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STOCHASTICITY

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"STOCHASTICITY"*

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Detailed examination of computed particle trajectories has revealed a complexity and disorder that is of increasing interest to accelerator specialists. To introduce this topic, I would like you to consider for a moment the analysis of synchrotron oscillations for a particle in a coasting beam, regarded as a problem in one degree of freedom. A simple analysis replaces the electric field of the RF-cavity system by a traveling wave, having the speed of a synchronous reference particle, and leads to a pair of differential equations of the form

$$dy/dn = -K \sin \pi x, \qquad (1a)$$

where y measures the fractional departure of energy from the reference value, πx measures the electrical phase angle at which the particle traverses the cavity, and K is proportional to the cavity voltage; and

$$dx/dn = \lambda' y, \qquad (1b)$$

in which λ' is proportional to the change of revolution period with respect to particle energy. It will be recognized that these equations can be derived from a Hamiltonian function

$$H = (1/2)\lambda' y^2 - (K/\pi) \cos \pi x . \qquad (2)$$

Because this Hamiltonian function does not contain the independent variable explicitly, it will constitute a constant of the motion and possible trajectories in the x,y phase space will be just the curves defined by H = Constant, namely the familiar simple curves in phase space that are characteristic of a physical (non-linear) pendulum.

If we note, however, that a localized cavity can affect the energy of a particle only when the particle encounters the cavity, it is natural to replace the differential equations by difference equations. Thus, measuring energy y_n at the n^{th} entry to the cavity, we write the transformation

$$\left.\begin{aligned} y_{n+1} &= y_n - K \sin \pi x_n \\ x_{n+1} &= x_n + \lambda' y_{n+1} \end{aligned}\right\} \qquad (3a,b)$$

(which can readily be shown to be area-preserving). Although alternatively the motion in this case could again be expressed by differential equations derivable from a Hamiltonian function, the Hamiltonian now would contain a periodic δ-function of the independent variable as a factor multiplying the term $-(K/\pi)\cos \pi x$ and hence could not be taken as a constant of the motion. (The differential equations, moreover, would be non-linear, so that Floquet theory could not be applied.) The use of such a Hamiltonian formulation nonetheless can be helpful in analytic work, but difference equations of course are attractive for computational investigations.

It is of interest to take a quick look at some computational results obtained through use of a transformation equivalent to (3a,b) but written in terms of working variables $Y = y - (K/2) \sin \pi x$, $X = x$, so that the transformation assumes the form

$$\left.\begin{aligned} X_{n+1} &= X_n + \lambda' [Y_n - (K/2) \sin \pi X_n] \\ Y_{n+1} &= Y_n - (K/2)[\sin \pi X_n + \sin \pi X_{n+1}] \end{aligned}\right\}, \qquad (3a',b')$$

with the result that the resulting phase diagrams will necessarily have a desirable symmetry about both the X- and Y-axes. With $K/\pi = 0.1$ and $\lambda' = 0.1$ we find what appear to be conventional bucket diagrams with buckets separated in Y by $2/\lambda'$ for successive harmonic modes, although we may wish to return to the question of whether the bucket boundaries are as simple and definite as appears on Fig. 1.

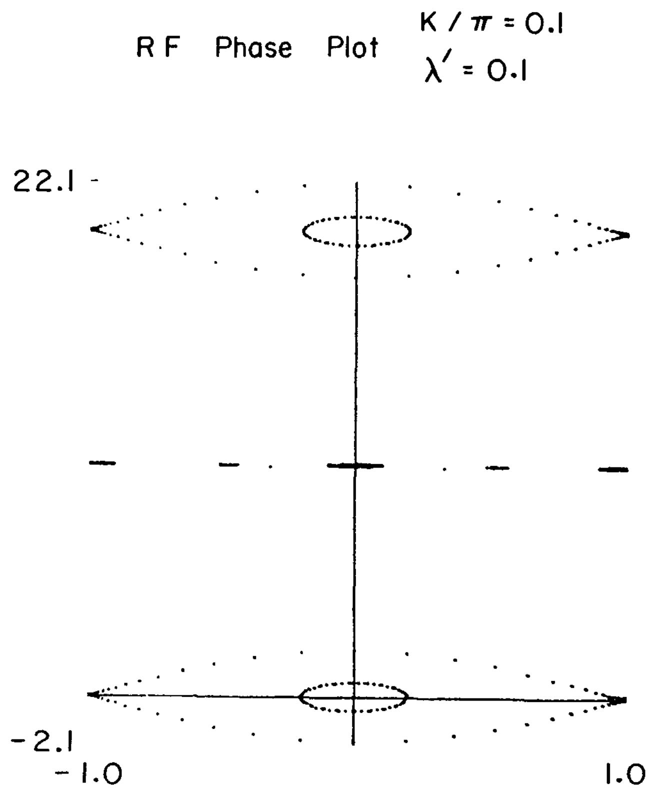

Fig. 1. - X,Y phase plot for a coasting beam under the influence of an R.F. cavity with $K/\pi = 0.1$, $\lambda' = 0.1$ -- as computed by Eqns. (3a',b'). X is plotted mod. 2.

