

An abbreviated version of this paper  
has been submitted to The Proceedings  
of the VIIIth International Conference  
on High Energy Accelerators, CERN (1971)

LBL-42  
Preprint

THE USE OF RF-KNOCKOUT FOR DETERMINATION OF THE  
CHARACTERISTICS OF THE TRANSVERSE COHERENT  
INSTABILITY OF AN INTENSE BEAM

Dieter Møhl and Andrew M. Sessler

August 1971

AEC Contract No. W-7405-eng-48

An abbreviated version of this paper has been submitted to The Proceedings of the VIIIth International Conference on High Energy Accelerators, CERN (1971).

LRL-42  
ERAN-161

### Introduction

Transverse coherent beam instabilities<sup>1</sup> have been observed in virtually all high intensity accelerators and storage rings. The theory of these instabilities is well established. However, it is generally difficult to make an accurate estimate of the wall and beam properties that enter into the theory. This difficulty results from the complex nature of the beam environment.

The present note gives an analysis of a technique by which transverse stability as a function of beam and wall properties can be measured. For simplicity we restrict the treatment to the case of dipole oscillations of a single-species beam. The basic idea is to observe the response of a beam to a deflecting rf-field. This technique was first used by the Mura group<sup>2</sup> several years before the detailed nature of the instability was understood. More recently, similar experiments have been performed on the Bevatron.<sup>3</sup> The Mura group<sup>2</sup> also gave a simplified analysis of the method, based on a single particle dynamics.

Although the model considered by the Mura group explains some of the important features of the instability, it does not include the interaction of the particles through both local and wake fields, nor does it give a quantitative description of the effect of Landau damping. The present note gives an analysis of beam response to a driving force in the presence of both self-field interaction and frequency spread. We find that a single analytic function (of complex frequency) enters into the dispersion relation that determines beam stability and into the response function that describes a driven beam oscillation. Thus analytic continuation permits, at least in principle, the determination from measured data on beam response to a driving force, of all the relevant parameters

## THE USE OF RF-KNOCKOUT FOR DETERMINATION OF THE CHARACTERISTICS OF THE TRANSVERSE COHERENT

### INSTABILITY OF AN INTENSE BEAM \*

Dieter M<sup>†</sup>chl and Andrew M. Sessler

Lawrence Berkeley Laboratory<sup>††</sup>  
University of California  
Berkeley, California

August 1971

#### Abstract

The response of an intense beam of interacting particles to a deflecting rf-signal is computed theoretically and shown to be closely related to transverse coherent beam stability. It is shown that the beam response to sinusoidal excitation provides a direct measure of the stability of beam modes for given machine conditions (beam intensity, octupole current, sextupole current, momentum spread, etc.). This measurement includes the properties of the beam surroundings as well as the frequency spread effective for Landau damping. Since it is generally difficult to evaluate theoretically the wall and beam properties that enter into stability calculations, the information which can be obtained from rf excitation experiments should be very valuable; especially in devising practical procedures for reducing the severity of coherent transverse instabilities.

---

\* Work supported by the U.S. Atomic Energy Commission.

<sup>†</sup> Permanent address: CERN, Geneva 23, Switzerland.

<sup>††</sup> Formerly Lawrence Radiation Laboratory.

describing beam stability.

Alternatively, one can regard a beam as a harmonic oscillator with a frequency-dependent "damping constant". This damping constant, which results from Landau damping and wake-field antidamping, when evaluated at the frequency of a natural mode of the system, is a measure of stability. The response of a beam to a driving force provides a measure of the damping constant at neighboring frequencies and by analytic continuation the damping at the mode-frequency can be obtained. Similarly, other information such as the stability coefficients  $U$ ,  $V$  and the frequency spread  $\Delta S$ , [see Ref. (1)], can be deduced from the response function.

### 1. Equations of Motion

The equation of motion of the  $i$ -th particle may, in linear approximation, be written as

$$\frac{d^2 x_i}{dt^2} + Q_i^2 \Omega_i^2 x_i + A x_i = -B \bar{x} + G e^{-i\omega_{rf} t} \quad (1.1)$$

where  $x_i$  is the position of the  $i$ -th beam particle and  $\bar{x} = \frac{1}{N} \sum x_i$  is the position of the beam center of mass.

