LBL-28395

Lawrence Berkeley Laboratory
UNIVERSITY OF CALIFORNIA

Accelerator & Fusion Research Division

Presented at the Bloch Symposium,
Stanford, CA, October 27, 1989

Understanding Modern Magnets through Conformal Mapping

K. Halbach

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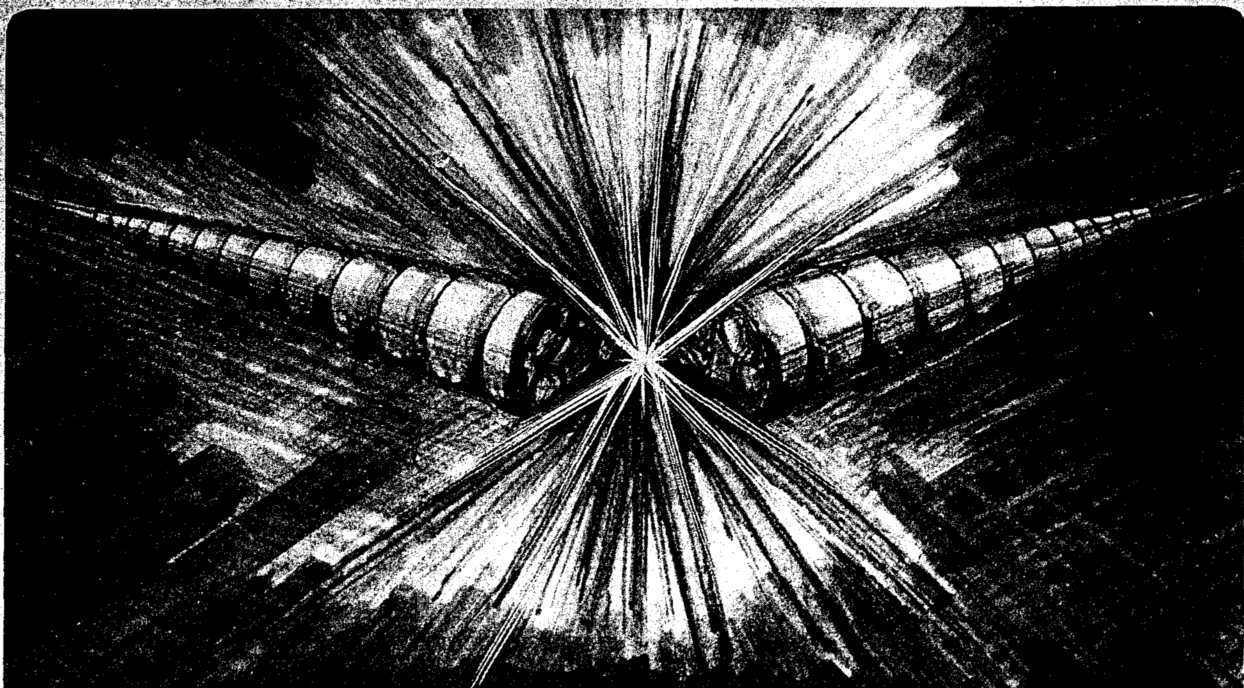

Prepared for the U.S. Department of Energy under Contract Number DE-AC03-76SF00098.

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This work was supported by the Director, Office of Energy Research,
Office of Basic Energy Sciences, Materials Sciences Division
of the U. S. Department of Energy under Contract No. DE-AC03-76SF00098.

Understanding Modern Magnets through Conformal Mapping

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A talk given at the
Bloch Symposium
at Stanford University on Oct.27, 1989.

Reminiscences

When Prof. Little invited me to present a paper at this symposium, I was very excited not only because it would allow me to talk to a knowledgeable audience about a subject about which I have very strong feelings, but even more so because it would give me the opportunity to talk about my interactions with Prof. Bloch during my work with him during my visit at Stanford from 1957 to 1959. While we all know that he was one of the great physicists of this century, it is not nearly as well known that he also was exceptionally generous, gentle and sensitive in his interactions with men that were less than his intellectual equals. I will recall here only a few of the many characteristic exchanges that showed his interest in helping a young man become a better physicist, but doing this in such a way that his emotional well-being was not harmed.

Even though Prof. Bloch knew that I was an experimentalist, he agreed to give me guidance in carrying out some theoretical work that I wanted to do. To this end we met 2-3 times per week in his office, discussing for typically 2 hours both "my" problem as well as the problem he was working on. Characteristic for all these discussions was the fact that he never ever used an argument that I could not fully understand, having calibrated me, and my shortcomings, in a very short time. Our discussions usually became, as most discussions do, a debate in which the participants took opposite views of the topic under discussion. While it is quite clear who would "lose" most of these discussions, it happened on some occasions that my view prevailed. Whenever that happened, he would come to me, pat me on my shoulder, and would say something like: "Halbach, I am very glad we discussed this to the end, because you were right and I was wrong, and I really learned something from this discussion!". While this obviously made me feel very good, he probably never really understood that it is not depressing when one "loses" (I use this word only because of the lack of a better word, since in this kind of a debate there are clearly no losers) a debate with a giant.- The fact that he, Prof. Bloch, was still learning things from others was a subject that he touched on every now and then, like when he told me that when he started his present work, he did not really understand how to use the density matrix, "Leonard Schiff taught me how to use it".- In his attempts to make me feel less inadequate, he would also occasionally tell me stories that showed an inadequacy of his own, as he did when he professed that he was very lucky that he was not present at the APS meeting in which Overhauser was strongly attacked by many prominent people when he first proposed the experiment that would show what is now called the Overhauser Effect, because "I would have said exactly the same fool things that everybody else said". While it is well known that Prof. Bloch was a wonderful teacher, it is less well known how concerned he was with the well being of individuals, and I am very happy to have this opportunity to shed some light on this side of his personality.

