be from Equation (73): Compute the tensor of the effective conductivity and find the eigenvalue and eigenvectors and $\boldsymbol{\beta}_*$-a matrix of transition from original coordinate system to eigen system. Then we need to find the diagonal tensor of the effective conductivity $\hat{\bar{\mathbf{B}}}_*$ within "eigen" system and, from the system of Equations (83) find components for the diagonal tensor $\bar{\mathbf{B}}\left(\mathbf{k}\right)$. Finally, we return to the initial coordinate system using the matrix $\boldsymbol{\beta}_*^{-1}$.

As we showed in Section 5, this method in the case of global symmetry leads to exact, unique, and reversible solution. The difference here lies in the fact that in the case of global symmetry, the components of the diagonal tensor are dependent on invariants, which are related to the type of symmetry. It is evident that finding the solutions

$\bar{\mathbf{B}}(\mathbf{k})$ and $\bar{\mathbf{R}}(\mathbf{k})$, we cannot assume that for any **k,** these tensors must be elliptic when $\mathbf{k} \neq 0$.

Now let us compare the approaches discussed above and estimate their adequacy and utility for describing averaged flow and studying appropriate direct and inverse problems. We shall call the approach presented in the paper ‘’P’’and the different approach ‘’D’’.

1. Describing averaged flow in Fourier-space **k** leads to linear algebraic equations for fields $\bar{\mathbf{V}}(\mathbf{k})$ and $\bar{U}(\mathbf{k})$ that contain the vector $\bar{\mathbf{\Pi}}(\mathbf{k})$ in P-approach and tensor $\bar{\mathbf{B}}(\mathbf{k})$ in D-approaches.

2. Under P, for each conductivity field there exists a unique vector $\bar{\mathbf{\Pi}}(\mathbf{k})$ or vector-operator $\mathbf{\Pi}(\mathbf{x})$ in **x** -space.

3. Under D, for each conductivity field an unlimited set of tensors $\bar{\mathbf{B}}(\mathbf{k})$ or operators $\mathbf{B}(\mathbf{x})$ in **x**-space exists. Any of them can be used to compute the exact mean velocity vector. Actually for this operation, we utilize the P.

4. In x-space, P leads to the equation that relates the mean velocity field to the pressure (head) field. But D relates the mean velocity field with the gradient of mean pressure (head).

5. These equations are nonlocal, contain convolutions, and are valid if the convolutions converge. The kernels are a vector-operator $\mathbf{\Pi}(\mathbf{x})$ in P and a tensor-operator $\bar{\mathbf{B}}(\mathbf{x})$ in D.

6. In P, to solve the inverse problem for finding the vector $\bar{\mathbf{\Pi}}(\mathbf{k})$ exactly and uniquely, it is sufficient to know a pair of fields, $\bar{U}(\mathbf{k})$ and $\bar{\mathbf{V}}(\mathbf{k})$**,** which are consistent with the density $\bar{f}(\mathbf{k})$.

7. In D, the inverse problem for finding tensor $\bar{\mathbf{B}}(\mathbf{k})$ in general is ill-posed.

7. Two-Dimensional and One-Dimensional Steady-State Flow in an Unbounded Domain

In Section 2, we briefly discussed the case of two-dimensional and one-dimensional flow. Now we will examine it in detail.

We consider the system of equations, that does not contain the function $u(\mathbf{x})$:

$$\frac{\partial v_l(\mathbf{x})}{\partial x_l} = f(\mathbf{x}) \tag{86}$$

$$v_l(\mathbf{x}) = \sigma_{lm}(\mathbf{x}) h_m(\mathbf{x}) \tag{87}$$

$$rot\,\mathbf{h}(\mathbf{x}) = 0 \tag{88}$$

In two-dimensional space, this system includes four scalar independent equations for four scalar functions and is closed. In an unbounded domain, we consider conditions

$$\lim v_l(\mathbf{x}) = 0 \text{ if } x \to \infty \tag{89}$$

and therefore we have from equation (87)

$$\lim h_l(\mathbf{x}) = 0 \text{ if } x \to \infty \tag{90}$$

It is evident that if function $f(\mathbf{x})$ is integrable and has bounded support, the system has a unique solution.

Now we consider an auxiliary system for Green`s velocity $\mathbf{v}_*(\mathbf{x},\mathbf{y})$ and intensity $\mathbf{h}_*(\mathbf{x},\mathbf{y})$:

$$\frac{\partial v_{*l}(\mathbf{x},\mathbf{y})}{\partial x_l} = \delta(\mathbf{x}-\mathbf{y}) \tag{91}$$

$$v_{*l}(\mathbf{x},\mathbf{y}) = \sigma_{lm}(\mathbf{x}) h_{*m}(\mathbf{x},\mathbf{y}) \tag{92}$$

$$\frac{\partial h_{*1}(\mathbf{x},\mathbf{y})}{\partial x_2} - \frac{\partial h_{*2}(\mathbf{x},\mathbf{y})}{\partial x_1} = 0 \tag{93}$$

$$\lim v_{*l}(\mathbf{x},\mathbf{y}) = 0 \;, \quad \lim h_{*l}(\mathbf{x},\mathbf{y}) = 0 \quad \text{if } x \to \infty \tag{94}$$

and we can write

$$\mathbf{h}(\mathbf{x}) = \int \mathbf{h}_*(\mathbf{x},\mathbf{y}) f(\mathbf{y}) dy^2 \quad , \quad \mathbf{v}(\mathbf{x}) = \int \mathbf{v}_*(\mathbf{x},\mathbf{y}) f(\mathbf{y}) dy^2 \tag{95}$$

After averaging the vector-functions $\mathbf{h}_*(\mathbf{x},\mathbf{y}), \mathbf{v}_*(\mathbf{x},\mathbf{y})$, and taking into account that the tensor-field $\boldsymbol{\sigma}(\mathbf{x})$ is assumed to be stochastically homogeneous, we have

$$\langle \mathbf{h}_*(\mathbf{x},\mathbf{y}) \rangle = \mathbf{H}_*(\mathbf{x}-\mathbf{y}), \langle \mathbf{v}_*(\mathbf{x},\mathbf{y}) \rangle = \mathbf{V}_*(\mathbf{x}-\mathbf{y}) \tag{96}$$

Thus, after averaging the equations in (95) we can write

$$\mathbf{H}(\mathbf{x}) = \int \mathbf{H}_*(\mathbf{x}-\mathbf{y}) f(\mathbf{y}) dy^2 , \mathbf{V}(\mathbf{x}) = \int \mathbf{V}_*(\mathbf{x}-\mathbf{y}) f(\mathbf{y}) dy^2 \tag{97}$$

Although the fields $\mathbf{H}_*(\mathbf{x}), \mathbf{H}(\mathbf{x})$ and $\mathbf{V}_*(\mathbf{x}), \mathbf{V}(\mathbf{x})$ vanish at infinity, they are not absolute integrable in all space. Nevertheless the convolutions in (97) converge because the function $f(\mathbf{x})$ has bounded support. Now, using the generalized Fourier transform (19) for equations (97) we have

$$\bar{\mathbf{H}}(\mathbf{k}) = \bar{\mathbf{H}}_*(\mathbf{k}) \bar{f}(\mathbf{k}), \bar{\mathbf{V}}(\mathbf{k}) = \bar{\mathbf{V}}_*(\mathbf{k}) \bar{f}(\mathbf{k}) \tag{98}$$

Because the components of the vectors $H_{*l}(\mathbf{x})$ and $V_{*l}(\mathbf{x})$ are odd functions of $\boldsymbol{x}$, the components $\bar{H}_{*l}(\mathbf{k})$ and $\bar{V}_{*l}(\mathbf{k})$ are imaginary functions of k **.** Then,

$$\bar{\mathbf{H}}_*(\mathbf{k}) = j\hat{\bar{\mathbf{H}}}_*(\mathbf{k}), \bar{\mathbf{V}}_*(\mathbf{k}) = j\hat{\bar{\mathbf{V}}}_*(\mathbf{k}) \tag{99}$$

Here $\hat{\bar{\mathbf{H}}}_*(\mathbf{k})$ and $\hat{\bar{\mathbf{V}}}_*(\mathbf{k})$ are real vectors.