Space-charge forces acting between the particles are described by  $A$  and  $B$ . Only linear space-charge forces are included, so that higher order terms in  $x_i$  and  $\bar{x}$  are neglected. Actually, the term  $B\bar{x}$  contains both the local space-charge field as well as wake fields left by particles which are located at a different azimuthal position in the beam. However, for the coherent oscillation,  $\bar{x}$  is the same at every azimuth except for a phase factor. We take the influence of this phase factor to be included in  $B$ .  $Q_i^2 \Omega_i^2 x_i$  presents the external focusing ( $\Omega_i$  is the  $i$ -th particle revolution frequency). The action of the knock-out electrode on particle  $i$

is described by the  $G$ -term (Appendix 1). Only the harmonic of the electrode field with  $\omega_{rf} \approx (n \pm Q_0) \Omega_0$  is retained ( $\Omega_0$  is the average revolution frequency and  $Q_0$  is the small-amplitude tune of a particle of average energy). The time derivative occurring in (1.1) is the "hydrodynamic derivative"

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \Omega_i \frac{\partial}{\partial \theta} \quad (1.2)$$

The coefficients  $A$  and  $B$  can be interpreted in terms of the familiar coherent and incoherent frequency shifts<sup>4</sup>, and also in terms of the stability coefficient<sup>1</sup>  $U + V + iV$ . The relevant relations, which were first obtained by L.J. Laslett (private communication), are derived in Appendix 2 and summarized in Table 1. We note that the "single particle frequency shift"  $A$  is real, whereas the "coherent shift"  $B$  includes contributions from resistive walls and, in general, is a complex quantity and different for different modes.

TABLE 1

Relation between the quantities A and B [Eq. (1.1)]; the Laslett Q-shifts  $\Delta Q_{ic}$ ,  $\Delta Q_c$ ; and the LNS-coefficients U and V.

$U + V + iV = - \frac{B}{2\Omega_o^2 Q_o}$	: LNS-stability coefficient
$\Delta Q = - \frac{A + B}{2\Omega_o^2 Q_o}$	: Coherent betatron frequency depression due to space-charge (coherent Laslett Q-shift)
$\Delta Q_{ic} = - \frac{A}{2\Omega_o^2 Q_o}$	: Incoherent Q-shift due to space-charge (single-particle Q-shift)

By combining these relations,

$$U + V + iV = \Omega_o (\Delta Q_c - \Delta Q_{ic}).$$

The above  $\Delta Q_c$  includes in-phase (resistive) components of the self-field.

## 2. Solution of the Basic Equation

In solving (1.1) we must take the effect of frequency spread into account. The case in which this spread is due to momentum spread is simple and will be considered first. The response of the l.h.s. of (1.1) to the driving force  $Ge^{-i\omega_{rf}t}$  is simply

$$x_i = \frac{Ge^{-i\omega_{rf}t}}{Q_i^2 \Omega_i^2 + A - (\omega_{rf} - n\Omega_i)^2} \quad (2.1)$$

Therefore, following the procedure first outlined by E. Courant<sup>5</sup>, we insert a trial solution

$$x_i = G f_i e^{-i\omega_{rf}t} \quad (2.2)$$

$$\bar{x} = GF e^{-i\omega_{rf}t}$$

The complex functions  $f_i(\omega_{rf})$  and  $F(\omega_{rf})$  include the phases of the oscillation and we assume that A and B are independent of frequency, for frequencies near the mode frequency.

Using the fact that

$$\bar{x} = \frac{1}{N} \sum x_i \approx \int n(p) x(p) dp \quad (2.3)$$

we find the beam response

$$F(\omega_{rf}) = \frac{1}{1/I_p + B} \quad (2.4)$$

where

$$I_p(\omega_{rf}) = \int_0^\infty \frac{n(p) dp}{(Q\Omega)^2 + A - (\omega_{rf} - n\Omega)^2} \quad (2.5)$$

The function  $n(p)$  is the energy distribution function of the particles, and  $\int_0^\infty n(p) dp = 1$ . Both  $Q$  and  $\Omega$  are functions of  $p$ .

Equations (2.2), (2.4), and (2.5) describe the response of the beam. In addition to the particular solution (2.2) the oscillation of a particle contains the free betatron oscillation which is of random phase and therefore does not contribute to the average motion (2.3). If the beam is unstable there will be growing collective oscillations at the frequencies of the beam normal modes. These terms don't contribute to the response at frequency  $\omega_{rf}$ , although they usually would preclude observation of beam response to the rf. This point is elaborated upon in Section 5.