Introduction

When I had to choose, within some narrow range, the topic of this paper, I received great help from a colleague in Berkeley and from Prof. Little when it was suggested that I should pick among the possible subjects of my talk the subject that Prof. Bloch would have enjoyed most. Since Prof. Bloch would prefer a scalpel over a sword every time, I hope and think that most people will approve my choice.

When one intends to talk about a subject that is as old as conformal mapping and one does not want to lose the audience in a very short time, it is advisable to start by explaining both the motivation for the talk as well as the goals one has in mind when giving the talk. This particular talk has been motivated by the increasing frequency with which one hears, from people that ought to know better, statements like: "Conformal mapping is really a thing of the past because of all the marvelous computer programs that we now have". Even though, or more likely because, I have been intimately involved in the development of some large and widely used computer codes, I am deeply disturbed by such statements since they indicate a severe lack of understanding of the purpose of conformal mapping techniques, computers, and computer codes. In my view, conformal mapping can be an extremely powerful computational technique, and the easy availability of computers has made that aspect even more important now than it has been in the past. Additionally, and more importantly, conformal mapping can give very deep and unique insight into problems, giving often solutions to problems that can not be obtained with any other method, in particular not with computers. Wanting to demonstrate in particular the latter part, I set myself two goals for this talk:

1) I want to show with the help of a number of examples that conformal mapping is a unique and enormously powerful tool for thinking about, and solving, problems. Usually one has to write

down only a few equations, and sometimes none at all!

When I started getting involved in work for which conformal mapping seemed to be a powerful tool, I did not think that I would ever be able to use that technique successfully because it seemed to require a nearly encyclopedic memory, an impression that was strengthened when I saw H.Kober's Dictionary of Conformal Representations (ref.1). This attitude changed when I started to realize that beyond the basics of the theory of a function of a complex variable, I needed to know only about a handful of conformal maps and procedures. Consequently, my second goal for this talk is to:

2) Show that in most cases conformal mapping functions can be obtained by formulating the underlying physics appropriately. This means particularly that encyclopedic knowledge of conformal maps is <u>not</u> necessary for successful use of conformal mapping techniques.

To demonstrate these facts I have chosen examples from an area of physics/engineering in which I am active, namely accelerator physics. In order to do that successfully I start with a brief introduction into high energy charged particle storage ring technology, even though not all examples used in this paper to elucidate my points come directly from this particular field of accelerator technology. This is followed by a brief summary of the most important properties of functions of a complex variable. When reading this introduction into the relevant mathematics, the reader needs to keep in mind that this is not a mathematics essay, but a demonstration how beautiful and powerful, but not always appreciated, mathematics can be if used by a physicist or engineer to solve some real life problems.

<u>High Energy Charged Particle Storage Rings</u>

High energy in this context means that the particles move with a velocity that is very close to the velocity of light. Storing particles at that velocity for something like ten hours means that they travel a distance of the order of the diameter of the orbit of the planet farthest from the sun, Pluto. This means that the trajectories of the particles must be bent so that they follow a closed path, and that the beam needs to be focused and refocused all the time in order to assure a long life time of the beam in the storage ring.

If one exposes a charged particle that moves with the velocity v through a magnetic field B in the direction perpendicular to its trajectory, it experiences a force that is equal in strength to that caused by an electric field of strength E=vB. This means that in order to generate the same force on a high energy particle as that produced by a magnetic field of the order of one Tesla, one would need to apply an electric field of the order of $3*10^8$ V/m. Since such large DC fields can not be generated without electrical breakdown, one uses exclusively magnetic fields to focus and bend (into a closed trajectory) high energy charged particle beams.

In order to focus a high energy charged particle, it is desirable for a number of reasons to provide a restoring (i.e. focusing) force that is proportional to the distance of the particle from the desired orbit, the same as in a harmonic oscillator. If a particle with charge e travels in the z-direction through a magnet that provides a magnetic field $B_y(x,y,z)$, that field component causes the x-component of the momentum to change by

$$\Delta p_x = e\int B_y(x,y,z)\cdot v\,dt = e\int B_y(x,y,z)\,dz \tag{1}$$

From this equation it is clear that one is mostly interested in the integral of fields over the length of magnets, and that one wants those integrals to be linear functions of the Cartesian coordinates for the magnets that are used to focus the particles. For that reason we study now the mathematical properties of the integrals over three-dimensional vacuum fields taken over the whole length of a magnet, i.e. from the field-free region on one side of the magnet to the field-free region on the other side of the magnet. Applying this integration to the two magnetostatic equations

$$\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = 0 \quad ; \quad \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0 \tag{2}$$

leads (because integration and differentiation commute) to

$$\frac{\partial \underline{B}_y}{\partial x} - \frac{\partial \underline{B}_x}{\partial y} = 0 \quad ; \quad \frac{\partial \underline{B}_x}{\partial x} + \frac{\partial \underline{B}_y}{\partial y} = 0 \tag{3}$$

with the integral over the fields indicated by underlining the letter B. These are clearly the same differential equations that are valid for two-dimensional fields, and I will be dealing exclusively with such fields from now on, making the use of the underlining unnecessary. An additional change in notation is convenient because of the two-dimensionality of all subsequent equations: from now on, z is defined by $z=x+iy$, with $i^2=-1$.