For any function $f(\mathbf{x})$ we can write $f(\mathbf{x}) = f_1(\mathbf{x}) + f_2(\mathbf{x})$ where $f_1(\mathbf{x}) = [f(\mathbf{x}) + f(-\mathbf{x})]/2$ is an even function and $f_2(\mathbf{x}) = [f(\mathbf{x}) - f(-\mathbf{x})]/2$ is an odd function. Then we have $\bar{f}(\mathbf{k}) = \bar{f}_1(\mathbf{k}) + j\hat{\bar{f}}_2(\mathbf{k})$ where $\bar{f}_1(\mathbf{k})$ is a real even function and $\hat{\bar{f}}_2(\mathbf{k})$ is an odd real function. Now we have the system

$$\bar{\mathbf{H}}(\mathbf{k}) = j\hat{\bar{\mathbf{H}}}^{(1)}(\mathbf{k}) + \bar{\mathbf{H}}^{(2)}(\mathbf{k}), \bar{\mathbf{V}}(\mathbf{k}) = j\hat{\bar{\mathbf{V}}}^{(1)}(\mathbf{k}) + \bar{\mathbf{V}}^{(2)}(\mathbf{k}) \tag{100}$$

$$\hat{\bar{\mathbf{H}}}^{(1)}(\mathbf{k}) = \hat{\bar{\mathbf{H}}}_*(\mathbf{k}) \bar{f}_1(\mathbf{k}) \quad , \quad \hat{\bar{\mathbf{H}}}^{(2)}(\mathbf{k}) = -\hat{\bar{\mathbf{H}}}_*(\mathbf{k}) \hat{\bar{f}}_2(\mathbf{k}) \tag{101}$$

$$\hat{\bar{\mathbf{V}}}^{(1)}(\mathbf{k}) = \hat{\bar{\mathbf{V}}}_*(\mathbf{k}) \bar{f}_1(\mathbf{k}) \quad , \quad \hat{\bar{\mathbf{V}}}^{(2)}(\mathbf{k}) = -\hat{\bar{\mathbf{V}}}_*(\mathbf{k}) \hat{\bar{f}}_2(\mathbf{k}) \tag{102}$$

According to the plus or minus sign of the function $\bar{f}_1(\mathbf{k})$, the vectors $\hat{\bar{\mathbf{H}}}_*(\mathbf{k})$ and $\hat{\bar{\mathbf{H}}}^{(1)}(\mathbf{k})$ and (similarly) vectors $\hat{\bar{\mathbf{V}}}_*(\mathbf{k})$ and $\hat{\bar{\mathbf{V}}}_*^{(1)}(\mathbf{k})$ are parallel or antiparallel.

Therefore, for any **k,** the angle $\beta(\mathbf{k})$ between the vectors $\hat{\bar{\mathbf{H}}}_*(\mathbf{k})$ and $\hat{\bar{\mathbf{V}}}_*(\mathbf{k})$ is the same as the angle $\beta^{(1)}(\mathbf{k})$ between vectors $\hat{\bar{\mathbf{H}}}^{(1)}(\mathbf{k})$ and $\hat{\bar{\mathbf{V}}}^{(1)}(\mathbf{k})$. From the equations (101) and (102) we then have

$$\hat{\bar{V}}^{(1)}(\mathbf{k})/\hat{\bar{H}}^{(1)}(\mathbf{k}) = \hat{\bar{V}}_*(\mathbf{k})/\hat{\bar{H}}_*(\mathbf{k}) = \bar{\lambda}(\mathbf{k}) \tag{103}$$

where $\bar{\lambda}(\mathbf{k})$ is a positive scalar function.

Now we introduce the right-hand rotation from vector $\hat{\bar{\mathbf{H}}}_*(\mathbf{k})$ to vector $\hat{\bar{\mathbf{V}}}_*(\mathbf{k})$ - real tensor $\bar{\boldsymbol{\alpha}}(\mathbf{k})$:

$$\bar{\boldsymbol{\alpha}}(\mathbf{k}) = \begin{pmatrix} \cos\beta(\mathbf{k}) & -\sin\beta(\mathbf{k}) \\ \sin\beta(\mathbf{k}) & \cos\beta(\mathbf{k}) \end{pmatrix} \tag{104}$$

where $\cos\beta(\mathbf{k}) = \left[\hat{\bar{H}}_{*1}(\mathbf{k})\hat{\bar{V}}_{*1}(\mathbf{k}) + \hat{\bar{H}}_{*2}(\mathbf{k})\hat{\bar{V}}_{*2}(\mathbf{k})\right]\hat{\bar{H}}_*^{-2}(\mathbf{k})\hat{\bar{V}}_*^{-2}(\mathbf{k})$ and

$$\sin\beta(\mathbf{k}) = \left[\hat{\bar{H}}_{*2}(\mathbf{k})\hat{\bar{V}}_{*1}(\mathbf{k}) - \hat{\bar{H}}_{*1}(\mathbf{k})\hat{\bar{V}}_{*2}(\mathbf{k})\right]\hat{\bar{H}}_*^{-2}(\mathbf{k})\hat{\bar{V}}_*^{-2}(\mathbf{k})$$

Multiplying $j\bar{\lambda}(\mathbf{k})\alpha(\mathbf{k})$ by the vector $\hat{\bar{\mathbf{H}}}^{(1)}(\mathbf{k})$, we find

$$\hat{\bar{\mathbf{V}}}^{(1)}(\mathbf{k}) = \hat{\bar{\mathbf{B}}}(\mathbf{k})\hat{\bar{\mathbf{H}}}^{(1)}(\mathbf{k}) \quad , \qquad \hat{\bar{\mathbf{B}}}(\mathbf{k}) = \bar{\lambda}(\mathbf{k})\boldsymbol{\alpha}(\mathbf{k}) \tag{105}$$

Repeating a similar operation for fields $\hat{\bar{\mathbf{H}}}^{(2)}(\mathbf{k})$ and $\hat{\bar{\mathbf{V}}}^{(2)}(\mathbf{k})$, we have

$$\hat{\bar{\mathbf{V}}}^{(2)}(\mathbf{k}) = \hat{\bar{\mathbf{B}}}(\mathbf{k})\hat{\bar{\mathbf{H}}}^{(2)}(\mathbf{k}) \tag{106}$$

Combining (105) and (106) with (100), we have two equivalent forms of averaged equations

$$\bar{\mathbf{V}}(\mathbf{k}) = \hat{\bar{\mathbf{B}}}(\mathbf{k})\bar{\mathbf{H}}(\mathbf{k}) \quad , \quad \bar{\mathbf{R}}(\mathbf{k})\bar{\mathbf{V}}(\mathbf{k}) = \bar{\mathbf{H}}(\mathbf{k}),\ \bar{\mathbf{R}}(\mathbf{k}) = \left[\hat{\bar{\mathbf{B}}}(\mathbf{k})\right]^{-1} \tag{107}$$

It should be noted that using the left hand coordinate system and left hand rotation tensor leads to different tensors $\breve{\bar{\mathbf{B}}}(\mathbf{k})$ and $\breve{\bar{\mathbf{R}}}(\mathbf{k})$, but does not affect calculation of $\bar{\mathbf{V}}(\mathbf{k})$ or $\bar{\mathbf{H}}(\mathbf{k})$. Consequently, we can write the two equivalent forms of averaged equations as

$$2\pi j\mathbf{k}\bar{\mathbf{V}}(\mathbf{k}) = \bar{f}(\mathbf{k}) \text{ or } 2\pi\mathbf{k}\hat{\bar{\mathbf{B}}}(\mathbf{k})\bar{\mathbf{H}}(\mathbf{k}) = \bar{f}(\mathbf{k}) \tag{108}$$

and add an equation- that is corollary to equations (93), (95), (96) and (97):

$$k_1\bar{H}_2(\mathbf{k})-k_2\bar{H}_1(\mathbf{k})=0 \tag{109}$$

We then have a closed system of averaged equations.

Eliminating $\bar{V}_l(\mathbf{k})$ from the system, we find $\bar{H}_l(\mathbf{k})$ and then $\bar{V}_l(\mathbf{k})$;

$$\bar{H}_l(\mathbf{k})=-2\pi k_l\bar{\Phi}(\mathbf{k})\,\bar{f}(\mathbf{k}) \quad , \quad \bar{\Phi}^{-1}(\mathbf{k})=4\pi^2 k_l\bar{B}_{lm}(\mathbf{k})k_m \tag{110}$$

$$\bar{V}_l(\mathbf{k})=-2\pi j\bar{B}_{lm}(\mathbf{k})k_m\bar{\Phi}(\mathbf{k})\,\bar{f}(\mathbf{k}) \tag{111}$$

Using the inverse Fourier transformation for the vector $\bar{\mathbf{H}}(\mathbf{k})$, we can find vector $\mathbf{H}(\mathbf{x})$, and for any two points A and B, we can write

$$\Delta_{BA}=U(\mathbf{B})-U(\mathbf{A})=\int_{AB}\mathbf{H}(\mathbf{x})\,d\mathbf{x} \tag{112}$$

It is well known that the difference Δ_{AB} does not depend on the path that connect **A** and **B.** It easy to see that if $\bar{f}(\mathbf{k})=1$, we have $\bar{\mathbf{H}}(\mathbf{k})=\bar{\mathbf{H}}_*(\mathbf{k})$, and from Equation (110) we obtain the following conditions for the tensors $\bar{\mathbf{R}}(\mathbf{k})$ and $\mathbf{R}(\mathbf{x})$**:**

$$\bar{R}_{ml}(\mathbf{k})=2\pi jk_r\bar{H}_{*r}(\mathbf{k})\delta_{ml} \quad , \qquad R_{ml}(\mathbf{x})=\frac{\partial H_{*r}(\mathbf{x})}{\partial x_r}\delta_{ml} \tag{113}$$

If an appropriate convolution exist we can write the averaged non-local equation

$$\int\frac{\partial H_{*r}(\mathbf{y})}{\partial y_r}V_m(\mathbf{x}-\mathbf{y})\,dy^2=H_m(\mathbf{x}) \tag{114}$$