Next, we proceed to include the frequency spread due to nonlinearities in the external focusing. It was pointed out by Hereward<sup>6</sup> that in this case the response of the l.h.s. of (1.1) to the driving force  $Ge^{-i\omega_{rf}t}$  is

$$x_i = \frac{1 - K}{(Q\Omega)^2 + A - (\omega_{rf} - n\Omega)^2} - K \frac{(Q\Omega)^2 + A + (\omega_{rf} - n\Omega)^2}{[(Q\Omega)^2 + A - (\omega_{rf} - n\Omega)^2]^2} Ge^{-i\omega_{rf}t} \quad (2.6)$$

Here  $K = \frac{1}{2} \frac{d(Q\Omega)}{da} \frac{a}{Q\Omega}$  is determined by the amplitude-dependence of the external betatron frequencies  $Q\Omega$ . Eq. (2.6) is correct to first order in  $K$  and  $G$ .

Using (2.6) we obtain in a similar fashion as was used to derive (2.4), [see Ref. (6)]:

$$F(\omega_{rf}) = \frac{1}{1/I_a + B} \quad (2.4a)$$

where

$$I_a(\omega_{rf}) \approx - \int_0^\infty \frac{g'(a) a^2 Q\Omega da}{(Q\Omega)^2 + A - (\omega_{rf} - n\Omega)^2} \quad (2.5a)$$

Here,  $g(a)$  is the amplitude distribution function of the particles, and we have normalized  $g(a)$  such that

$$\int_0^\infty g(a) d(a^2 Q\Omega) = 1.$$

both  $Q$  and  $\Omega$  are functions of  $a$ .

Finally we consider the combined effect of momentum spread and nonlinearity. We first consider a group of particles with the same momentum, but of different betatron amplitude; and thereafter sum over all groups. If there is no correlation between betatron amplitude and momentum of the particles we may write

$$F(\omega_{rf}) = \frac{1}{1/I + B} \quad (2.4b)$$

we now

$$I(\omega_{rf}) \approx - \int_0^\infty \int_0^\infty \frac{g'(a) a^2 Q\Omega n(p) da dp}{(Q\Omega)^2 + A - (\omega_{rf} - n\Omega)^2} \quad (2.5b)$$

$$\int_0^\infty n(p) dp = 1$$

$$\int_0^\infty g(a) d(a^2 Q\Omega) = 1.$$

Now  $Q$  and  $\Omega$  are functions of  $a$  and  $p$ .

The observation which forms the basis for this paper is that  $I(\omega_{rf})$ , Eq. (2.5b), is the same dispersion integral that appears in the LNS theory<sup>1</sup> and is used there to determine the complex mode-frequency  $\omega_c$  from the relation\*

$$1/I(\omega_c) = 2 Q_0 \Omega_0 (U + V + iV) \quad (2.6)$$

In the present notation, we write (2.6) as

$$1/I(\omega_c) + B \equiv 1/F(\omega_c) = 0 \quad (2.7)$$

### 3. Measurement

#### 3.1 Sinusoidal Excitation

The fact that beam response (2.4b) and mode stability (2.7) are governed by the same function  $F(\omega)$  suggests determining the mode-frequency  $\omega_c$  by analytic continuation of the function  $F(\omega_{rf})$  as measured for real frequencies  $(\omega_{rf})$ .

To elucidate the procedure, let us introduce the inverse of  $F(\omega)$ :

$$X(\omega) = 1/F(\omega).$$

Now, because  $X(\omega)$  [as defined by (2.4b), (2.5b), and (3.1)] is analytic, we can expand around some frequency  $\omega_1$  to obtain  $X(\omega_c)$ :

\* The "single particle shift"  $A$  does not appear explicitly in Ref. 1 but is incorporated into the  $\nu$ -value. The function  $h(a)$  of Ref. 1 is related to  $g(a)$  used above by  $h' = 2g'Q\Omega$  and the  $a$ - and  $p$ -dependence of  $Q\Omega$  is neglected in the numerator of the dispersion integral.

$$X(\omega_c) = X(\omega_1) + \sum_{n=1}^{\infty} \frac{X^{(n)}(\omega_1)}{n!} (\omega_c - \omega_1)^n. \quad (3.2)$$

$X(\omega_1)$  and  $X^{(n)}(\omega_1)$  can be determined for  $\omega_1$  real from the measured response curve  $X(\omega)$  and hence  $\omega_c$  can be determined from (2.7) and (3.2).