If one writes down the equation for the change in the y component of the momentum due to B_x, one finds that if the field is focusing the particles in one direction, it always defocuses them in the other direction, seemingly making focusing in both directions impossible. However Fig.1 shows that if one separates two linearly focusing/defocusing magnets with equal strength but opposite polarity appropriately one can obtain a system that is focusing in both directions, and all modern accelerator structures use, in one form or another, this so called strong focusing principle.

Summary of properties of a function of a complex variable

Fig.2 summarizes (and derives some of) the most important properties of a function of a complex variable. For our purposes, it is sufficient to define an analytical function of a complex variable by any kind of an algorithm that allows, in any number and sequence, only the four fundamental operations listed on line 2. While we use the complex conjugate of complex numbers and indicate it by *, it is specifically forbidden to use that operation in the definition of an analytical function of a complex variable. It is clear that under these conditions all the standard rules of differentiation are valid. Therefore lines 3 and 4 (and the specific example in line 8) are straight forward, as are the consequences, line 5 (the Cauchy-Riemann-condition) and line 6, the latter expressing that both the real and the imaginary part of a function of a complex variable satisfy the Laplace equation.

While we often use functions whose real and imaginary parts are non-geometric quantities, like the x and y components of fields, or vector and scalar potentials, a geometric interpretation is also possible and useful. The function $w(z)=u+iv$ can be used to map the x-y plane onto the u-v plane. If $w(z)$ is an analytical function as defined above, line 7 shows that the angle between two intersecting lines in the x-y plane is the same as the angle between their maps in the u-v plane (provided neither of the derivatives w' or z' are zero), i.e. the map is conformal. Line 9 states, without proof, how one can calculate from the knowledge of the real or imaginary part of an analytical function of z on the circumference of a circle the value of the complete function in the interior of the circle. This solution to the Dirichlet problem in a circle is valid only if the function has no singularity inside the circle.

Switching now from mathematics to physics, line 10 shows the relationships between the x- and y- components of the magnetic field in vacuum on one hand, and the component A of the vector potential that is perpendicular to the x-y plane, and the scalar potential V on the other hand. Having fortuitously chosen the appropriate notation in the mathematics part, comparison with line 5 shows that A and V can be considered as real and imaginary part of the analytical function $F(z)$, customarily called the complex potential. From line 10 follow directly lines 11 and 12.

Since the result that it is H_x-iH_y and not H_x+iH_y that is an analytical function of z has been derived here in a rather abstract way, lines 13 and 14 show how one can come to the same result in a more direct and elementary way. Line 15 shows the complex potential and its constituent parts for the case of the ubiquitous quadrupole, the magnet used for linear focusing in accelerators. Also shown are the scalar potential surfaces that will produce such a field distribution.

Design of non-dipole magnets in dipole geometry

Fig.3 shows schematically the cross section of an iron dominated dipole magnet, with the region within which one wants to produce a uniform field indicated by an ellipse. As the name implies, in this type of magnet the field distribution in the region to be used is dominated by the iron configuration in the vicinity of that region. Since for most accelerator applications the fields are symmetrical with respect to the midplane, one usually discusses in detail only the fields in one half of the magnet, and assumes for the major design decisions that the coils are sufficiently far away that they do not have a major effect on the overall field distribution, thus leading to a sketch as shown in Fig.4. In this magnet the lateral sides are not perpendicular to the midplane in order to keep the field more uniform at high field levels, and so called shims at the ends of the pole face are also indicated. If properly designed, such shims increase the size of the good field region for a given width of the pole face. It is clear that this type of magnet is fairly easy to design, and the "recipes" for such a design are well known and understood. One of the important uses of conformal mapping makes it possible to apply the knowledge and understanding of dipole design to the design of any non-dipole.

If one were to apply an arbitrary coordinate transformation u(x,y), v(x,y) to the geometry of the field-producing entities (potential surfaces and/or current filament locations), one would get a new geometry of the field-producing entities, but one would also have to solve transformed magnetostatic differential equations that, in that new geometry, would look very different from the equations shown in Fig.2. That difficulty is completely avoided if one restricts oneself to conformal transformations $w(z)=u+iv$. In order to find the conformal map that transforms a magnet with the ideal desired field distribution into an ideal dipole, one obviously needs to know what the ideal desired field distribution is. Through line 11 in Fig.2 one therefore also knows the complex potential $F_{id}(z)$ that contains the information about the geometry of the ideal field-producing entities. Applying the conformal transformation $w(z)$ to the field-producing entities leads to the complex potential $F_{id}(z(w))$ in the w-plane. Knowing that one wants in that plane a perfect dipole, characterized by the complex potential cw (c=const.), the desired conformal transformation that turns the ideal non-dipole into an ideal dipole is therefor established through the simple expression

$$w(z) = \text{const.} \cdot F_{id}(z) \tag{4}$$

In order to make proper design decisions in dipole geometry, one needs to know the relationship between (non-ideal) fields, and field errors, in the w- and z-planes. From the equations in Fig. 2 follows directly

$$H_w^* = i \cdot dF/dw = i \cdot dF/dz \cdot dz/dw \tag{5}$$

$$H_z^* = w' \cdot H_w^* \tag{6}$$

It should be noted that w'=dw/dz comes from the chosen conformal map and has nothing to do with the actual fields. From equ.6 follows for the field errors, i.e. actual deviations from the ideal field, that the relative field errors are identical in the two planes:

$$\Delta H_z/H_z = \Delta H_w/H_w \tag{7}$$

After the conformal mapping function w(z) has been established, the detailed procedure used to design a non-dipole typically consists of a number of steps that, in addition to their general formulation, are elucidated by applying them to the specific example of the design of a sextupole characterized by