We also find evidence of some "sub-harmonic" structure (with higher order fixed points) that, if enlarged some 60X, has the appearance shown in Fig. 2.[3]

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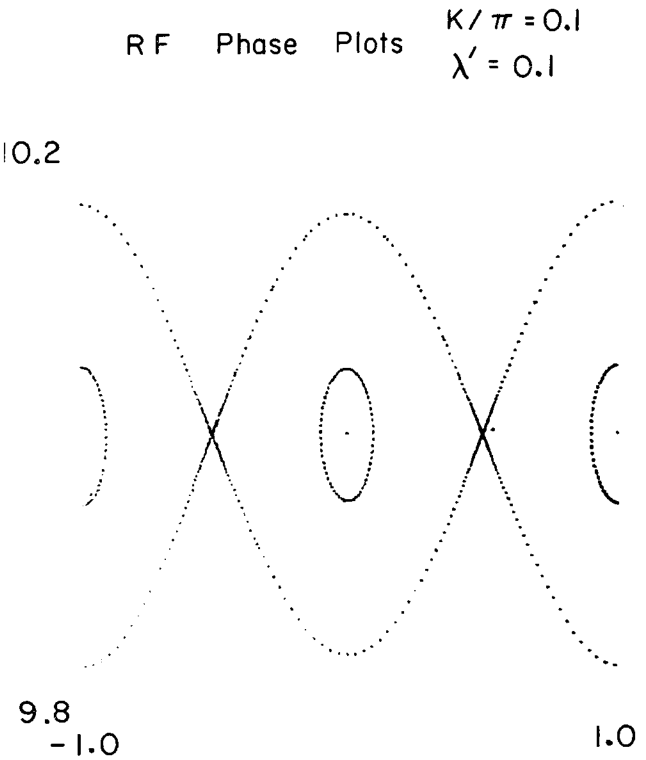

Fig. 2. - Circa 60-fold vertical enlargement of central portion of Fig. 1, near Y = 10.0, showing sub-harmonic structure.

If the cavity voltage is increased eight-fold (so $K/\pi = 0.8$), the bucket areas are expected to become larger, and we indeed find this to be the case (Fig. 3), with an accompanying very marked increase of complexity

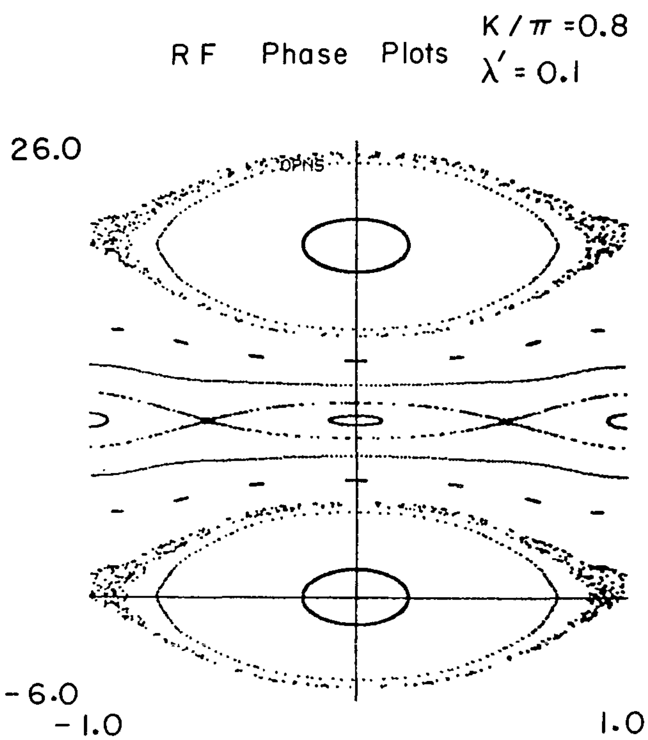

Fig. 3. - Phase plot similar to Fig. 1, but for operation with $K/\pi = 0.8$, showing the obvious development of complex structure.

that is immediately apparent in the phase plot. Of particular interest is the evident diffuse character of phase trajectories generated by points launched close to the first-order unstable fixed points situated at $X = \pm 1$, since the bucket boundary in consequence no longer appears clearly defined.

In the first example ($K/\pi = 0.1$), on the other hand, where the bucket width is some two and one-half times smaller in relation to the bucket separation, the presence of structure in the separatrix can be revealed computationally only with considerable care.[4] To do this, one can extend from the unstable fixed points the eigenvector directions of the transformation linearized about these fixed points, and examine whether such curves intersect smoothly. One finds in fact that they do not quite do so, but generate loops (of a nature to be illustrated later) that in this instance ($K/\pi = 0.1$) have a very small area that amounts to only about $1/(5 \times 10^{11})$ of the area of the bucket itself.

Similar questions concerning the character of phase trajectories and the possible erratic or stochastic behavior of canonical mappings can arise in problems with more than one degree of freedom. As an example, Hénon and Hiles[5] and subsequently Walker and Ford[6] studied a model of an astronomical system, for which the Hamiltonian function was taken to be

$$H = \frac{1}{2}(p_1^2 + p_2^2 + q_1^2 + q_2^2) + q_1^2 q_2 - \frac{1}{3} q_2^3 . \qquad (4)$$

The cubic terms appearing here as coupling terms become increasingly significant for increasingly large values of H -- which is itself a constant of the motion. With the coupling terms present, however, and in the absence of any simple constant of the motion other than H, a given phase trajectory might be expected to wander (ergodically) over virtually all of a three-dimensional surface specified by H = Constant (and that will be a closed surface for values of H below the dissociation energy). If, on the other hand, some additional integral of the motion were in fact also acting, the phase points of a given trajectory then would be constrained to lie on a two-dimensional surface, and graphs of the intersection of such surfaces with some selected plane or other surface (a "surface-of-section") would lead to simple curves in this plane rather than to a scattering of points. Computations of this nature indicated that for sufficiently small values of energy (e.g., $H \leqslant 1/12$) only curves that to computer accuracy were smooth (and relatively simple) were formed by intersection with the plane $q_1 = 0$ (and $p_1 \geqslant 0$). Examples in which the energy of the particles was successively raised, however, resulted in the development of ragged island structures or of apparent stochastic behavior over increasingly large portions of this surface-of-section (Fig. 4).