If  $\omega_1$  is close to  $\omega_c$  we can neglect higher order terms in (3.2) and obtain, from (2.7),

$$(\omega_c - \omega_1) = - \frac{X(\omega_1)}{X'(\omega_1)} \quad (3.3)$$

Of great interest is the imaginary part  $\text{Im}(\omega_c)$  which is a direct measure of the mode stability. This quantity can, for example, be deduced from the phase response

$$\alpha(\omega) = \tan^{-1} \left\{ \frac{\text{Re}[X(\omega)]}{\text{Im}[X(\omega)]} \right\} \quad (3.4)$$

Thus, let us assume that we measure the slope of the quantity  $\tan \alpha$

$$s = \frac{d}{d\omega} (\tan \alpha) \quad (3.5)$$

at a frequency  $\omega_a$  where  $\frac{d}{d\omega} \text{Im}[X(\omega)] = 0$ . Then we have, by virtue of (3.3), (3.4), and (3.5),

$$s(\omega_a) = \frac{\text{Re}[X'(\omega_a)]}{\text{Im}[X(\omega_a)]} \quad (3.6)$$

and

$$\text{Im}(\omega_c) \approx -1/s(\omega_a). \quad (3.7)$$

In other words, the slope  $s(\omega_a)$  is a direct measure of beam stability.

The evaluation of measured data is simplified if the functional form of  $X(\omega)$  is known;  $X(\omega)$  in turn is determined by the dispersion

integral (2.5b). This integral is evaluated for various distribution functions in Refs. (1) and (7). For convenience we include results from some representative examples in Table 2.

In addition to the effective damping (3.7) the quantities  $A$ ,  $B$  and  $\Delta S$  can be deduced from the measured response curve  $F(\omega_{rf})$  if we can anticipate the shape of the distribution function. Let us take the function 2 of Table 2 as an example. We find the frequency  $\omega_a$  as defined above given by

$$\omega_a = (n - Q_o)\Omega_o - \frac{A}{2Q_o\Omega_o}. \quad (3.8)$$

At  $\omega_{rf} = \omega_a$ :

$$X(\omega_a) = B + i |\Delta S| Q_o \Omega_o \quad (3.9)$$

One may also e.g. measure the frequency  $\omega_b$  where  $\text{Re}[X(\omega_b)] = 0$ .

For  $|\Delta S| > |\text{Re}(B)/Q_o\Omega_o|$ :

$$\omega_b = (n - Q_o)\Omega_o - \frac{\text{Re}(B) + \frac{A}{2}}{Q_o\Omega_o} \quad (3.10)$$

$$X(\omega_b) = \text{Im}(B) + i Q_o\Omega_o \sqrt{(\Delta S)^2 - \left[ \frac{\text{Re}(B)}{Q_o\Omega_o} \right]^2} \quad (3.11)$$

The unknown quantities  $A$ ,  $\Delta S$ ,  $\text{Re}(B)$ ,  $\text{Im}(B)$  can then be derived from the measured quantities  $\omega_a$ ,  $\omega_b$ ,  $X(\omega_a)$ ,  $X(\omega_b)$ . Obviously, this is only one example of how  $A$ ,  $B$  and  $\Delta S$  can be deduced from the measured data.

TABLE 2

The dispersion integral  $I(\omega)$ : some examples

Assumptions:  $|\omega - (n - Q_0)\Omega_0| \ll \Omega_0$ ,  $|\Delta S| \ll \Omega_0$ ,  $\omega$  has a positive imaginary part, and the  $a$ -dependence of  $Q\Omega$  in the numerator of  $I(\omega)$  is neglected. Results for fast waves,  $\omega \approx (n + Q_0)\Omega_0$ , are obtained by replacing  $Q_0$  with  $-Q_0$ ,  $\frac{\partial Q}{\partial p}$  with  $-\frac{\partial Q}{\partial p}$ , and  $\frac{\partial Q}{\partial a^2}$  with  $-\frac{\partial Q}{\partial a^2}$ .