$$H^* = i \cdot \text{const.} \cdot z^2 \text{ and } w(z) = z^3 \tag{8}$$

1) Establish both the allowed region (i.e. the region outside a boundary into which no part of the magnet can penetrate, usually the vacuum chamber) as well as the good field region(s). These regions, incorporating for our example two good field regions with different demands on field quality, are given in Fig.5. From this geometry and the symmetry of the desired fields one must establish the minimum space in the z-plane in which one has to design the magnet, the rest of the magnet being then determined by symmetry. In the case of the example used here, one clearly needs to design the magnet only in the first quadrant of the x-y plane, i.e. for x>0, y>0.
2) Using w(z), map, only in the above established region of the x-y-plane, both the boundary of the allowed region and the good field region(s) from the z-plane into the w-plane. The result of those mappings are given for the sextupole in Fig.6a. This Figure also shows the maps of points that are equidistant on the line that represents the vacuum chamber wall, and of approximately equidistant points on the outer good field region boundary, thus giving a good qualitative indication of the value of w' at these locations.
3) Design the polefaces of the dipole in w-geometry. These poles are, for our specific example, also shown in Fig.6a. While the

upper pole is a full pole, the lower pole is only 1/2 pole, the other half, as well as all the other poles of the complete magnet, being established from symmetry in the z-plane at the end of the design process. In the chosen example the lower half pole has the same absolute distance from the u-axis as the upper pole, leading to a symmetric magnet with all poles excited by the same absolute number of ampere turns. Since one has a good understanding of the relation between pole width and field quality in dipole geometry, that understanding can be transferred to the non-dipole geometry by marking equidistant points on the poles in dipole geometry and then using them as explained in the next step.

4) Map the straight lines representing the poles in dipole geometry with their marker points from the w-plane to the z-plane. For our example this leads to Fig.6b. Knowing from considerations in the w-plane to what marker on the pole face line a pole has to go in order to achieve a particular field quality transfers that understanding via the maps of those markers directly to the z-plane. The distance between the markers in the z-plane can also be used to get a feeling for z' and, with this, the field strength on the pole face.

5) Beyond this point, hard rules can not be given. Usually one designs after step 4) the rest of the magnet (for instance coils and yoke) in z-geometry, and often iterates between the representations in the z-and w-planes as the detailed design proceeds.

One of the most interesting insights obtained from the design of the example-sextupole in w geometry is the great difference in the size of the two good field regions that are not that much different in size in the z-plane, thus showing that a much greater effort is required to get a good field in the larger good field region than it is to achieve the same field quality in the smaller good field region.

For some applications it is advantageous to use sextupoles that have a field that increases less strongly than an ideal sextupole on the x-axis, and Fig.7a and Fig.7b are equivalent to Fig.6a and Fig.6b for the case

$$H^{*} = i \cdot z^2 \cdot \exp(K \cdot z^2) = i \cdot \text{const.} \cdot w' \text{ for } K = -.1 \text{ cm}^{-2}. \tag{9}$$

It is noticeable how different the shapes of the poles are compared to the straight sextupole.

Both of these examples are special cases of the most frequently occurring case of symmetrical field specifications that can best be formulated like this: For y=0 the field is specified by

$$H_x = 0, \quad H_y = g(x) \tag{10}$$

From this follows that $H^*(z)$ and $w'(z)$ must be given by

$$H^*(z) = -i \cdot g(z) = i \cdot \text{const.} \cdot w'(z) \tag{11}$$

thus leading to w(z) in a straight forward way. It should be noticed that it can be quite advantageous to execute not only the basic design in w-geometry but also the detailed design, a task that can be accomplished by at least one computer code (POISSON, ref.2) even when the non-linearity of the iron has to be taken into account.

Design of an electrostatic extraction system

Some years ago I was asked for advice on an electrostatic extraction system that did not work as hoped for, and here is the story with the question and the answer.

An existing electrostatic extraction system with two parallel electrodes as shown in Fig.8a did not give good extraction efficiency since linear focusing was needed in order to clear all apertures. It was therefore replaced by a system that would give linear focusing. An ideal system should look as the one shown in Fig.8b, representing one quarter of a quadrupole system (as depicted at the bottom of Fig.2) with the hyperbolae going to infinity. Since that is not a realizable solution and one has to reduce the largest electric field in order to avoid electrical breakdown, the system shown in Fig.8c was chosen, consisting of the large electrode and a hyperbolic electrode with circular extensions at the ends for field strength reduction. The problem was that this system did not work well because of the large optical aberrations suffered by the particles, residing in the circular region that is shown in Fig.8c. The reason for these aberrations, and the "fix", became clear through the following consideration.

The complex potential, and with it the appropriate conformal transformation, for the desired ideal field for linear focusing is given by cz^2 (c=const.), and Fig.8d shows the electrode system and the beam cross section when subjected to that map. It should be noticed that map of the circle in Fig.8c has, in Fig.8d, a nearly flat area at the bottom. Needing in this geometry a uniform field in the beam region, it is clear from what has been said above that the horizontal extent of the good field region equals the flat part of the electrode (in w-geometry) minus a certain fraction of the half gap of the system. However, in contrast to the magnet shown in Fig.4, what appears to be the midplane in Fig.8d is in fact the map of a material electrode that can be shaped or deformed essentially at will. Understanding that, it becomes clear without any computation that a symmetrical system (in w-geometry) as shown in Fig.8e will produce better fields for the beam, thus leading to the system in the z-plane shown in Fig.8f.