Now, as in the case of d=3, we transform the tensor $\bar{R}_{ml}(\mathbf{k})$ in the form $\bar{R}_{ml}(\mathbf{k})=2\pi j\bar{H}_{*r}(\mathbf{k})\delta_{rp}k_p\delta_{ml}$ and write the convolution equation that equally matches the equation (114):

$$-\int H_{*l}(\mathbf{y})\frac{\partial V_m(\mathbf{x}-\mathbf{y})}{\partial y_l}dy^2=H_m(\mathbf{x}) \tag{115}$$

Taking into account the estimation $H_{*l}(\mathbf{y})\sim y_l/2\pi\sigma_* y^2$ for small y , and the differentiability of the flow velocity $\mathbf{V}(\mathbf{x}-\mathbf{y})$, we see that the integral converges at $\mathbf{y}=0$. For very large y , we use the estimate $H_{*l}(\mathbf{y})\sim y_l/2\pi\sigma^* y^2$ and $V_m(\mathbf{x}-\mathbf{y})\sim qy_m/2\pi y^2$ for finite $\mathbf{x}$. Thus we have the estimation for very large y :

$H_l\ \mathbf{y}\ \dfrac{V_m\ \mathbf{x}\quad \mathbf{y}}{y_l} \sim y_m/2 \qquad y^4$, and the integral converges at infinity. The convolutions in (114) and (115) exist.

Finally, we will examine the simple case of one-dimensional flow in an unbounded domain with non-random sources. Above, in Section 2, we noted that for existence of a unique solution in one-dimensional case the total intensity of compactly supported density $f(x)$ must be zero.

Thus, the system of the flow equations is

$$\frac{dv(x)}{dx} = f(x), \quad -\infty < x < \infty \tag{116}$$

$$v(x) = -\sigma(x)\frac{du(x)}{dx} \tag{117}$$

with the condition $\int_{-\infty}^{\infty} f(x)dx = 0$.

Let us assume that at infinity $v(-\infty) = 0$, so that

$$v(x) = \int_{-\infty}^{x} f(y)dy \tag{118}$$

and $v(+\infty) = 0$. Therefore, the velocity $v(x)$ is non-random and hence $v(x) = V(x)$.

We write the conservative form of stochastic Darcy`s equation

$$\sigma^{-1}(x)V(x) = -\frac{du(x)}{dx} \tag{119}$$

and by averaging this equation, we have a closed averaged system

$$\frac{dV(x)}{dx} = f(x), \quad V(x) = -\sigma_* \frac{dU(x)}{dx}, \ \sigma_* = \left\langle \sigma^{-1}(x) \right\rangle^{-1} \tag{120}$$

with conditions $V(\pm\infty) = 0$.

Finally, for any x_1 and x_2 we can find the difference

$$U(x_2) - U(x_1) = -\frac{1}{\sigma_*}\int_{x_1}^{x_2}\int_{-\infty}^{y} f(z)dzdy \tag{121}$$

8.Non-Steady Transient Flow with Sources

Let us consider the stochastic system of equations in a three-dimensional unbounded domain:

$$\frac{\partial m(\mathbf{x},t)}{\partial t} + \frac{\partial v_l(\mathbf{x},t)}{\partial x_l} = f(\mathbf{x},t) \tag{122}$$

$$m(\mathbf{x},t) = \tilde{\alpha}(\mathbf{x})u(\mathbf{x},t) \tag{123}$$

$$\mathbf{r}(\mathbf{x})\mathbf{v}(\mathbf{x}) = -\nabla u(\mathbf{x},t) \tag{124}$$

$$u(\mathbf{x},t_0) = 0 \tag{125}$$

Here the scalar function $\tilde{\alpha}(\mathbf{x})$ and the tensor of $\mathbf{r}(\mathbf{x})$ are statistically homogeneous random fields of the storage capacity and resistance, respectively, and $u(\mathbf{x},t)$ is the pressure that tends to zero at infinity. We introduce Green's function $g(\mathbf{x},t,\mathbf{y},\tau)$, which also tends to zero at infinity, as the random solution of the system of Equations (122)-(125) for $f(\mathbf{x},t) = \delta(\mathbf{x}-\mathbf{y})\delta(t-\tau)$. Let us introduce, in the same way as before:

$$G(\mathbf{x}-\mathbf{y},t-\tau) = \langle g(\mathbf{x},t,\mathbf{y},\tau)\rangle \tag{126}$$

$$N(\mathbf{x}-\mathbf{y},t-\tau) = \langle \tilde{\alpha}(\mathbf{x})g(\mathbf{x},t,\mathbf{y},\tau)\rangle \tag{127}$$

$$\Gamma_l(\mathbf{x}-\mathbf{y},t-\tau) = -\left\langle \sigma_{lj}(\mathbf{x})\frac{\partial g(\mathbf{x},t,\mathbf{y},\tau)}{\partial x_j}\right\rangle \tag{128}$$

We consider $\bar{T}_{FL}$ and $\bar{T}_{FL}^{-1}$- the direct and inverse Fourier-Laplace transforms and use the following designations:

$$\bar{G}(\mathbf{k},\mu) = \bar{T}_{FL}G, \quad \bar{N}(\mathbf{k},\mu) = \bar{T}_{FL}N, \quad \bar{\Gamma}_l(\mathbf{k},\mu) = \bar{T}_{FL}\Gamma_l \tag{129}$$

and then introduce the following scalar function and vector:

$$\bar{S}(\mathbf{k},\mu) = \bar{N}(\mathbf{k},\mu)\bar{G}^{-1}(\mathbf{k},\mu), \quad \bar{\Pi}_l(\mathbf{k},\mu) = \bar{\Gamma}_l(\mathbf{k},\mu)\bar{G}^{-1}(\mathbf{k},\mu) \tag{130}$$

It easy to show that $\mu\bar{N}(\mathbf{k},\mu) + 2\pi i k_l\bar{\Pi}_l(\mathbf{k},\mu) = \bar{G}^{-1}(\mathbf{k},\mu)$. Thus, the averaged system is

$$\frac{\partial M(\mathbf{x},t)}{\partial t} + \frac{\partial V_l(\mathbf{x},t)}{\partial x_l} = f(\mathbf{x},t) \tag{131}$$

$$M(\mathbf{x},t) = \int\int_{t_0}^{t} S(\mathbf{x}-\mathbf{y},\ t-\tau) U(\mathbf{y},\tau)\ dy^3 d\tau \tag{132}$$

$$V_l(\mathbf{x},t) = -\int\int_{t_0}^{t} \Pi_l(\mathbf{x}-\mathbf{y},t-\tau) U(\mathbf{y},\tau) dy^3\, d\,\tau \tag{133}$$

$$U(\mathbf{x},t_0) = 0 \tag{134}$$

Here, $U(\mathbf{x},t) = \langle u(\mathbf{x},t)\rangle$, $V_l(\mathbf{x},t) = \langle v_l(\mathbf{x},t)\rangle$, $M(\mathbf{x},t) = \langle m(\mathbf{x},t)\rangle$, $S(\mathbf{x},t) = T_{FL}^{-1}\,\bar{S}(\mathbf{k},\mu)$, and $\mathbf{\Pi}(\mathbf{x},t) = T_{FL}^{-1}\,\bar{\mathbf{\Pi}}(\mathbf{k},\mu)$.

9. NON-REACTIVE SOLUTE TRANSPORT

We consider a stochastic system of equations in a three dimensional unbounded domain:

$$\frac{\partial a(\mathbf{x},t)}{\partial t} + \frac{\partial q_l(\mathbf{x},t)}{\partial x_l} = f(\mathbf{x},t) \tag{135}$$

$$a(\mathbf{x},t) = \theta(\mathbf{x}) c(\mathbf{x},t) \tag{136}$$

$$q_l(\mathbf{x},t) = v_l(\mathbf{x},t) c(\mathbf{x},t) - D_{lj} \frac{\partial c(\mathbf{x},t)}{\partial x_j} \tag{137}$$

$$c(\mathbf{x},t_0) = 0 \tag{138}$$

Here, $c(\mathbf{x},t)$ is the concentration of solute that tends to zero at infinity, and $\theta(\mathbf{x}), \mathbf{q}(\mathbf{x},t), \mathbf{D}$ are random porosity, solute flux, and non-random dispersion tensor, respectively.