1. Momentum spread from a distribution function with a long tail (Lorentz Line).

$$n(p) = \frac{(\Delta p)}{\pi[(\Delta p)^2 + (p - p_0)^2]}$$

$$I(\omega) = \frac{1}{(2Q_0\Omega_0)(x^2 + 1)} \left( \frac{x}{\Delta S} - \frac{i}{|\Delta S|} \right),$$

with

$$x = \left[ \omega - (n - Q_0)\Omega_0 + \frac{A}{2Q_0\Omega_0} \right] / \Delta S,$$

and

$$\Delta S = \Delta p \left[ (n - Q_0) \frac{\partial \Omega}{\partial p} - \Omega_0 \frac{\partial Q}{\partial p} \right].$$

$$X(\omega) = 1/I + B = B + A - 2Q_0\Omega_0 [(n - Q_0)\Omega_0 - \omega - i|\Delta S|].$$

Remark: The damping term  $\text{Im}(1/I)$  is frequency independent because the distribution has a long tail.

2. Momentum spread from a distribution function with a cut-off.

$$n(p) = \begin{cases} \frac{2}{\pi(\Delta p)^2} \sqrt{(\Delta p)^2 - (p - p_0)^2}, & |p - p_0| < \Delta p; \\ 0, & |p - p_0| \geq \Delta p. \end{cases}$$

$$I(\omega) = \frac{1}{Q_0\Omega_0\Delta S} \left( x - \sqrt{x^2 - 1} \right),$$

with  $x = \left[ \omega - (n - Q_0)\Omega_0 + \frac{A}{2Q_0\Omega_0} \right] / \Delta S,$

and  $\Delta S = \Delta p \left[ (n - Q_0) \frac{\partial \Omega}{\partial p} - \Omega_0 \frac{\partial Q}{\partial p} \right].$

For real  $\omega$ , ( $\text{Im}(\omega) = +0$ ):

$$X(\omega) = B + \frac{A}{2} - Q_0\Omega_0 \left\{ (n - Q_0)\Omega_0 - \omega + \delta \sqrt{(\Delta S)^2 - \left[ (n - Q_0)\Omega_0 - \frac{A}{2Q_0\Omega_0} - \omega \right]^2} \right\},$$

$$\text{where } \delta = \begin{cases} -\text{sign}(\Delta S), & x \leq 1; \\ -i, & -1 < x < 1; \\ \text{sign}(\Delta S), & x \geq 1. \end{cases}$$

Remark: The damping term  $\text{Im}[1/I(\omega_{rf})] = 0$ , if  $|x| \geq 1$  as a result of the sharp cut-off of the distribution.

3. Amplitude spread from a distribution with a cut-off.

$$g(a) = \frac{2}{Q_0\Omega_0(\Delta a)^4} \begin{cases} (\Delta a)^2 - a^2, & a < \Delta a; \\ 0, & a \geq \Delta a; \end{cases}$$

$$I(\omega) = \frac{-1}{\Delta S Q_0 \Omega_0} \left[ 1 + y \ln \left( \frac{y-1}{y} \right) \right],$$

with

$$\Delta S = (\Delta a)^2 \left[ (n - Q_0) \frac{\partial \Omega}{\partial a^2} - \Omega_0 \frac{\partial Q}{\partial a^2} \right],$$

and

$$y = \left[ \omega - (n - Q_0) \Omega_0 + \frac{A}{2Q_0 \Omega_0} \right] / \Delta S.$$

For real  $\omega$ : [ $\text{Im}(\omega) = +0$ ]:

$$X(\omega) = B - \Delta S Q_0 \Omega_0 \left[ \frac{1 + y \ln \left| \frac{1-y}{y} \right| - i \delta \pi y}{(1 + y \ln \left| \frac{1-y}{y} \right|)^2 + (\delta \pi y)^2} \right],$$

$$\delta = \begin{cases} 1, & |y| < 1; \\ 0, & |y| \geq 1. \end{cases}$$

Remark: the damping term  $\text{Im} [1/I(\omega_{rf})] = 0$ , for  $y > 1$  since the distribution has a cutoff.

### 3.2 Pulse Excitation

An alternative measuring technique which can be used is based on pulse exciting the beam and observing the transient behaviour of the modes<sup>2</sup>. Excitation of a given mode may be accentuated by choice of the pulse waveform. The transient behaviour of a mode is clearly

$$\bar{x} = \bar{x}(0) e^{-i\omega_c t}.$$

with  $\omega_c$  the mode frequency determined by (2.7). The decay rate of the transient,  $-\text{Im}(\omega_c)$ , is a direct measure of the "effective damping."