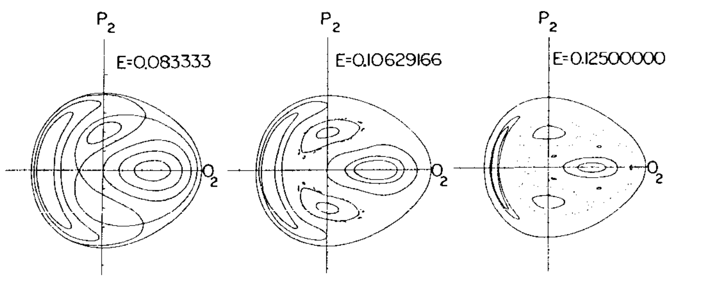

Fig. 4. - Phase plots, in the surface of section $q_1 = 0$, resulting from the equations implied by the Hamiltonian function (4) -- for increasing values of the energy. [After Walker and Ford.[6]]

Such behavior appears concordant with the "KAM" (Kolmogorov-Arnol'd-Moser) theory (see Refs. 58, 59, & 60 of our Ref. 1c), which suggests that many of the invariant curves or surfaces present in the absence of the perturbation will persist, with only minor distortion, in the presence of a sufficiently small perturbation (see, however, Note 7). It is of interest, of course, to determine or to estimate the circumstances (e.g., perturbation strength) at which the KAM theory becomes inapplicable and extended regions of erratic (or stochastic) behavior develop. As was suggested by our first examples, and has been expounded more extensevely by Zaslavskij and Chirikov,[1c,8] one means for obtaining such estimates may be by determining the ratio of resonance width $[\delta\omega = (d\omega/dI)_r \delta I]$ to the distance $(\Delta\omega)$ to the nearest neighboring resonance.

Additional tests (to be mentioned below) may be required to determine the degree of disorder associated with the movement of phase points in such stochastic regions. We may first note, however, that the existence of nested closed invariant curves in a plane -- as suggested by the KAM theorem for a problem in one degree of freedom -- prevents phase points from moving outward or inward to regions of substantially different "amplitude" (in the absence of noise). With more than one degree of freedom, however, stochastic layers may intersect, to form an intricate system of channels along which a phase point can slowly diffuse and result in instability. The possibility of such "Arnol'd diffusion" has been demonstrated by Arnol'd [Ref. 35 of our Ref. 1c; stated simply the example considered by Arnol'd is comprised of a physical pendulum and a simple-harmonic oscillator, with a time-dependent coupling (that also depends on the phases, or angle variables, of these oscillations)].

It should be pointed out that some non-linear transformations -- say for a system with one degree of freedom -- will not lead to the disappearance of some or all of the invariant phase curves at substantial amplitudes. Thus for transformations of the form

$$x_{n+1} = y_n; \quad y_{n+1} = -x_n + f(y_n), \qquad (5a,b)$$

McMillan[9] has shown that if $f(y)$ can be written as $\Phi(y) + \Phi^{-1}(y)$ (where Φ^{-1} denotes the function inverse to Φ), then the curves $y = \Phi(x)$ and $x = \Phi(y)$ will constitute invariant curves. Such curves will pass through the first-order fixed point(s) situated at the intersection(s) of $y = (1/2)f(x)$ with the principal diagonal. An enclosed area can thereby be formed from which phase points cannot escape even if the behavior in portions of the interior becomes highly stochastic. This is illustrated by an example (Fig. 5) in which

$$f(y) = \frac{1}{2}(3y-1) - \frac{1}{2}\frac{k^2}{y+1} + \sqrt{y^2+k^2} \qquad (6a)$$

and

$$\Phi(x) = x - 1 + \sqrt{x^2+k^2}\ . \qquad (6b)$$

Such a situation also can develop when $f(y)$ is a stepwise linear function of y with discontinuities of slope, as has been noted by Dr. Judd [see, for example, Figs. 13 and 14 (pp. 27-28) of Ref. 10]. If $f(y)$ is of the form

$$f(y) = -(By^2 + Dy)/(Ay^2 + By + C), \qquad (7)$$

moreover, the entire phase plane will be covered by a family of simple invariant curves -- see, for example, the cases[9] $f(y) = 2ky/(1+y^2)$, with the invariants $x^2y^2 + x^2 + y^2 - 2kxy = \text{Constant}$, and $f(y) = 2ky/(1-y^2)$, with the invariants $x^2y^2-x^2-y^2+2kxy = \text{Constant}$, illustrated by Figs. 6-8.