We introduce the random Green`s function $g_c(\mathbf{x},t,\mathbf{y},\tau)$, also tending to zero at infinity which is the solution to the system of Equations (135)-(138) for $f(\mathbf{x},t) = \delta(\mathbf{x}-\mathbf{y})\delta(t-\tau)$, and the following $\theta(x)\ v(x,t)\ D_{lj} = const\ \ g_c(x,t,y,\tau)\ \ f(x,t) = \delta(x-y)\delta(t-\tau)$ relations:

$$G_c(\mathbf{x}-\mathbf{y},t-\tau)=\langle g_c\rangle,\ H(\mathbf{x}-\mathbf{y},t-\tau)=\langle\theta(\mathbf{x})g_c\rangle,\ P_l(\mathbf{x}-\mathbf{y},t-\tau)=\langle v_l(\mathbf{x})g_c\rangle \tag{139}$$

$$\bar{G}_c(\mathbf{k},\mu)=T_{FL}G_c,\ \bar{H}(\mathbf{k},\mu)=T_{FL}H,\ \bar{P}_l(\mathbf{k},\mu)=T_{FL}P_l,\ \bar{\tilde{\theta}}(\mathbf{k},\mu)=\bar{H}\bar{G}^{-1},\ \bar{W}_l(\mathbf{k},\mu)=\bar{P}_l\bar{G}^{-1}$$

It easy to show that $\mu\bar{\tilde{\theta}}+2\pi i k_l\bar{W}_l=\bar{G}^{-1}$.

Thus, for the mean concentration $C(\mathbf{x},t)=\langle c(\mathbf{x},t)\rangle$ tending to zero at infinity, we have:

$$\frac{\partial A(\mathbf{x},t)}{\partial t}+\frac{\partial Q_l(\mathbf{x},t)}{\partial x_l}=f(\mathbf{x},t) \tag{140}$$

$$A(\mathbf{x},t)=\int\int_{t_0}^{t}\tilde{\theta}(\mathbf{x}-\mathbf{y},t-\tau)C(\mathbf{y},\tau)\,d\,y^3 d\,\tau \tag{141}$$

$$Q_l(\mathbf{x},t)=\int\int_{t_0}^{t}W_l(\mathbf{x}-\mathbf{y},t-\tau)C(\mathbf{y},\tau)\,dy^3 d\,\tau\ \ -D_{lj}\frac{\partial C(x,t)}{\partial x_j} \tag{142}$$

$$C(\mathbf{x},t_0)=0 \tag{143}$$

Here,

$$Q_l(\mathbf{x},t)=\langle q_l(\mathbf{x},t)\rangle,\ \tilde{\theta}(\mathbf{x},t)=T_{FL}^{-1}\bar{\tilde{\theta}}(\mathbf{k},\mu),\ W_l(\mathbf{x},t)=T_{FL}^{-1}\bar{W}_l(\mathbf{k},\mu) \tag{144}$$

10. SUMMARY

We have described a general form for the exactly averaged equations of flow and transport in a stochastically homogeneous unbounded field with sources. We examined the validity of the averaged descriptions for the given fields and the generalized law for some nonlocal models. A variant of the generalization for a given field, with a unique kernel-vector and with a unique kernel-tensor for some cases, was presented. We discussed the problem of uniqueness and the properties of the non-local averaged equations for three types of global symmetry: isotropic, transversal isotropic, and orthotropic. We analyzed the structure of non-local equations in the general case of stochastically homogeneous fields.

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APPENDIX

Here, we discuss the problem of the random flow in bounded domain with small radius wells. Earlier, we noted that this is a very difficult problem, because the flow-domain with wells is multiply connected. However, for the deterministic case, if the radii of the wells are small compared to scale of heterogeneity and the scale of domain, the wells can be modeled with singular source or sink. Some modifications of this approach are often applied to model flow using finite difference approximation and when the grid spacing is much larger then the wells radius (for example, see Shvidler, 1964; Aziz and Settari, 1979). In this variant of two-dimensional modeling case, the flow near the well (in the cell where a well located) is described as radial. However this is a rough approximation.

According to the classical theory of wells interactions (see for example Muskat, 1937; Shcelkachev and Pychacev, 1939; Charny, 1948; Polubarinova- Kochina, 1962; Bear, 1972), when the productions of some wells are preassigned and the radius of these wells are much less than the scale of heterogeneity, modeling the flow near the well with local singularities (local source or sink) is usually acceptable. If for some wells the pressure or head is preassigned, for singularities approximation we need to describe and preassign the productions of these wells. Evidently we need to solve for each realization the boundary value problem for a field with singularities: one part of them have preassigned productions and second part has desired productions. To find these productions we can use the preassigned pressures or heads. Only after finding or predicting these productions can we analyze the field with all singularities.

It should be noted that Indelman et al., (1996) presented some attempt to simplify the prediction of random flow toward a small radius well in cylindrical domain with finite radius and height. Authors represented the well with singular source (sink) and assumed that the density of production along the vertical singular line is proportional to the local

conductivity multiplied by the well production of a cylindrical domain with any constant conductivity, divided by this conductivity and the height of the domain. Later in this Appendix, we will analyze this assumption.

Another questionable points is their modification of the problem in describing the unbounded space and using the Fourier transformation. It remains not properly understood how authors transformed the bounded domain into unbounded without changing the basic equation. Moreover, as is well known, in the case where the function at infinity is finite, the classical Fourier transformation does not exist. For more generalized problem of multiple wells, in the later published paper (Indelman, 2002) explanation is missing for what was done to achieve the generalized equations for a multiwell system.

Now we will continue our analysis. Let for example for two-dimensional case we consider the steady-state flow in some bounded domain D with wells, using the equations

$$\nabla\left[\boldsymbol{\sigma}(\mathbf{x})\nabla u(\mathbf{x})\right] = f(\mathbf{x}) \qquad in\ D \tag{A1}$$

$$u(\mathbf{x}) = \bar{u} = const \quad if\ \mathbf{x} \in \partial D \tag{A2}$$

Let the function $f(\mathbf{x})$ model the set of N wells

$$f(\mathbf{x}) = \sum_{i=1}^{n_1} q_i \delta(\mathbf{x} - \mathbf{x}_i) + \sum_{j=n_1+1}^{N=n_1+n_2} \tilde{q}_j \delta(\mathbf{x} - \mathbf{x}_j) \tag{A3}$$

here n_1 wells have preassigned relative productions $q_i = Q_i / h$ where Q_i is the actual production of i-th well and $h = const$ is the thickness of flow domain, n_2 wells have preassigned pressure or head $\bar{u}_j$, the $\mathbf{x}_i$ or $\mathbf{x}_j$ -are the centers of i-th or j –th well.

For closing the system of Equations (A1)-(A3) and determination of productions $\tilde{q}_j$ we can use the relations

$$u(\mathbf{x}_j + r_j \mathbf{e}) = \bar{u}_j \tag{A4}$$

Here r_j is the radius of j-th well and $\mathbf{e}$ is any unit-module vector.

It should be noted that when D is an unbounded domain the problem (A1)-(A4) valid only if all the wells are compactly distributed in two-dimensional space with the condition $\sum_{i=1}^{n_1} q_i + \sum_{j=n_1+1}^{N=n_1+n_2} q_j = 0$.

Let $g(\mathbf{x},\mathbf{y})$ be the the random Green`s function for Equation (A1) in the bounded domain D, with condition $g(\mathbf{x},\mathbf{y}) = 0$ if $\mathbf{x} \in \partial D$. Then

$$u(\mathbf{x}) = \bar{u} + \sum_{i=1}^{n_1} q_i g(\mathbf{x}, y_i) + \sum_{j=n_1+1}^{N=n_1+n_2} \tilde{q}_j g(\mathbf{x}, \mathbf{y}_j) \tag{A5}$$

and for $\tilde{q}_i$ we have a closed linear algebraic system of n_2 equations:

$$\sum_{j=n_1+1}^{N=n_1+n_2} \tilde{q}_j g(\mathbf{x}_l + r_l \mathbf{e}, \mathbf{x}_j) = \bar{u}_l - \bar{u} - \sum_{i=1}^{n_1} q_i g(\mathbf{x}_l + r_l \mathbf{e}, \mathbf{x}_i), \quad l = n_1+1, ..., N = n_1+n_2 \tag{A6}$$

It is clear that the coefficients and the right part of the equations are random, and therefore the solution $\tilde{q}_j$ is random and is dependent on all preassigned nonrandom $\bar{u}_j$ and q_i. It is also clear that the solution $\tilde{q}_j$ is dependent on all components of $n_2 \times n_2$ random matrix $g^1_{l,j} = g(\mathbf{x}_l + r_l \mathbf{e}, \mathbf{x}_j)$, where $n_1+1 \le l, j \le n_1+n_2$ and $n_2 \times n_1$ random matrix $g^2_{l,i} = g(\mathbf{x}_l + r_l \mathbf{e}, \mathbf{x}_i)$, where $n_1+1 \le l \le n_1+n_2$ and $1 \le i \le n_1$.

Let the distances between-wells $d_{l,j} = |\mathbf{x}_l - \mathbf{x}_j|$ and the shortest distance for each wells to the external border ∂D be much more than the scale of correlation a. In this case the nondiagonal component of matrix $\mathbf{g}^1$ and all the components of matrix $\mathbf{g}^2$, in contrast to $g^1_{l,l}$-diagonal components of matrix $\mathbf{g}^1$, are low variable. However, it is the manifestation of self-averaging phenomena. Below, we show for the one-well example how significantly they depend on the variation coefficient of well production $\tilde{q}$, r and a.

Thus, returning to system of Equations (A1)-(A3), we need to find the random Green`s function $g(\mathbf{x},\mathbf{y})$ and solve the system (A6). Only after this can we discuss the problem of averaging the system of equations (A 1)-(A 3).