### 4. Bunched Beams

The generalization to a machine with equally shaped, equally spaced and equally populated bunches<sup>8</sup> is straight forward. The same measuring techniques that were discussed for a coasting beam can be used in this case to measure the stability of "coherent bunch modes".

The other limiting case, where the bunch to bunch spread is large enough to decouple the bunches<sup>8</sup> will need a somewhat modified measuring method. Since the bunches are largely decoupled, each bunch will resonate at a slightly different frequency. By observing the response of a bunch in the neighborhood of its resonance we may measure the "single bunch stability". At the same time the bunch to bunch frequency spread can be detected.

### 5. Discussion

The quantities  $1/s(\omega_a)$  or  $-\text{Im}(\omega_c)$ , which can be measured as described in Section 3, are measures of the effective stability of the mode under consideration. Thus, by measuring these quantities as a function of relevant machine parameters, such as intensity, octupole current, energy spread and wall properties, one can predict threshold conditions [ $1/s(\omega_a) \rightarrow 0$ ,  $\text{Im}(\omega_c) \rightarrow 0$ ] and presumably thus devise procedures for reducing the instability.

These measurements can only be performed in an intensity range such that the machine is stable since otherwise the driven response will be masked by spontaneously growing (or stimulated) coherent modes. However, measured data can be extrapolated to the threshold. If the wall impedances are not strongly frequency-dependent, one can often make measurements near the stable modes and extrapolate from there to the frequencies of the unstable modes, a technique employed in both Refs. (2) and (3).



It is noted that the measurement of  $1/s(\omega_a)$  or  $\text{Im}(\omega_c)$

will not give explicit information on the values of U and V, but rather a quantity related to  $V - \Delta S$ . However, U and V are only of interest for calculating the effective beam stability and this quantity is directly obtained from the measurement. If required, U, V, and  $\Delta S$  can also be derived from the beam response curve, as described in Section 3.1.

Finally we note from the examples given in Table 2 that in general, both the "coherent Q-shift"  $\Delta Q_c$  as well as the "single particle Q-shift"  $\Delta Q_{ic}$  enter into the coherent beam response. Traditionally, the coherent Q-shift is worked-out under the assumption that all particles respond to the driving force in the same way. This assumption is correct in the absence of frequency spread or more precisely if the external driving frequency  $\omega$  is such that

$$\left| \omega - (n - Q_0)\Omega_0 + \frac{A}{2Q_0\Omega_0} \right| \gg |\Delta S| .$$

In this case we find, for any distribution function, that the beam resonance frequency (defined by  $\text{Re}[X(\omega_r)] = 0$ ) is given by

$$\omega_r = \frac{A + \text{Re}(B)}{2Q_0\Omega_0} - (n - Q_0)\Omega_0$$

Hence, neglecting the imaginary part of B, in this case the Q-shift is  $(A + B)/2Q_0\Omega_0^2$  and this agrees with the usual coherent Q-shift (see Table 1).

If, however, the frequency spread is large, more precisely if

$$\left| \omega - (n - Q_0)\Omega_0 + \frac{A}{2Q_0\Omega_0} \right| < |\Delta S|$$

then

$$\omega_r = \frac{A + \eta B}{2Q_0\Omega_0} - (n - Q_0)\Omega_0$$

where typically  $1 < \eta \leq 2$ . We note, from Table 1, that for  $\eta \neq 1$  both  $\Delta Q_{ic}$  and  $\Delta Q_c$  enter into the "resonance frequency"; i.e., internal forces, as well as wall terms, contribute to  $\omega_r$ . The same phenomenon has been observed in the analysis of beam behavior, near an integer resonance, in the presence of magnetic guide field imperfections (In this case  $\omega_r \rightarrow 0$ ).<sup>10</sup>

#### Appendix 1: Rf Excitation

##### 1.1 Unbunched Beam

Assume a deflecting rf-field localized at  $s = 0$ . Write the deflecting force as

$$F = 2\pi R G e^{-i\omega_{rf}t} \delta(s). \quad (A1)$$

Here, R is the orbit radius. Fourier expansion of the  $\delta$ -function yields

$$\delta(s) = \frac{1}{2\pi R} \sum_{n=-\infty}^{\infty} e^{-i\frac{n}{R}s} .$$

Assume an unbunched beam. Let the test particle at time t be at

$$s_i = R (\alpha_i + \Omega_i t) . \quad (A2)$$

Then the force acting on this particle is

$$F_i = G \sum_{n=-\infty}^{\infty} e^{-i(\omega_{rf} - n\Omega_i)t + in\alpha_i} . \quad (A3)$$

If one of the frequencies  $(\omega_{rf} - n\Omega_i)$  is close to the betatron frequency  $Q_0\Omega_0$ , the response of the beam to this harmonic will predominate.