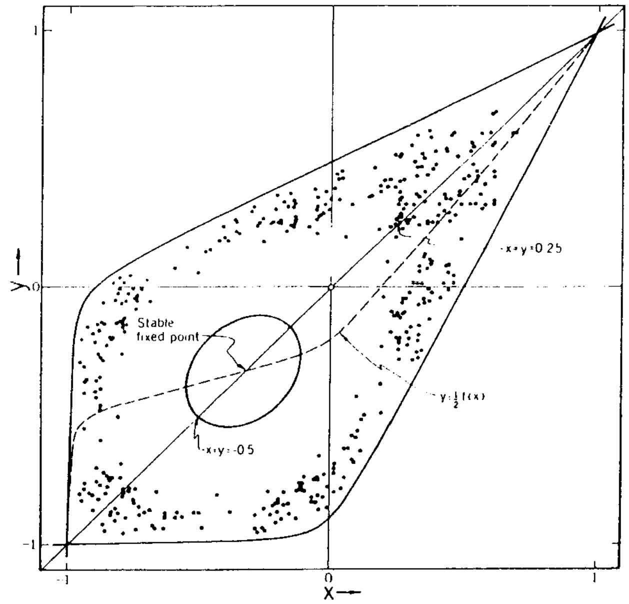

Fig. 5. - Phase diagram for the transformation (5a,b), with f(y) given by Eqn. (6a). The scattered points result from computations initiated with $x_0 = y_0 = 0.25$, but must remain within the separatrix defined by the function Φ [Eqn. (6b)].

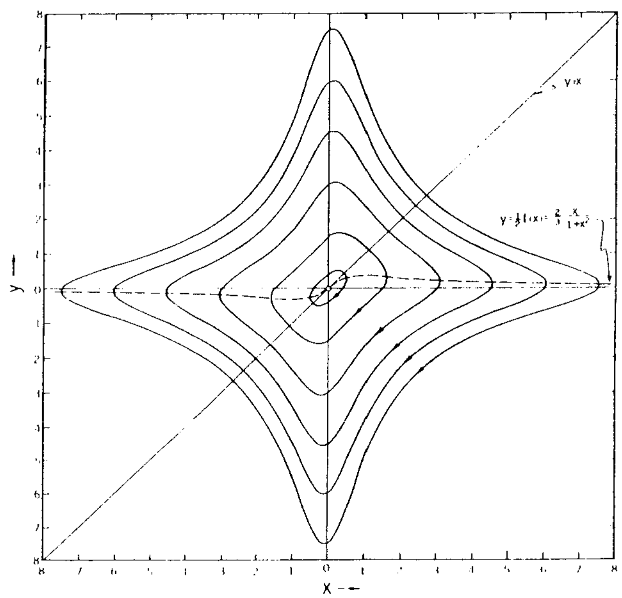

Fig. 6. - Invariant curves for the transformation (5a,b) with $f(y) = 2ky/(1+y^2)$ and $k = 2/3$. [Figs. 6-10 after McMillan.[9]]

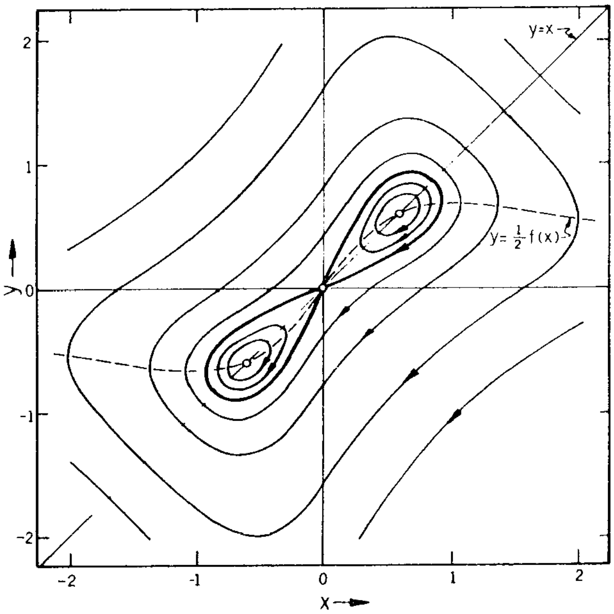

Fig. 7. - Invariant curves for the same transformation as in Fig. 6, but with k = 1.36.

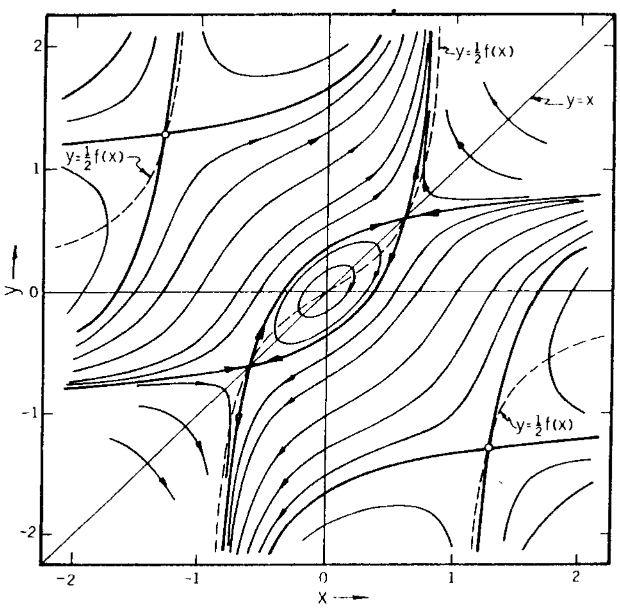

Fig. 8. - Invariant curves for the transformation (5a,b) with $f(y) = 2ky/(1-y^2)$ and k = 0.64.