It is (barely) visible in this Figure that the extensions of the ground electrode are not circular as the extensions of the hyperbolae, but are, as just stated, identically shaped only in w-geometry. It is worth noting that this obviously correct solution to a non-trivial problem was obtained by thinking about it in a particular way, without doing any extensive computations. It is also clear that this, to most people surprising, solution can not be found without conformal mapping techniques.

<u>Mapping of the interior of a perfect multipole onto a circular disk</u>

Line 9 of Fig.2 makes it possible to calculate from a change of a potential on a closed contour the change of the complex potential in the entire region enclosed by that contour if the region enclosed by that contour can be mapped onto a circular disk. Having to deal with that kind of a problem (ref.3) was the reason why I had to find in the 1960's the conformal transformation that mapped the interior of a symmetrical perfect multipole onto a circular disk. Formulated as a purely geometrical problem, it is very difficult to find that map. That is the reason why I like to use this problem as the example that shows how simple it can be to find a solution to a problem if one uses the underlying physics to re-formulate the problem. In this case I do this first for a quadrupole, and then generalize the answer for the multipole.

When I put the four poles of the perfect quadrupole, schematically shown in Fig.9a, alternatingly on the scalar potentials +1 and-1, the complex potential within that quadrupole is given by

$$F(z) = z^2 \tag{12}$$

Since the geometry to be mapped is invariant to rotation by 90 degrees, the geometry of the map into a circular disk with radius 1 in the w-plane can be expected to have the same property, i.e. the maps of the four poles will be 90 degree circular arcs, as shown in Fig.9b. Putting them alternatingly on scalar potentials +1 and -1 calls again for the solution to the Dirichlet problem in a circular disk given in line 9 of Fig.2. Using that equation leads after a few simple lines to

$$F(z(w)) = \frac{2}{\pi} \cdot \ln \frac{1+w^2}{1-w^2} \tag{13}$$

leading to

$$w = \left(\tanh(\frac{\pi}{4}z^2)\right)^{1/2} \tag{14}$$

With this understanding of the process, it this then easy to find the map for a perfect 2N-pole:

$$w = \left(\tanh\left(\frac{\pi}{4} z^N\right)\right)^{1/N} \tag{15}$$

This and similar procedures have been extremely successful for me, making consultation of either a beautiful book like Kober's or of a very good memory (that I do not possess) completely unnecessary.

A remarkable theorem about multipoles

Fig.10a shows a cross section of the (infinitely permeable) iron-vacuum interface of an octupole. The only symmetry that I assume to be satisfied is invariance of the geometry to rotation by 45 degrees. In contrast to what was assumed in the previous section, this magnet will, in general, not produce ideal octupole fields.

When this type of a magnet is excited in an unorthodox manner, it is sometimes necessary to know how much flux is induced in a particular pole when another pole is excited by its coil, for instance when iron saturation effects are important, or when one wants to know the mutual induction between coils on different poles. For the purpose of this discussion, I idealize these coils by filaments at the numbered locations in Fig.10a. Because of the singularity of the field at the locations of the filaments, this makes it impossible to make simple statements about the interaction between directly adjacent poles, but does allow the formulation of a very general theorem about the interaction between poles that are not immediate neighbors.

To prove this theorem, I assume that I have two current filaments of equal strength, but opposite sign, at locations 1 and 2, and I want to know, for instance, the flux entering the pole that extends from location 3 to location 4. To calculate this flux, I assume that I know the conformal transformation that maps the interior of the octupole onto a circular disk. Since the geometry of the octupole boundary is invariant to rotation by 45 degrees, I can assume (and prove if I have to do that) that the eight poles can be mapped onto eight 45 degree circular arcs. This transforms my original problem into a Dirichlet problem in a circular disk, i.e. from the knowledge of the scalar potential on the circumference of the circle (V=1 between points 1 and 2, and V=0 elsewhere on the circle) I have to calculate the flux between points 3 and 4, given by the difference between the vector potential at those points. Application of line 9 of Fig.2 gives immediately an explicit expression for this flux. I don't even give here that expression because the exciting aspect of this answer is the fact that this flux is also the flux entering between points 3 and 4 of the octupole, independent of what the conformal transformation was that mapped the interior of the octupole onto the circular disk. Generalized and expressed differently:

The flux induced by any pole onto another pole that is not its immediate neighbor depends only on the multipolarity of the multipole and the "distance" to that pole, not at all on the geometry of the poles.

Since I can never be quite sure that I expressed myself in a way that cannot possibly be misunderstood, it is worth stating that the octupole geometry could an octagon, or configuration as shown in Fig.10b, or whatever; as long as the geometry is invariant to rotation by 45 degrees, the flux induced into equivalently located poles is the same.

This theorem can be extended in an interesting way: By keeping the distance between adjacent poles constant and increasing simultaneously both the radius and multipolarity of the multipole, one obtains in the limit of an infinitely large radius an infinitely long linear array of poles, as shown in Fig.10c. The above stated theorem that the interaction between not immediate neighbors is independent of geometry clearly applies to this linear array as well. Fig.10d-Fig.10f show arrays that are in this regard equivalent to the array shown in Fig.10c, with Fig.10e showing an array of infinitely thin poles, and Fig.10f showing an array where each pole occupies all available space except for an infinitely thin sheet of vacuum between sheets. The array shown in Fig.10f makes the calculation of the interaction between poles trivially easy, and it turns out that this result is very useful for the prediction of certain properties of hybrid undulators.