However, it is possible in some cases to simplify the analysis, presupposing some corresponding conditions. If, for example, the scale of heterogeneity is small and

fluctuations of $\boldsymbol{\sigma}(\mathbf{x})$ are moderate, for estimation of $\tilde{q}_j$, we can use the solution of system (A6), where the random Green`s function $g(\mathbf{x},\mathbf{y})$ is replaced with non-random Green`s function $g_0(\mathbf{x},\mathbf{y})$ for equation (A1), in which $\boldsymbol{\sigma}(\mathbf{x}) = \langle \boldsymbol{\sigma}(\mathbf{x}) \rangle = \boldsymbol{\sigma}_0$. In this variant, the $\tilde{q}_j$ are non-random and the problem of averaging is as described earlier.

Moreover, if the wells are far apart in space, it is possible to ignore the interactions between them, that is, to neglect the sum on the right side of (A6) and all terms for $j \neq l$ on the left side. In this case, we have

$$\tilde{q}_l = (\bar{u}_l - \bar{u}) / g(\mathbf{x}_l + r_l \mathbf{e}, \mathbf{x}_l) \tag{A7}$$

Subsequent simplification is related to the assumption that for scalar field $\sigma(\mathbf{x})$ and for small r_l the random Green`s function can be presented in the form:

$$g(\mathbf{x}_l + r_l \mathbf{e}, \mathbf{x}_l) = \sigma^{-1}(\mathbf{x}) \sigma_0 g_0(\mathbf{x}_l + r_l \mathbf{e}, \mathbf{x}_l) \tag{A8}$$

In this case we have

$$\tilde{q}_l = \frac{(\bar{u}_l - \bar{u}) \sigma(\mathbf{x}_l)}{\sigma_0 g_0(\mathbf{x}_l + r_l \mathbf{e}, \mathbf{x}_l)} \quad , \qquad \langle \tilde{q}_l \rangle = (\bar{u}_l - \bar{u}) g_0^{-1}(\mathbf{x}_l + r_l \mathbf{e}, \mathbf{x}_l) \tag{A9}$$

$$\tilde{q}_l = \sigma(\mathbf{x}_l) \langle \tilde{q}_l \rangle / \sigma_0 \tag{A10}$$

Below we will designate this simple result, to be as s-prediction. We should emphasize that in the 3-dimensional case, for the productions density along the well a similar prediction was in fact utilized by Indelman et al. (1996).

Using (A10), we can write, for $S\tilde{q}_l$, the mean square deviation of $\tilde{q}_l$:

$$S\tilde{q}_l = \frac{|\langle \tilde{q}_l \rangle|}{\sigma_0} S\sigma \tag{A11}$$

Now we can describe positive ς_l -some degree of variability for random production $\tilde{q}_l$, as the ratio the coefficient of variation $\tilde{q}_l$ to coefficient of variation conductivity σ

$$\varsigma_l = \left(\frac{S\tilde{q}_l}{|\langle \tilde{q}_l \rangle|} \right) / \left(\frac{S\sigma}{\sigma_0} \right) \tag{A12}$$

and for the s- approximation (A10), we have $\varsigma_l^* = 1$.

Since the essence of the discussed s-prediction for $\tilde{q}_l$, is in reality the random functional from random function $\boldsymbol{\sigma}(\mathbf{x})$ in all domain, with only one random value at some point in space, it is important to examine the quality of this operation at least on independent examples.

It should be pointed out that Shvidler (1964), using the method of perturbation, examined flow in a two-dimensional domain-ring with a central well (in polar coordinates: $\rho < r < R, 0 < \theta < 2\pi$) with preassigned non-random pressures on the borders $u(\rho) = u_\rho = const$ and $u(R) = u_R = const$. In a first order approximation, this study presented the relationship between production fluctuations $\tilde{q}'_{(2)} = \tilde{q}_{(2)} - \langle \tilde{q}_{(2)} \rangle$ and field $\sigma'(\mathbf{x}) = \sigma(\mathbf{x}) - \sigma_0$

$$\tilde{q}'_{(2)} = \frac{\langle \tilde{q}_{(2)} \rangle}{2\pi \ln R/\rho} \int_\rho^R \int_0^{2\pi} \frac{\sigma'(r,\vartheta)}{r} dr d\vartheta \quad , \quad \langle \tilde{q}_{(2)} \rangle = \frac{2\pi\sigma_0}{\ln R/\rho}(u_R - u_\rho) \tag{A13}$$

In this case, for the Gaussian correlation function of conductivity

$K(r,\theta;r',\theta') = S^2\sigma \exp\left\{-a^{-2}\left[r^2 + r'^2 - 2rr'\cos(\theta - \theta')\right]\right\}$, the following was derived

$$\varsigma^2_{(2)}(R,\rho,a) = \frac{S^2\tilde{q}_{(2)}}{\langle \tilde{q}_{(2)} \rangle^2} / \frac{S^2\sigma}{\sigma_0^2} \tag{A14}$$

$$\varsigma^2_{(2)} = \frac{1}{4\ln^2(R/\rho)} \left\{ \left[Ei\left(-\frac{R^2}{a^2}\right) - Ei\left(-\frac{\rho^2}{a^2}\right) \right]^2 + \sum_{n=1}^{\infty} \frac{1}{n^2} \left[\varphi_n\left(\frac{R}{a}\right) - \varphi\left(\frac{\rho}{a}\right) \right]^2 \right\} \tag{A15}$$

$$-Ei(-x) = \int_x^\infty \frac{e^{-z}}{z}, \quad \varphi_n(x) = e^{-x^2} \sum_{k=0}^{n-1} \frac{x^{2(n-k-1)}}{(n-k-1)!}$$

and some partial examples were calculated. Much more detailed calculation and analysis were presented in the book (Shvidler, 1985).

It is plain to see that in the limit $R/a \to 0$, Equations (A 13) tends toward Equations (A 9) and (A 10), and in this limit $\varsigma_{(2)} = \varsigma^*_{(2)} = 1$. In Table A, we present the improved variant of calculation $\varsigma_{(2)} = \varsigma_{(2)}(R,\rho,a) = \left(S\tilde{q} / \left|\langle \tilde{q} \rangle\right|\right) / (S\sigma/\sigma_o)$.

Comparing $\varsigma_{(2)}$ from Table A with $\varsigma_{(2)}^{*}=1$, in accordance with the s-prediction (A10), we see a significant difference; they are close only if $a>R$, the case in which the random conductivity is practically constant in the ring-domain.

It is interesting to find the reciprocal correlation moment for the random well production $\tilde{q}_{(2)}$ and in some point random conductivity $\sigma\left(r_1,\vartheta_1\right)$. Following Shvidler (1985), we have

$$H_{(2)}^{\oplus}=\left\langle\sigma'\left(r_1,\theta_1\right)\tilde{q}'_{(2)}\right\rangle=\frac{\left\langle\tilde{q}_{(2)}\right\rangle}{2\pi\sigma_0\ln R/\rho}\int_{\rho}^{R}\int_{0}^{2\pi}\frac{K\left(r,\vartheta;r_1,\vartheta_1\right)}{r}drd\vartheta \tag{A16}$$

If $r_1=0$, using the Gaussian correlation function after simple transformation, we have

$$H_{(2)}^{\oplus}=\frac{\left\langle\tilde{q}_{(2)}\right\rangle S^2\sigma}{2\sigma_0\ln R/\rho}\left[Ei\left(-\frac{R^2}{a^2}\right)-Ei\left(-\frac{\rho^2}{a^2}\right)\right] \tag{A17}$$

and for $\lambda_{(2)}\left(R,\rho,a\right)=H_{(2)}^{\oplus}/S\tilde{q}_{(2)}S\sigma$ - the coefficient of correlation we find

$$\lambda_{(2)}\left(R,\rho,a\right)=\frac{sgn\left\langle\tilde{q}_{(2)}\right\rangle}{2\varsigma_{(2)}\ln R/\rho}\left[Ei\left(-\frac{R^2}{a^2}\right)-Ei\left(-\frac{\rho^2}{a^2}\right)\right] \tag{A18}$$

The results of computing $\left|\lambda_{(2)}\left(R,\rho,a\right)\right|$ are presented in Table B. Clearly, for s-approximation (A 10), we have $\left|\lambda_{(2)}^{*}\left(R,\rho,a\right)\right|\equiv 1$ and the same follows from (A18) in the limiting case of $R/a\to 0$. It may appear that the closeness of the coefficient of correlation $\left|\lambda_{(2)}\right|$ to unity, is an indication of the proportionality between random functional $\tilde{q}$ and random conductivity $\sigma_{*}=\sigma\left(0,\theta\right)$. But we need to consider that the significant deviation $\varsigma_{(2)}$ from 1 (see Table A) contradict this assumption. Now, we analyze this situation in more detail.