It is of interest to examine the mechanism whereby irregular behavior can develop in the neighborhood of unstable fixed points, taking as an illustration an example suggested by Progessor deVogelaere that [when generalized and rewritten in variables leading to the form (5a,b) advocated by McMillan] employs

$$f(y) = 2[Ty + (1 - T)y^2]. \tag{8}$$

First-order fixed points appear at (0,0) and at (1,1). For T = 0, this transformation, when linearized about the unstable fixed point at (1,1), can be represented by the matrix $\begin{pmatrix} 0 & 0 \\ -1 & 4 \end{pmatrix}$, with eigenvalues and eigenvector slopes

$$\lambda = 2 \pm \sqrt{3}, \quad dy/dx = \lambda. \tag{9}$$

A line segment extending downward from the fixed point (1,1) with the slope $2 + \sqrt{3}$, if subjected to repeated applications of the transformation, generates the loops shown in Fig. 9; similarly a line segment of slope $2 - \sqrt{3}$, if extended by the inverse transformation, generates the mirror-image curve (mirrored about the principal diagonal). Points such as A, B, C ··· progress toward the fixed point in smaller and smaller steps and, since the transformation is area-preserving, the associated loops clearly must become increasingly elongated as they become increasingly narrow from repeated applications of the forward transformation. The evolution of such loops clearly will become quite intricate (Fig. 10),

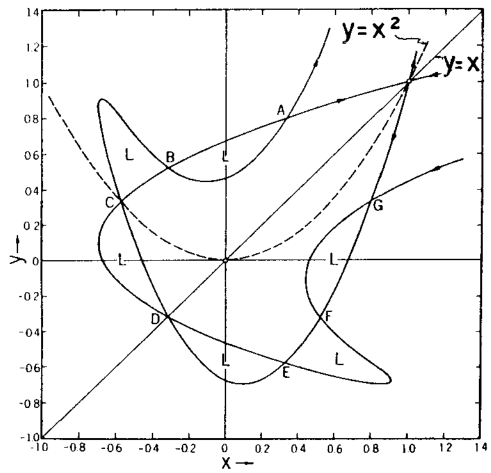

Fig. 9. - Plot of the extensions of the eigenvector directions from the unstable fixed point at (1,1), for the deVogelaere transformation expressed in McMillan's variables [Eqns. (5a,b) and (8), with T = 0]. The areas of the loops marked L are all equal, by virtue of the area-preserving character of the transformation and the inherent symmetry about the principal diagonal.

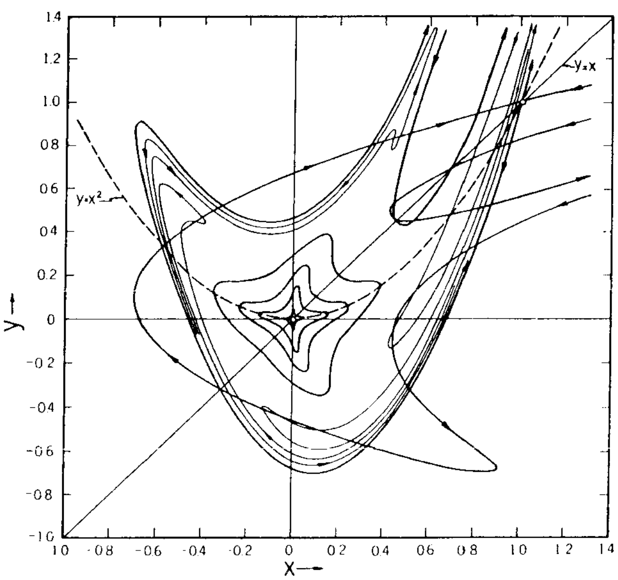

Fig. 10. - A partial extension of the curves shown on Fig. 9.

but the loops apparently need not permeate the entire "interior" -- portions of an inward loop can, in fact, enter, on a later iteration, into the interior of an outward-lying loop (as indicated on Fig. 10).[11] It is clear, however, that the development of such a loop system can readily give rise to an apparent stochastic motion of phase points in portions of the phase diagram - - most particularly near an unstable fixed point such as that mentioned here.

The existence of a firm separatrix, or of an extensive family of invariant curves generally, can be extremely sensitive to the exact form of the transformation.[12] A case of some physical interest arises in computational studies relating to the Toda Lattice.[13] This one-dimensional lattice consists of particles interacting through exponential pair potentials and can propagate certain non-linear wave forms ("solitons") without change of shape. One computational investigation[14] of stability for a three-particle lattice (with periodic boundary conditions) has commenced with a Hamiltonian function

$$H = \frac{1}{2}(P_1^2+P_2^2+P_3^2)+e^{-(Q_1-Q_3)}+e^{-(Q_2-Q_1)}+e^{-(Q_3-Q_2)}. \quad (10)$$

By a canonical transformation of variables, in recognition of the invariance of this system to translation -- so that $I_1 = P_1 + P_2 + P_3$ constitutes a constant of the motion -- the Hamiltonian (10) becomes expressible as a function of two pair of conjugate variables in the form

$$H = \frac{1}{2}(p_1^2+p_2^2)+ \frac{1}{24}[e^{(2q_2+2\sqrt{3}q_1)}+e^{(2q_2-2\sqrt{3}q_1)}+e^{-4q_2}], \quad (11)$$

which is identical to the Hénon-Heiles Hamiltonian function (4) through terms of third order. It is of interest to examine whether in the present case constants of the motion other than H act to restrict the motion. Computationally it was found -- again using the surface-of-section $q_1 = 0(p_1 > 0)$ -- that in this case simple invariant curves apparently continued to exist in the q_2p_2, plane, even for very large values of H. Stimulated by this result, Hénon[15] has directed attention to an additional integral of the motion that is valid in this case; the constants of the motion for the three-particle lattice then can be written in a form that we may express as[16]

$$H = \text{Constant} \quad (12a)$$

$$P_1 + P_2 + P_3 = \text{Constant, and} \quad (12b)$$

$$P_1P_2P_3 - P_1e^{-(Q_3-Q_2)} - P_2e^{-(Q_1-Q_3)} - P_3e^{-(Q_2-Q_1)} = \text{Constant}. \quad (12c)$$

[Evidently[15] further analytic work in fact has now established that the n-particle Toda lattice with periodic boundary conditions (or with fixed ends) is a "completely integrable" system.]