It is probably fair to characterize this section in the following way:

1) I have proven a very general theorem.
2) Most people consider this theorem very counter-intuitive.
3) It was not necessary to write down a single formula in this section.
4) It is very hard to conceive of a method to prove this theorem without using the theory of a function of a complex variable and thinking that comes from conformal mapping considerations.

Concluding remarks

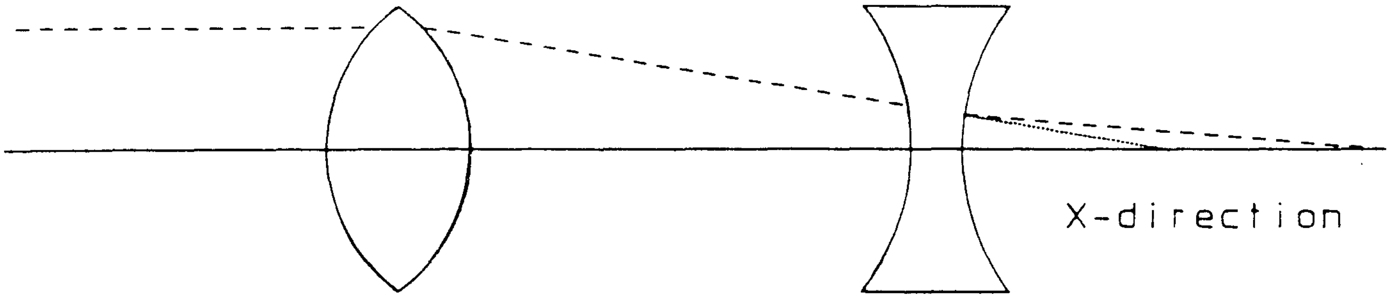
I hope that I have demonstrated to everyone's satisfaction that promise #1 made in the technical introduction has been fulfilled. Full delivery on promise #2 can of course not be proven here in the same sense, complete delivery of that proof is ultimately up to every individual user.

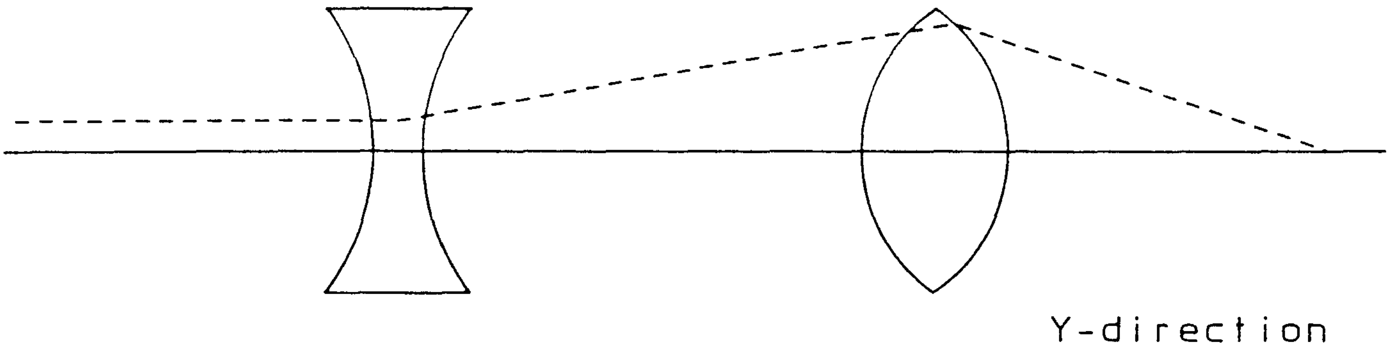
Acknowledgment

This work was supported by the Director, Office of Energy Research, Office of Basic Energy Sciences, Materials Sciences Division, of the U.S. Department of Energy, under Contract No. DE-AC03-76SF00098.

References

1) H.Kober, Dictionary of Conformal Representations, Dover (1957)
2) K.Halbach, Nucl. Instr. and Meth. 64 (1968) 278
3) K.Halbach, Nucl. Instr. and Meth. 74 (1969) 147





1) Strong focusing optics

Mathematics

$$z = x + iy \quad ; \quad F(z) = A(x,y) + iV(x,y) \tag{1}$$

Only $+, -, \times, \div$ allowed in definition of $F(z)$ (2)

$$\frac{\partial F}{\partial x} = \frac{dF}{dz} \cdot \frac{\partial z}{\partial x} = \frac{dF}{dz} = F' = \frac{\partial A}{\partial x} + i\frac{\partial V}{\partial x} \tag{3}$$

$$\frac{\partial F}{\partial y} = \frac{dF}{dz} \cdot \frac{\partial z}{\partial y} = iF' = \frac{\partial A}{\partial y} + i\frac{\partial V}{\partial y} \tag{4}$$

$$\frac{\partial A}{\partial x} = \frac{\partial V}{\partial y} \quad : \quad \frac{\partial V}{\partial x} = -\frac{\partial A}{\partial y} \qquad \text{Cauchy Riemann conditions} \tag{5}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) F = 0 \rightarrow \nabla^2 A = 0 \ : \ \nabla^2 V = 0 \tag{6}$$

$$w(z) = u + iv \ ; \ \Delta w = \Delta z \cdot dw/dz \rightarrow w(z) = \text{conformal map if } dw/dz \neq 0 \text{ and } dz/dw \neq 0 \tag{7}$$

$$\text{Example}: \ F(z) = z^2 = \underbrace{x^2 - y^2}_{A} + \underbrace{i2xy}_{V} \ ; \ \frac{\partial F}{\partial x} = 2(x+iy) \ : \ \frac{\partial F}{\partial y} = i \cdot 2(x+iy) \tag{8}$$

$$F(z_0) = A(0)\cdot(1-\alpha) + iV(0)\cdot\alpha + \frac{1}{2\pi}\int_0^{2\pi} \frac{re^{i\phi} + z_0}{re^{i\phi} - z_0}\left(A(r,\phi)\cdot\alpha + iV(r,\phi)(1-\alpha)\right) d\phi$$