Like the relation (A 13s) between fluctuations $\tilde{q}'_{(2)}$ and $\sigma'\left(\mathbf{x}\right)$ we can, in the same order of approximation, write the equation

$$\tilde{q}_{(2)}=\frac{\left\langle\tilde{q}_{(2)}\right\rangle}{2\pi\sigma_0\ln R/\rho}\int_{\rho}^{R}\int_{0}^{2\pi}\frac{\sigma\left(r,\vartheta\right)}{r}drd\vartheta \tag{A19}$$

Using identity $\sigma(r,\vartheta)=\sigma_*+\left[\sigma(r,\vartheta)-\sigma_*\right]$ and splitting the integral into two parts we find

$$\tilde{q}_{(2)}=\tilde{q}^*_{(2)}+\tilde{q}^{\otimes}_{(2)}\ ,\ \tilde{q}^*_{(2)}=\left\langle\tilde{q}_{(2)}\right\rangle\frac{\sigma_*}{\sigma_0} \tag{A 20}$$

where the term $\tilde{q}^*$ is the known s-prediction (A10) and $\tilde{q}^{\otimes}_{(2)}$ is a random residual functional:

$$\tilde{q}^{\otimes}_{(2)}=\frac{\left\langle\tilde{q}_{(2)}\right\rangle}{2\pi\sigma_0\ln R/\rho}\int_{\rho}^{R}\int_{0}^{2\pi}\frac{\sigma(r,\vartheta)-\sigma_*}{r}drd\vartheta \tag{A 21}$$

Averaging equation (A21), we find $\left\langle\tilde{q}^{\otimes}_{(2)}\right\rangle=0$. Using the relation (A20) again, we can find significant parameters: $S\tilde{q}^{\otimes}_{(2)}$ - mean square deviation of residual functional $\tilde{q}^{\otimes}_{(2)}$, and $C^*_{(2)}$ and $C_{(2)}$ - the correlation coefficients of residual functional $\tilde{q}^{\otimes}_{(2)}$ with prediction $\tilde{q}^*_{(2)}$ and functional $\tilde{q}_{(2)}$ respectively. In addition, we find ratios $\omega_{(2)}=S\tilde{q}_{(2)}/S\tilde{q}^{\otimes}_{(2)}$ and $\omega^*_{(2)}=S\tilde{q}^*_{(2)}/S\tilde{q}^{\otimes}_{(2)}$.

$$S\tilde{q}^{\otimes}_{(2)}=\left|\left\langle\tilde{q}_{(2)}\right\rangle\right|\frac{S\sigma}{\sigma_0}\eta_{(2)}\ ,\ \eta_{(2)}=\sqrt{1+\varsigma^2_{(2)}-2\varsigma_{(2)}\left|\lambda_{(2)}\right|} \tag{A 22}$$

$$C^*_{(2)}=\frac{\varsigma_{(2)}\left|\lambda_{(2)}\right|-1}{\eta_{(2)}}\ ,\ C_{(2)}=\frac{\varsigma_{(2)}-\left|\lambda_{(2)}\right|}{\eta_{(2)}} \tag{A 23}$$

$$\omega_{(2)}=\varsigma_{(2)}/\eta_{(2)},\ \omega^*_{(2)}=1/\eta_{(2)} \tag{A 24}$$

In Tables C, D, and E, we present the results of calculation $\eta_{(2)}$, $-C^*_{(2)}$, and $C_{(2)}$, and in Tables F and G, the ratios $\omega^*_{(2)}$, $\omega_{(2)}$ respectively as a function of ρ and a when $R=250m$.

We can see from Table C that for $S\ \ /\ \ _0\sim 1$ and $a<R$, the mean square deviation of residual functional $\tilde{q}^{\otimes}_{(2)}$ has the same degree as the functional $\left|\left\langle\tilde{q}_{(2)}\right\rangle\right|$. In addition, for

small a (compare with ς_0 from Table A and $\omega_{(2)}$ from Table G), the mean square deviation of $\tilde{q}^{\otimes}_{(2)}$ is larger than the mean square deviation of functional $\tilde{q}_{(2)}$.

The Table D demonstrates that residual functional $\tilde{q}^{\otimes}_{(2)}$ is relatively well correlated with conductivity σ_*. From Table E, we see that the module of correlation coefficient C_2 is small if $\rho > a$, and when a increases, the module reaches its maximum and then slowly decreases.

In the Tables F and G, we can see the significance of $S\tilde{q}^{\otimes}_{(2)}$ - the measure of variability residual functional $\tilde{q}^{\otimes}_{(2)}$, compared to $S\tilde{q}_{(2)}$ and $S\tilde{q}^{*}_{(2)}$ - the measures of variability the production $\tilde{q}_{(2)}$ and s-prediction $\tilde{q}^{*}_{(2)}$.

We describe the three-dimensional flow in domain $D = \{\rho \le r \le R, 0 \le \vartheta \le 2\pi, 0 \le z \le h\}$, and in the first order approximation of conductivity fluctuation $\sigma'(r,\vartheta,z)$, we have $\tilde{q}'_{(3)}$ - the fluctuation of the well production and appropriate $S^2\tilde{q}_{(3)} = \langle \tilde{q}'^2_{(3)} \rangle$

$$\tilde{q}'_{(3)} = \frac{\langle \tilde{q}_{(3)} \rangle}{2\pi\sigma_0 h \ln R/\rho} \int_{\rho}^{R}\int_{0}^{2\pi}\int_{0}^{h} \frac{\sigma'(r,\vartheta,z)}{r} dr d\vartheta dz \quad , \quad \langle \tilde{q}_{(3)} \rangle = \frac{2\pi\sigma_0 h}{\ln R/\rho}(u_R - u_\rho) \qquad \text{(A 25)}$$

$$S^2 q_{(3)} = \frac{\langle \tilde{q}_{(3)} \rangle^2}{4\pi^2\sigma_0^2 h^2 (\ln R/r)^2} \int_{\rho}^{R}\int_{0}^{2\pi}\int_{0}^{h}\int_{\rho}^{R}\int_{0}^{2\pi}\int_{0}^{h} \frac{K(r,\vartheta,z;r',\vartheta',z')}{rr'} dr d\vartheta dz dr' d\vartheta' dz' \qquad \text{(A 26)}$$

Using the correlation function for conductivity in three-dimensional space $K = S^2\sigma \exp\left\{-\left[r^2 + r'^2 - 2rr'\cos(\vartheta - \vartheta')\right]a^{-2} - (z - z')^2 a_{\perp}^{-2}\right\}$, we have:

$$\zeta_{(3)} = \zeta_{(2)}\sqrt{f^{\square}\left(\frac{h}{a_{\perp}}\right)}, \; x^2 f^{\square}(x) = \int_0^x\int_0^x \exp\left[-(z - z')^2\right] dz dz' = x\sqrt{\pi}\, erf x + e^{-x^2} - 1 \qquad \text{(A 27)}$$

The function $\zeta_{(3)}$ and the function $\sqrt{f^{\square}(h/a_{\perp})}$ are presented in Table H. Since for any $h/a_{\perp}$, the function $f^{\square} \le 1$, the degree of variability $\varsigma_{(3)} \le \varsigma_{(2)}$.

Similarly we calculated $H_{(3)}^{\otimes}$ - the correlation moment between $\tilde{q}_{(3)}$ and random $\sigma_* = h^{-1}\int_0^h \sigma(0,\vartheta,z)dz$, and the appropriate coefficient of correlation $\lambda_{(3)} = H_{(3)}^{\otimes} / S\tilde{q}_{(3)} S\sigma$:

$$\lambda_{(3)}(R,\rho,a,a_\perp) = \lambda_{(2)}(R,\rho,a)\sqrt{f^{\square}(h/a_\perp)} \tag{A 28}$$

In the three-dimensional case, the analog for Equation (A10) is $\tilde{q}_{(3)}^* = \sigma_* \left\langle \tilde{q}_{(3)}^* \right\rangle \sigma_0^{-1}$ or the equivalent relation for fluctuations $\tilde{q}_{(3)}^{*/} = \sigma_*' \left\langle \tilde{q}_{(3)}^* \right\rangle \sigma_0^{-1}$. After simple calculations, we have

$$\varsigma_{(3)}^* = \sqrt{f^{\square}(h/a_\perp)} \quad , \quad \zeta_{(3)} / \zeta_{(3)}^* = \zeta_{(2)}, \quad \lambda_{(3)}^* = \sqrt{f^{\square}(h/a_\perp)} \tag{A 29}$$

It is clear that as for $a/R \to \infty$ we have $\varsigma_{(2)} \to 1$, $\lambda_{(2)} \to 1$, and therefore, from Equations (A 27) and (A 28), we have the same result.