It is of some interest to seek means for anticipating whether stochastic behavior will occur in various portions of a phase diagram and to examine the character of such stochastic behavior as does occur. What we here have loosely termed stochastic behavior can be catalogued with respect to a hierachy of properties (ergodicity, mixing, ···), indicative of increasing disorder, that are fundamentally significant for statistical mechanics.[1a,e] Of particular interest to the accelerator designer, of course, is the determination of a threshold beyond which stochastic behavior will set in and may act to carry a phase point to unacceptably large amplitudes. As noted earlier, stochastic behavior appears to be associated with overlapping resonances,[1c] and this concept has served as the basis for some analytic estimates of stochasticity limits.[1c,17] It has been noted by René deVogelaere and confirmed in subsequent computations[18] that for a particular class of fixed-point families -- say those with rotation of the form m/(4m+1) -- there is a closely linear relationship between the order of the resonance (4m+1) and $\ln|1 - \frac{1}{2}\text{ Trace}|$ through many decades ("Trace" denoting the trace of the tangential-mapping or differential matrix associated with the 4m+1 iterations required to map a given fixed point onto itself). Such regularities, and others relating to the apparent size of the stable areas about high-order fixed points (e.g., as estimated from the intersection angle of eigenvectors), have been considered useful indicators of the change in character of a mapping at certain amplitudes.[19,8,20]

A computational procedure of considerable interest for recognizing stochasticity is that in which one follows the evolution of the distance between two initially very close points in phase space. In practice it can prove desirable to reduce the separation from time to time by a recorded factor whenever the separation becomes excessive during the computations, or, perhaps preferably, to evaluate the growth of an infinitesimal vector through use of the cumulative tangential-mapping matrix. A high degree of stochasticity can be ascribed to the behavior of the transformation if there are such vectors whose length generally grows beyond the first iteration by a factor greater than unity (while others may similarly contract). (Ref. 1a, p. 55; for examples, see Ref. 21.) An analogous procedure -- that can be more attractive, although possibly of a less direct basic significance -- is an investigation of the growth of the eigenvalue(s) of the cumulative tangential mapping. Such eigenvalues can change sign repeatedly during the course of many iterations, and hence will be seen to decrease from time to time, but an exponentially increasing trend in eigenvalue magnitude is likely to be associated with a similar type of increase for the lengths of the vectors mentioned previously. The nature of eigenvalue growth has been illustrated by Froeschlé[22] for the transformation[23]

$$\left.\begin{aligned} x_{n+1} &= x_n \cos\alpha - (y_n - x_n^2)\sin\alpha \\ y_{n+1} &= x_n \sin\alpha + (y_n - x_n^2)\cos\alpha \end{aligned}\right\}. \quad (13a,b)$$

The general characteristics of this transformation, expressed in variables such that the transformation has the symmetry of McMillan's form, is seen on Fig. 11. On an expanded scale (X10), we see (Fig. 12) the sudden onset of erratic behavior as the starting values for the transformation are successively increased (in steps $\Delta x_o = 0.0025$, for $y_o = 0$), and on a scale expanded by a further factor 100/6 we see (Fig. 13) the presence of a great deal of additional structure within a portion of this "stochastic" region. Associated with the transition to the stochastic region there appears to be a marked change in the manner of growth of $\psi_n = \log|\lambda_n|$ (linear, vs. n, in the stochastic case -- indicative of an exponential trend for $|\lambda_n|$) or of the "Cesaro mean" $\mu_n = \frac{1}{n}\sum_{m=1}^{n}\frac{1}{m}\psi_m$ (constancy in the stochastic case, monotonically decreasing otherwise -- Fig. 14).[24] Such methods indeed may prove useful in investigating computationally the possible development of stochastic motion in storage-ring devices. Extended computations of this nature can present challenging problems with respect to computer accuracy.[25]

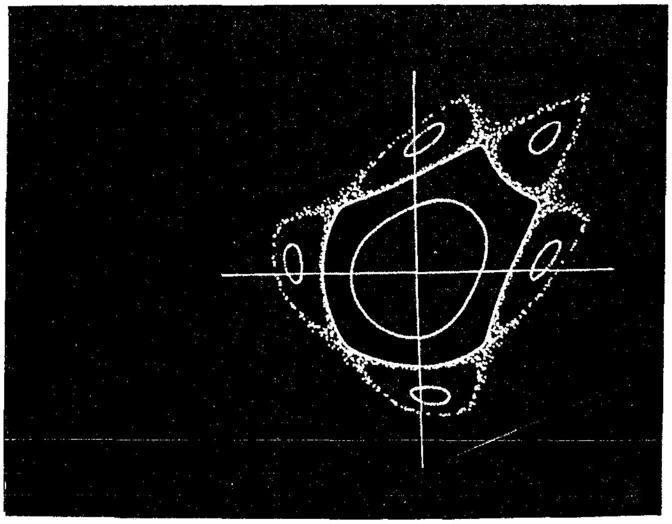

Fig. 11. - Apparently-smooth phase curves and a scattering of points resulting from iteration of the transformation (11a,b), with $\cos\alpha = 0.22$ and coordinates X,Y appropriate to expressing the transformation in the form (5a,b).[23] Five islands of stability (containing stable fixed points of order 5) are seen surrounding the area associated with the order-1 fixed point at the origin. The outermost smooth curve, shown as bounding this inner area, resulted from the starting values $x_0 = 0.5350$, $y_0 = 0$ (Froschlé notation), and the scattered points result from $x_0 = 0.5375$, $y_0 = 0$. Scale (as indicated by the coordinate axes): -1.0 to 1.0