For an observer at a fixed azimuth  $\theta$  the oscillation of the beam

is characterized by harmonics with  $x_n = \xi_n e^{i(n\theta - \omega_{rf} t)}$  and for  $\omega_{rf} \approx (n - Q_0)\Omega_0$  the pattern is similar to the n-th dipole mode of a coasting beam.

### 1.2 Bunched Beam

Let the center of bunch  $\ell$  move in the longitudinal direction according to

$$s_\ell = R \left( \frac{2\pi}{h} + \Omega_0 t \right) \quad (A4)$$

( $h = \text{rf harmonic} = \text{number of bunches}$ ).

The force (A2) acting on the bunch is

$$F_\ell = G \sum_{n=-\infty}^{\infty} e^{-i(\omega_{rf} - n\Omega_0)t + ins_\ell/R} \quad (A5)$$

We assume that the bunch length  $2\Delta\phi$  is small ( $n2\Delta\phi \ll 2\pi$ ) so that the bunch is deflected coherently.

Again taking only the response of bunch  $\ell$  to the harmonic of (A5) with  $(\omega_{rf} - n\Omega_0) \approx Q_0\Omega_0$  we write the equation of motion of particle  $i$  in bunch  $\ell$  as

$$\begin{aligned} \frac{d^2}{dt^2} x_{i,\ell} + Q_i^2 \Omega_i^2 x_{i,\ell} + A_\ell x_{i,\ell} + B_\ell \bar{x}_\ell \\ = \sum_{k=1}^h W_{k\ell} \bar{x}_k + G e^{-i\omega_{rf} t} \end{aligned} \quad (A6)$$

Here  $A_\ell$  and  $B_\ell$  are the local Coulomb and image forces acting within bunch  $\ell$  and the elements  $W_{k\ell}$  represent the wake fields acting from bunch  $k$  on bunch  $\ell$ ;  $W_{\ell\ell}$  describes the wake fields from the previous turns which act on bunch  $\ell$  due to its own motion.

Using the same procedure employed to derive (2.5) we assume

$$x_{i,\ell} = f_{i,\ell} G e^{-i\omega t} \quad (A7)$$

$$x_\ell = F_\ell G e^{-i\omega t}$$

Then from (A6) we obtain the following system of linear equations for the response,  $F_\ell$ , of bunch  $\ell$ :

$$(1/I_\ell + B_\ell) F_\ell - \sum_{k=1}^h W_{k\ell} F_k = 1 \quad (A8)$$

Here  $I_\ell$  is the integral (2.5b) for bunch  $\ell$ . For coherent bunch modes<sup>1</sup> all  $F_k(\omega)$  are equal. For single bunch modes<sup>8</sup>, and near the resonance of bunch  $\ell$ , all terms of the sum except the term with  $W_{\ell\ell}$  can be neglected. In either case only  $F_\ell(\omega)$  remains in (A8).

### Appendix 2: Various Coefficients

The coefficients A and B which occur in Eq. (1.1) can be interpreted in terms of the familiar coherent and incoherent frequency shifts.<sup>4</sup> Assuming, as in Ref. (4), that all particles oscillate coherently ( $x_i = \bar{x}$ ) Eq. (1.1) yields

$$\frac{d^2 \bar{x}}{dt^2} + (Q^2 \Omega^2 + A + B) \bar{x} = G e^{-i\omega t} \quad (A2.1)$$

This equation suggests that the coherent frequency is

$$\omega_c^2 = (Q\Omega)^2 + A + B$$

and we identify

$$\Delta\omega_c = \Omega_0 \Delta Q_c \approx - \frac{A + B}{2Q_0 \Omega_0} \quad (A2.2)$$

as the coherent frequency depression due to space-charge image forces<sup>4</sup>. Similarly, when the beam center is at rest ( $\bar{x} = \frac{1}{N} \sum x_i = 0$ ) and in the absence of a driving force

$$\frac{d^2 x_i}{dt^2} + (Q_i^2 \Omega^2 + A) x_i = 0 \quad (A2.3)$$

Thus,

$$\Delta \omega_{ic} = \Omega_o \Delta Q_{ic} \approx - \frac{A}{2Q_o \Omega_o} \quad (A2.4)$$

may be interpreted as the familiar single particle frequency shift<sup>4</sup>.