Now, as in the two-dimensional case, we write the random well production $\tilde{q}_{(3)}$

$$\tilde{q}_{(3)} = \frac{\left\langle \tilde{q}_{(3)} \right\rangle}{2\pi\sigma_0 h \ln R/\rho} \int_\rho^R \int_0^{2\pi} \int_0^h \frac{\sigma(r,\vartheta,z)}{r} dr d\vartheta dz \tag{A 30}$$

and present $\tilde{q}_{(3)}$ as the sum $\tilde{q}_{(3)} = \tilde{q}_{(3)}^* + \tilde{q}_{(3)}^{\otimes}$, where

$$\tilde{q}_{(3)}^{\otimes} = \frac{\left\langle \tilde{q}_{(3)} \right\rangle}{2\pi\sigma_0 h \ln R/\rho} \int_\rho^R \int_0^{2\pi} \int_0^h \frac{\sigma(r,\vartheta,z) - \sigma_*}{r} dr d\vartheta dz \tag{A 31}$$

Averaging Equation (A 31), we find $\left\langle \tilde{q}_{(3)}^{\otimes} \right\rangle = 0$, and after this we derive $S\tilde{q}_{(3)}^{\otimes}$-mean square deviation of functional $\tilde{q}_{(3)}^{\otimes}$ and $C_{(3)}^*$ and $C_{(3)}$ - the correlation coefficients functional $\tilde{q}_{(3)}^{\otimes}$ with functionals $\tilde{q}_{(3)}^*$ and $\tilde{q}_{(3)}$ respectively. Finally, we find the ratios $\omega_{(3)} = S\tilde{q}_{(3)} / S\tilde{q}_{(3)}^{\otimes}$ and $\omega_{(3)}^* = S\tilde{q}_{(3)}^* / S\tilde{q}_{(3)}^{\otimes}$

$$S\tilde{q}_{(3)}^{\otimes} = \left| \left\langle \tilde{q}_{(3)} \right\rangle \right| \frac{S\sigma}{\sigma_0} \eta_{(3)} \quad , \quad \eta_{(3)} = \eta_{(2)} \sqrt{f^{\square}(h/a_\perp)} \tag{A 32}$$

$$C_{(3)}^* = C_{(2)}^* \, , \, C_{(3)} = C_{(2)} \, , \, \omega_{(3)}^* = \omega_{(2)}^* \, , \, \omega_{(3)} = \omega_{(2)} \tag{A 33}$$

Information presented here enables some model estimation: whereby the production of a well in a random field is proportional to the conductivity at the center of the well.

It is evident that the productions at the wells can be determined for each realization of random conductivity field only after solving appropriate boundary value problem for the

elliptic equation with variable coefficients. However, such a method is practically unrealizable.

On the other hand, our estimation must take into consideration the goal for the sake of accepting similar models. Since the random well productions are necessary for solving the averaging problem, in some cases we sufficiently know the mean productions, and mean square deviations. If the number of wells with preassigned pressure or head is more than one, it is important to know the matrix correlation coefficients of productions, that is, the functional of random conductivity field and the parameters that describe the geometry of the well set.

As we showed in this paper, the variation coefficient for one well production obtained in our analysis and that discussed s-prediction are significantly different.

We can see that the discussed s-prediction used oversimplification in many cases.

References

Aziz, K and A.Settari.:1979, *Petroleum Reservoir Simulation*. Elsevier Applied Science Publisher.London

Bachvalov, N.S., G.P.Panasenko.: 1989, *Homogenization: Averaging Processes in Periodic Media; Mathematical Problems in the Mechanics of Composite Material*, Kluver Academic, Dordrecht, Boston MA

Bear, J.:1972, *Dynamics of fluids in porous media*, Dover Publications. New York

Charny, I.A.:1948, *Underground hydrodinamics*, Gostechisdat. Moscow

Courant, R.:1962, *Partial Differential Equations*, Interscience. New York

Cvetcovic, V.D., Dagan, G. and Shapiro, A.M.:1991, An exact solution of solute transport by one-dimensional random velocity fields, *Stochastic Hydrol., Hydraulics*, **5,** 45-54

Dagan, G.;1989*, Flow and Transport in Porous Formations*, Springer-Verlag, New York

Dykhne, A.M..:1971, Conductivity of two-dimensional two-phase systems. *Soviet Physics. JETP* **32** ,63

Gelfand, I.M. and G.E.Shilov.:1964, *Generalized Functions*, vol. **1-2**, Academic Press , New York

Herring, C.:1960, Effect of random inhomogeneities on electrical and galvanomagnetic measurments. *J.Appl.Phys*. **31**, 1939

Indelman, P.V. and M.I.Shvidler.:1985, Averaging of stochastic evolution equation of transport in porous media, *Fluid Dyn*., 2**0** (5), 775-784

Indelman, P.V.:1986, Statistical characteristics of the moving liquid particle in porous media with random porosity, *Fluid Dyn.,* 2**1** (6), 59-65

Indelman, P., and B.Abramovich.:1994, Nonlocal properties of nonuniform averaged flows in heterogeneous media, *Water Resour.Res*.,3385-3393

Indelman, P.:1996, Averaging of unsteady flows in heterogeneous media of stationary conductivity, *J. Fluid Mech*.,**310,** 39-60

Indelman, P., Fiori, A. and Dagan, G.:1996, Steady flow toward wells in heterogeneous formations; mean head and equivalent conductivity, *Water Resour. Res*. **32** ,1975-1983

Indelman, P.:2002, On mathematical models of average flow in heterogeneous formations; *Transport in Porous Media*, **45:105-127**

Keller, J.B..:1964, A theorem on the conductivity of a composite media*. J. Math.Phys.* **: 5,** 548-599

King, P.R..:1987, The use of field theoretic methods for study of flow in a heterogeneous porous media. *J. Phys. A, Math*., Gen., **20**, 3935-3947

Klyatskin,V.I.:1980, *Stochastic Equations and Wave in Random Heterogeneity Media*.(In Russian), Nauka, Moscow

Kudinov,V.A. and B.Ya.Moizhes.:1972, Effective conductivity of an inhomogeneous isotropic medium. *Soviet Physics. Tech. Physics*. **17, 3**, 467-472

Lifshitz, I.M., S.A.Grodeskuul, L.A.Pastur .:1988, *Introduction to theory of disordered solids*, John Wiley, New York

Matheron, G.: 1967, *Elements Pour Une Theorie Des Mileeux Poreux*. Masson & Cie, Editeurs, Paris

Matheron, G. and de Marsily,G.: 1980, Is transport in porous media always diffusive? A counterexample. *Water Resour. Res*., **16** (5), 901-917

Monin, A.S. and A.M. Yaglom.:1965-1967, *Statisticheskaia Gidromekhanika*: *Mekhanika Turbulentnosti*, (In Russian), M. Nauka,2v. English:1971-1975 *Statistical Fluid Mecanics; Mechanics of Turbulence*, Cambridge, MA, MIT Press ,2v.

Muskat, M.:1937, *Flow of homogeneous fluids through porous media*. McGraw-Hill, New York

Polubarinova-Kochina, P.Ya.:1962, *Theory of Groundwater Movement*. Princeton Press. Princeton

Saffman, P.G..:1971, On the boundary conditions at the surface of a porous medium, *Stud.. Appl.Math*., 93-101

Schwartz, L.: 1961, *Methods Mathematiques Pour Les Sciences Physiques*, Herman & Cie, Paris

Shcelkachev,V.N. and B.G.Pychachev.:1939, *Interference of the wells and theory of reservoir waterdrive system*. AzGonty-Bakhu (in Russian).

Shermergor, T.D.: 1979, *Theory of Elasticity in Micro-heterogeneous Media* (In Russian), Nauka, Moscow

Shvidler, M.I.:1964*, Filtration Flows in Heterogeneous Media (A Statistical Approach)*, Consultants Bureau, New York

Shvidler, M.I.: 1966, Source-solution of problem for unsteady flow trough random porous medium*, Fluid Dyn*.**1**, (4)

Shvidler, M.I.: 1985, *Statistical Hydrodynamics of Porous Media.* (In Russian), Nedra, Moscow

Shvidler, M. and K.Karasaki.: 1999, Investigation of the exactly averaged equations of flow and transport in random porous media, In Transactions, *AGU Fall Meeting*, San Francisco, 1999.

Shvidler, M. and K. Karasaki.:2001, Exactly Averaged Stochastic Equations for Flow and Transport in Random Media. *Earth Sciences Division Ernest Orlando Lawrence Laboratory, University of California, Berkeley*, LBNL-49529, 22

Shvidler, M. and K. Karasaki.: 2003a, Exact averaging of stochastic equations for transport in random velocity field, *Transport in Porous Media*, **50** (3), 246-266

Shvidler, M. and K. Karasaki.: 2003b, Probability density functions for solute transport in random Field*, Transport in Porous Media*, **50** (3), 243-266

Tatarsky, V.I.: 1967, *Wave Propagation in Turbulence Atmosphere* (In Russian), Nauka, Moscow

Teodorovich, E.V.:1997, Calculation of the effective permeability of randomly inhomogeneous medium, *JETP,* **85**(1)

Yosida, K.:1978, *Functional Analysis,* Fifth Edition, Springer- Verlag, Berlin

Zemanian, A.H.: 1987, *Distribution Theory and Transform Analysis*, Dover Publications, INC, New York

Zhikov, V.V., S.M.Kozlov, O.A.Oleinik.:1993, *Homogenization of Differential Operators*. (In Russian).Nauka, Moscow, English:1994,*Homogenization of Differential Operetors and Integral Functionals*, Springer-Verlag, Berlin