Fig. 13. - Detailed multiple-island structure in the immediate neighborhood of an order-65 stable fixed point (shown here just below the center of the diagram) of which mention has been made in the caption to Fig. 12. Scales: 0.470 to 0.482 for X, 0.516 to 0.528 for Y.

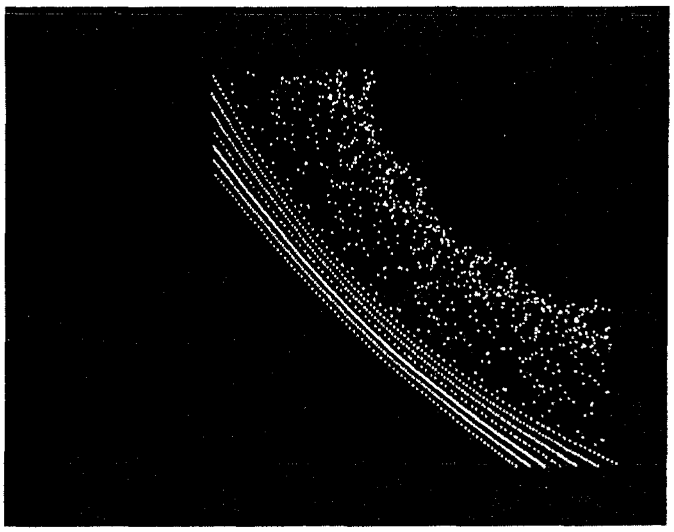

Fig. 12. - Enlarged portion (10X) of Fig. 11, showing seven smooth phase trajectories resulting from starting values $x_0 = 0.5200, 0.5225, \cdots 0.5350$ (and $y_0 = 0$) and a scattering of points resulting from $x_0 = 0.5375$, $y_0 = 0$. Note the occurrence of open areas within the region covered by the scattered points -- for example the area surrounding an (unplotted) stable fixed point of order 65 at $X \cong 0.476$, $Y \cong 0.521$
Scale: 0.38 to 0.58

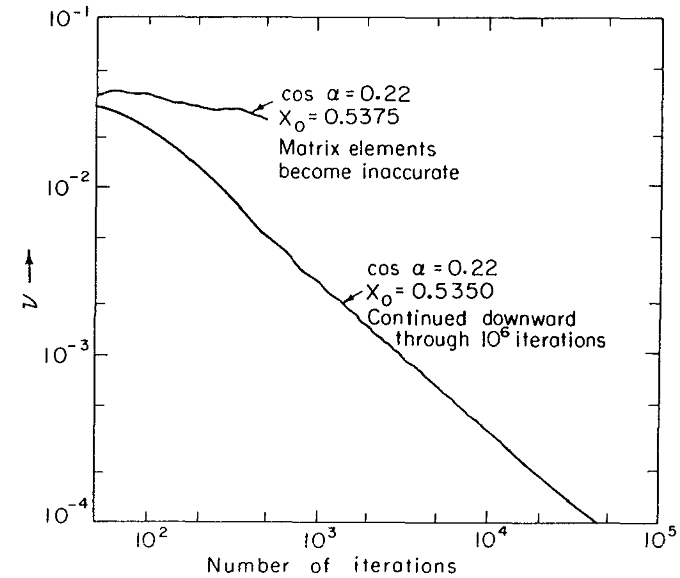

Fig. 14. - Plots of the "sliding mean", ν_n (Note 24), vs. n, obtained from computations begun (i) with initial conditions leading to the last smooth curve of Fig. 12 ($x_0 = 0.5350$) and (ii) with initial conditions leading to the scattered points on that Figure ($x_0 = 0.5375$), of which only the results for the latter case indicate a general exponential upward trend of $|\lambda_n|$.

References and Notes

1. (a) An authorative treatment of the mathematical aspects of the problems discussed here is given by V.I. Arnol'd and A. Avez, "Ergotic Problems of Statistical Mechanics" (Benjamin, New York, N.Y.; 1968) and (b) by Jürgen Moser, "Stable and Random Motions in Dynamical Systems" (Princeton Univ. Press, Princeton, N.J.; 1973); (c) an extended discussion of which portions relate more immediately to those of interest to an accelerator designer is presented, with many references, by G.M. Zaslavskij and B.V. Chirikov, Uspekhi Fizicheskikh Nauk 105 and Engl. Transl. Sov. Phys. Usp. 14, 549-568 (1972), with further discussion and examples (d) by B.V. Chirikov in Nucl. Phys. Inst. Report 267 (Novosibirsk, USSR) with Engl. Transl. by A.T. Sanders, CERN Trans. 71-40 (CERN, Geneva, Switzerland; October 1971); and (e) related questions of ergotic theory in statistical mechanics are summarized by J.L. Lebowitz and O. Penrose, Physics Today, pp. 23-29 (February 1973).