$\alpha =$ arbitrary, but usually $\alpha = 0$ or $\alpha = 1$. (9)

Physics

Fields in vacuum:

$$H_x = -\partial V/\partial x = \partial A/\partial y \ : \ H_y = -\partial V/\partial y = -\partial A/\partial x \ : \ (C-R \text{ for } A, V) \tag{10}$$

$$H^* = H_x - iH_y = -\partial V/\partial x + i\partial A/\partial x = i\partial F/\partial x = iF' \tag{11}$$

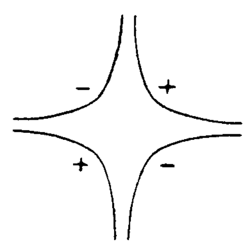
$$\text{Dipole}: \ iF' = H_0^* = \text{const.} \rightarrow F = -iH_0^* z \tag{12}$$

H from current filament at $z = 0$:

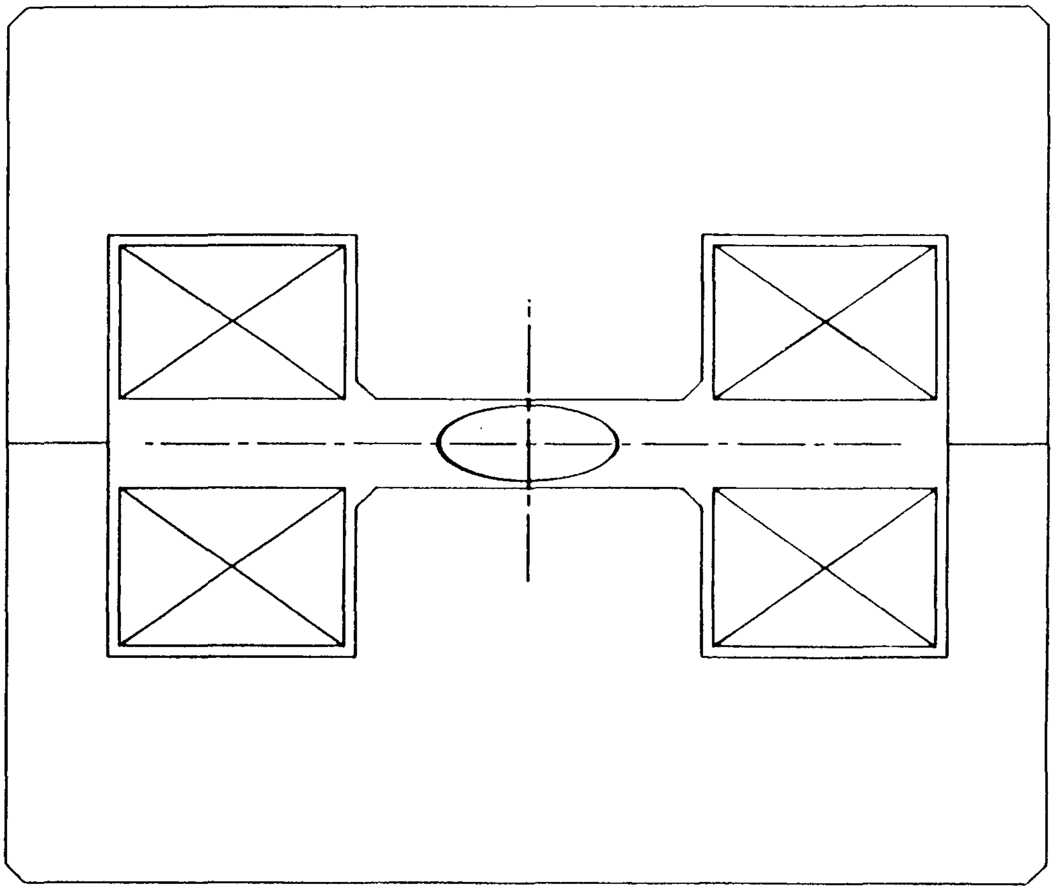
$$H = H_x + iH_y = \frac{I}{2\pi r}\cdot e^{i\phi}\cdot i = -\frac{I}{2\pi i z^*} \ : \ H^* = \frac{I}{2\pi i z} \tag{13}$$

$$\text{Filament at } z = z_0: \ H^* = \frac{I}{2\pi i(z - z_0)} = iF' \ : \ F = \frac{-I}{2\pi}\ln(z - z_0) \tag{14}$$