Finally, we may identify  $\frac{-B}{2Q_o \Omega_o}$  as the coefficient  $U + V + iV$  of Ref. (1). In fact, assuming no external driving forces ( $G = 0$ ) we write Eq. (1.1) as

$$\frac{d^2 x_i}{dt^2} + Q_i^2 \Omega_i^2 x_i + A x_i + B \bar{x} \quad (A2.5)$$

Following the procedure used to derive Eq. (2.5) we find the dispersion relation Eq. (2.7), and by comparison with the "standard notation" we identify

$$U + V + iV = - \frac{B}{2Q_o \Omega_o} \quad (A2.6)$$

By combining (A2.2), (A2.4) and (A2.6) we may express the LNS-coefficient  $U + V + iV$  in terms of the Laslett Q-shifts as indicated in Table 1. Note that the coherent Q-shift defined this way includes the effect of finite wall conductivity. It is therefore, in general, complex and different for different modes of oscillation.

The coefficient  $U$  is determined by the Q-shifts in their usual definition (perfectly conducting vacuum chamber). From Ref. (4) we have -- assuming that the mode frequency  $\omega_c \approx (n - Q_o) \Omega_o \neq 0$  --

$$\Delta Q_c = \frac{NRr}{\pi Q \gamma b^2} \left[ \epsilon_1 + \frac{\xi_1}{\gamma^2 - 1} + \epsilon_2 \frac{b^2}{g^2} \right] \quad (A2.7)$$

$$\Delta Q_{ic} = \frac{NRr}{\pi Q \gamma b^2} \left[ \epsilon_1 + \frac{\epsilon_1}{\gamma^2 - 1} + \epsilon_2 \frac{b^2}{g^2} + \frac{b^2}{2a^2(\gamma^2 - 1)} \right]$$

and thus

$$U = \frac{Ncr}{\pi a^2 Q \beta \gamma^3} \left[ (\xi_1 - \epsilon_1) \frac{a^2}{b^2} - \frac{1}{2} \right]. \quad (A2.8)$$

Here  $\xi_1$ ,  $\epsilon_1$ , and  $\epsilon_2$  are the Laslett image coefficients,  $a$  the beam radius,  $b$  the half-height of the chamber (for vertical oscillations),  $R$  the orbit radius, and  $g$  the half-height of the magnet gap.

#### References

1. L.J. Laslett, V.K. Neil, and A.M. Sessler, Rev. Sci. Instr. 36, p. 436 (1965).
2. C.D. Curtis, et al., "Beam Experiments with the MJRA 50 MeV FFAG-Accelerator", in Proceedings of the Vth International Conference on High Energy Accelerators, Dubna (1963) p. 620.
3. H. Grunder and G. Lambertson, "Transverse Beam Instabilities at the Bevatron", in the Proceedings of the VIIIth International Conference on High Energy Accelerators, CERN (1971) (to be published).
4. L.J. Laslett, "On Intensity Limitations Imposed by Transverse Space-charge Effects in Circular Particle Accelerators", in Proceedings 1963 Summer Study on Storage Rings, Accelerators and Experimentation at Super High Energies, BNL 7534, p. 324 (1963);  
L.J. Laslett and L. Resegotti, "The Space-Charge Intensity Limit Imposed by Coherent Oscillation of a Bunched Synchrotron Beam" in Proceedings of the Vth International Conference on High Energy Accelerators, CEAL-2000, p. 150 (1967).

5. E. Courant, "Beam Instabilities in Circular Accelerators", BNL-AADD-69, (March, 1965).
6. H.G. Hereward, "Landau Damping by Non-linearity", CERN/MPS/DL-69-11 (July, 1969).
7. K. Hübner and V.G. Vaccaro, "Dispersion Relations and Stability of Coasting Particle Beams", CERN/ISR/TH/70-44 (August, 1970).
8. E.D. Courant and A.M. Sessler, Rev. Sci. Inst. 37, p. 1579 (1966).
9. C. Pellegrini and A.M. Sessler, Nucl. Inst. and Methods 84, 109 (1970).