2. CNRS Internat. Conf. on Point Transformations and their Applications, Laboratoire d'Automatique et d'Analyse des Systèmes, Toulouse, France (10-14 September 1973).

3. K.R. Symon and A.M. Sessler, Proc. CERN Symposium on High Energy Accelerators and Pion Physics 1, 44-58 (CERN, Geneva, Switzerland; 1956).

4. L. Jackson Laslett, ERAN-57 (Lawrence Berkeley Laboratory; 1970).

5. M. Henón and C. Heiles, Astron. J. 69, 73-79 (1964).

6. G.H. Walker and J. Ford, Phys. Rev. 188, 416-432 (1969).

7. The present example in fact is exceptional in that the unperturbed frequencies, being equal, are "rationally connected" and the analysis requires special treatment -- see F.G. Gustavson, Astron. J. 71, 670-686 (1966); J. Moser, "Lectures on Hamiltonian Systems", Memoirs Amer. Math. Soc., No. 81, 1-60 (1968).

8. See also V.K. Mel'nikov, Soviet Math. 4, 266-270 (1963).

9. Edwin M. McMillan, "A Problem in the Stability of Periodic Systems", in "Topics in Modern Physics -- A Tribute to Edward U. Condon", pp. 219-244 (Colorado Asso. University Press, Boulder, Colorado; 1971). A transformation written in the form (5a,b) is convenient for the study of area-preserving transformations in the plane because of the "double symmetry" pointed out by McMillan (p. 225). The transformation can be interpreted as describing the effect of a simple linear focusing system supplemented by a periodic sequence of thin non-linear lenses that introduce at such points a Δy specified by $f(x)$.

10. Laslett, McMillan, and Moser, Courant-Institute Report NYO-1480-101 (New York University, N.Y.; 1 July 1968).

11. A wealth of island structure of course can develop throughout the area of such phase diagrams. In some instances a family of unstable fixed points for which the eigenvalues are negative may arise (in place of a stable family, for which λ is purely imaginary), and the appearance of phase trajectories can thereby be drastically affected -- see Ref. 10, esp. Fig. 2 (p. 35) of the Appendix, drawn for $T = -1/8$. For discussion of the occurrence and consequences of loop systems, see S. Smale, "Diffeomorphisms with Many Periodic Points ...", (Princeton Univ. Press, Princeton, N.J.; 1965); E. Zehnder, Comm. Pure Appl. Math. 26, 131-182 (1973); Ref. 1c, Sect. 6.1; and Ref. 1d, Sect. 2.6.

12. The loss of a firm separatrix can be illustrated computationally for the transformation (5a,b) by modifying the function $f(y)$ of (6a) so as to introduce the quantity $1 - b$ as a factor multiplying the second term on the right and setting $b \neq 0$ (for example, $b = 0.05$) -- L. Jackson Laslett, ERAN-239 (in preparation; 1974).

13. See, for example, M. Toda, Prog. Theoret. Phys. (Kyoto) Suppl. 45, 174-200 (1970); references cited therein; and related papers in this issue of the Supplement.

14. J. Ford, S.D. Stoddard, and J.S. Turner, Prog. Theoret. Phys. (Kyoto) 50, 1547-1560 (1973).

15. Cited in Ref. 13, p. 1558.

16. The validity of these (time-independent) expressions as constants of the motion of course can be confirmed directly by forming their Poisson-bracket expressions with the Hamiltonian function (10).

17. B.V. Chirikov, E. Keil, and A.M. Sessler, CERN Report ISR-TH/69-59 (CERN, Geneva, Switzerland; 15 October 1969).

18. E.g., Ref. 10, pp. 42-43, where is also listed a quantity $\lambda_I = \lambda - 1$ for fixed-point families that have rotation $\frac{m}{4m+1}$.

19. John M. Greene, J. Math. Phys. 9, 760-768 (1968).

20. James H. Bartlett, "Stability of Area-Preserving Mappings", Paper III-3 of Ref. 2.

21. J. Ford and G.H. Lunsford, Phys. Rev. A1, 59-70 (1970).

22. C. Froeschlé, Astron. and Astrophys. 9, 15-23 (1970); C. Froeschlé and J.-P. Scheidecker, ibid. 22, 431-436 (1973); and other references cited therein.

23. This transformation can be put into McMillan's form[9] by the change of variables $x = \sqrt{\sin\alpha}\, Y$, $y = (X - Y\cos\alpha)/\sqrt{\sin\alpha}$, with $f(Y)$ then becoming $2Y\cos\alpha + Y^2 \sin^{3/2}\alpha$.

24. The curves of Fig. 14 are plots of

$$\nu_n \equiv \sum_{m=1}^{n} \frac{1}{m}\,\Psi_m \exp\left(-\frac{n-m}{\tau}\right) \Big/ \sum_{m=1}^{n} \exp\left(-\frac{n-m}{\tau}\right) \text{ with}$$

$1/\tau = 0.015$, the sliding exponential factor being designed to provide some smoothing of the results (L. Jackson Laslett, unpublished LBL Report). Extended computations of this nature can present challenging problems with respect to computer accuracy. (Refs. 24 & 25).

25. C. Froeschlé and J.-P. Scheidecker, J. Comp. Phys. 11, 423-439 (1973); ____, Astrophys. & Space Sci. 25, 373-386 (1973).

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