Table A: $\zeta_0 = \zeta_0(\rho, a)$, R = 250m

ρ (m)	a (m)									
	10^{-3}	10^{-2}	10^{-1}	0.5	1	5	10	50	100	500
10^{-3}	0.0332	0.1703	0.3511	0.4795	0.535	0.664	0.7196	0.8488	0.9043	0.9906
10^{-2}	0.0001	0.0408	0.209	0.3634	0.4309	0.5886	0.6567	0.815	0.883	0.9884
10^{-1}	0	0.0002	0.0528	0.189	0.2705	0.4703	0.5577	0.7617	0.8494	0.9851
0.5	0	0	0.0127	0.0665	0.1225	0.3406	0.4477	0.702	0.8118	0.9812
1	0	0	0.0003	0.0394	0.0748	0.2678	0.3833	0.6662	0.7892	0.9789
5	0	0	0	0.0004	0.0202	0.1056	0.1946	0.5405	0.7088	0.9704
10	0	0	0	1E-81	0.0005	0.0676	0.1284	0.4585	0.6536	0.9641

Table B: $\lambda_0 = \lambda_0(\rho, a)$, R = 250m

((m)	a (m)									
	10-3	10-2	10-1	0.5	1	5	10	50	100	500
10-3	0.2655	0.9539	0.9892	0.9942	0.9954	0.997	0.9974	0.9982	0.9986	1
10-2	0	0.2655	0.9539	0.9848	0.9892	0.9942	0.9954	0.9971	0.9978	0.9999
10-1	0	0	0.2655	0.9067	0.9539	0.9848	0.9892	0.9944	0.996	0.9999
0.5	0	0	3E-12	0.2655	0.686	0.9539	0.9733	0.9894	0.993	0.9998
1	0	0	0	0.0087	0.2655	0.9067	0.9539	0.9851	0.9906	0.9998
5	0	0	0	0	3E-12	0.2655	0.686	0.9548	0.9767	0.9996
10	0	0	0	0	0	0.0087	0.2655	0.9084	0.9595	0.9994

Table C: $\chi\left(\varsigma_0, \lambda_o\right) = 1 - \lambda_0 \varsigma_0$, R = 250m

ρ (m)	a (m)									
	10^{-3}	10^{-2}	10^{-1}	0.5	1	5	10	50	100	500
10^{-3}	0.9912	0.8376	0.6527	0.5232	0.4675	0.3380	0.2822	0.1527	0.09695	0.0095
10^{-2}	1	0.9892	0.8006	0.6422	0.5737	0.4148	0.3464	0.1874	0.1190	0.01161
10^{-1}	1	1	0.9860	0.8287	0.7420	0.5369	0.4483	0.2426	0.1540	0.01503
0.5	1	1	1	0.9823	0.9160	0.6751	0.5642	0.3054	0.1939	0.01892
1	1	1	1	0.9997	0.9801	0.7572	0.6343	0.3437	0.2182	0.02130
5	1	1	1	1	1	0.9720	0.8665	0.4839	0.3077	0.03005
10	1	1	1	1	1	0.9994	0.9659	0.5835	0.3728	0.03647

Table D: $\eta\left(\varsigma_0, \lambda_0\right) = \dfrac{\varsigma_0 \sqrt{1 - \lambda_0^2}}{\chi}$, R = 250m

ρ (m)	a (m)									
	10^{-3}	10^{-2}	10^{-1}	0.5	1	5	10	50	100	500
10^{-3}	0.0323	0.0610	0.0788	0.0983	0.1100	0.1521	0.1822	0.3332	0.4952	0.9421
10^{-2}	0.0001	0.0398	0.0783	0.0983	0.1100	0.1521	0.1822	0.3332	0.4952	0.9421
10^{-1}	0	0.0002	0.0516	0.0962	0.1094	0.1521	0.1822	0.3332	0.4952	0.9421
0.5	0	0	0.0127	0.0653	0.0973	0.1514	0.1820	0.3331	0.4952	0.9421
1	0	0	0.0003	0.0394	0.0736	0.1492	0.1814	0.3331	0.4952	0.9421
5	0	0	0	0.000381	0.020249	0.1048	0.1634	0.3320	0.4948	0.9421
10	0	0	0	1.01E-81	0.000463	0.0676	0.1281	0.3284	0.4938	0.9421

Table E: $C_{(2)} = C_{(2)}(\rho, a)$, R = 250m

ρ (m)	a (m)									
	10^{-3}	10^{-2}	10^{-1}	0.5	1	5	10	50	100	500
10^{-3}	-0.2342	-0.9338	-0.9747	-0.9790	-0.9789	-0.9740	-0.9685	-0.9281	-0.8713	-0.7216
10^{-2}	0.0001	-0.2270	-0.9275	-0.9631	-0.9673	-0.9668	-0.9620	-0.9217	-0.8645	-0.7202
10^{-1}	0.0000	0.0002	-0.2154	-0.8621	-0.9156	-0.9475	-0.9469	-0.9098	-0.8527	-0.7179
0.5	0.0000	0.0000	0.0127	-0.2021	-0.6123	-0.8982	-0.9166	-0.8929	-0.8374	-0.7153
1	0.0000	0.0000	0.0003	0.0307	-0.1940	-0.8345	-0.8850	-0.8803	-0.8270	-0.7136
5	0.0000	0.0000	0.0000	0.0004	0.0202	-0.1636	-0.5597	-0.8126	-0.7801	-0.7075
10	0.0000	0.0000	0.0000	0.0000	0.0005	0.0588	-0.1408	-0.7327	-0.7357	-0.7030

Table F: $\eta_1 = \sqrt{\chi + \varsigma_0^2 - \varsigma_0 \lambda_0}$, R = 250m

ρ (m)	a (m)									
	10^{-3}	10^{-2}	10^{-1}	0.5	1	5	10	50	100	500
10^{-3}	0.9917	0.8391	0.6547	0.5257	0.4703	0.3419	0.2868	0.1610	0.1082	0.0130
10^{-2}	1.0000	0.9899	0.8031	0.6453	0.5772	0.4196	0.3521	0.1976	0.1328	0.0160
10^{-1}	1.0000	1.0000	0.9873	0.8325	0.7464	0.5430	0.4557	0.2557	0.1719	0.0207
0.5	1.0000	1.0000	1.0001	0.9844	0.9203	0.6828	0.5735	0.3219	0.2164	0.0260
1	1.0000	1.0000	1.0000	1.0004	0.9828	0.7656	0.6447	0.3623	0.2435	0.0293
5	1.0000	1.0000	1.0000	1.0000	1.0002	0.9773	0.8780	0.5099	0.3433	0.0413
10	1.0000	1.0000	1.0000	1.0000	1.0000	1.0017	0.9738	0.6142	0.4158	0.0501

Table G: $\lambda_1 = \frac{\chi}{\eta_1}$, R = 250m

((m)	a (m)									
	10-3	10-2	10-1	0.5	1	5	10	50	100	500
10-3	0.9995	0.9981	0.9969	0.9952	0.9940	0.9886	0.9838	0.9487	0.8962	0.7278
10-2	1.0000	0.9992	0.9969	0.9952	0.9940	0.9886	0.9838	0.9487	0.8962	0.7278
10-1	1.0000	1.0000	0.9987	0.9954	0.9941	0.9886	0.9838	0.9487	0.8962	0.7278
0.5	1.0000	1.0000	0.9999	0.9979	0.9953	0.9887	0.9838	0.9487	0.8962	0.7278
1	1.0000	1.0000	1.0000	0.9992	0.9973	0.9891	0.9839	0.9487	0.8962	0.7278
5	1.0000	1.0000	1.0000	1.0000	0.9998	0.9946	0.9869	0.9491	0.8963	0.7279
10	1.0000	1.0000	1.0000	1.0000	1.0000	0.9977	0.9919	0.9501	0.8967	0.7279

Table H: $\lambda_2 = \frac{\varsigma_0 - \lambda_0}{\eta_1}$, R = 250m

ρ (m)	a (m)									
	10^{-3}	10^{-2}	10^{-1}	0.5	1	5	10	50	100	500
10^{-3}	-0.2342	-0.9338	-0.9747	-0.9790	-0.9789	-0.9740	-0.9685	-0.9281	-0.8713	-0.7216
10^{-2}	0.0001	-0.2270	-0.9275	-0.9631	-0.9673	-0.9668	-0.9620	-0.9217	-0.8645	-0.7202
10^{-1}	0.0000	0.0002	-0.2154	-0.8621	-0.9156	-0.9475	-0.9469	-0.9098	-0.8527	-0.7179
0.5	0.0000	0.0000	0.0127	-0.2021	-0.6123	-0.8982	-0.9166	-0.8929	-0.8374	-0.7153
1	0.0000	0.0000	0.0003	0.0307	-0.1940	-0.8345	-0.8850	-0.8803	-0.8270	-0.7136
5	0.0000	0.0000	0.0000	0.0004	0.0202	-0.1636	-0.5597	-0.8126	-0.7801	-0.7075
10	0.0000	0.0000	0.0000	0.0000	0.0005	0.0588	-0.1408	-0.7327	-0.7357	-0.7030