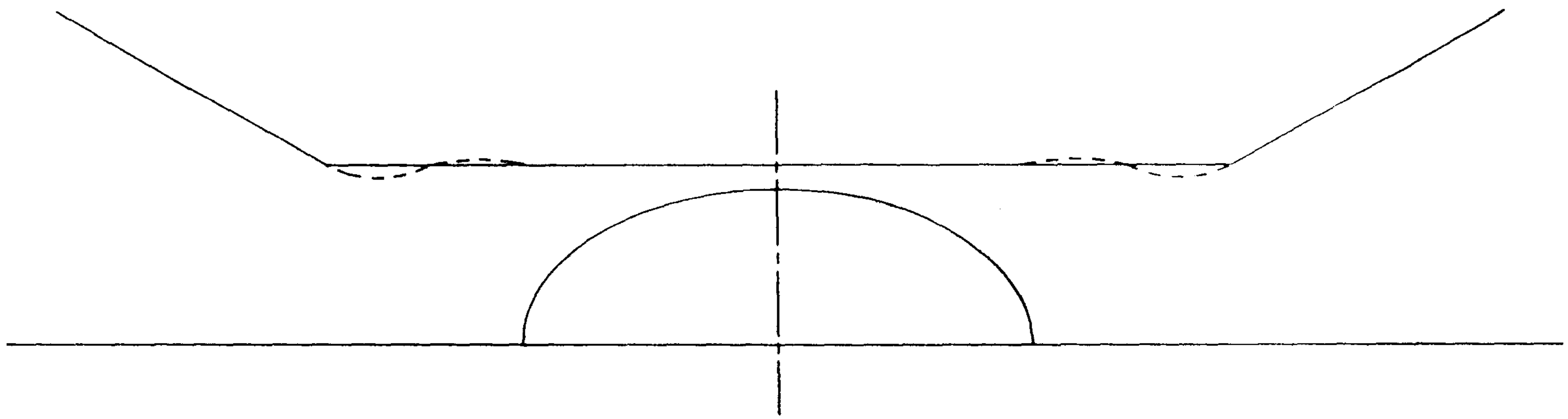
$$F(z) = z^2 = \underbrace{x^2 - y^2}_{A} + i\cdot\underbrace{2xy}_{V} \ ; \ x\cdot y = \pm V_0 \text{ gives hyperbolic iron poles} \tag{15}$$



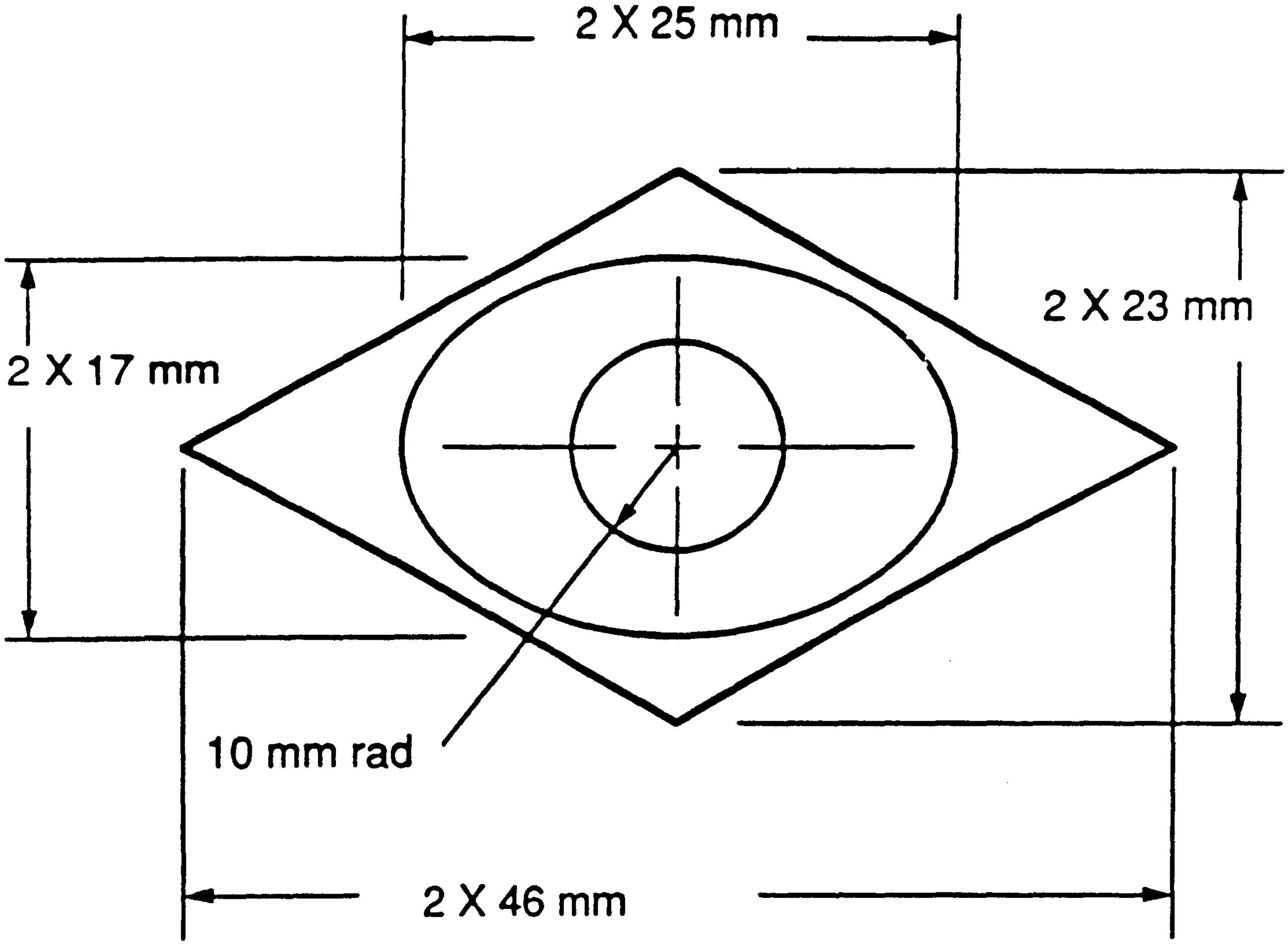
2) Summary of properties of functions of a complex variable



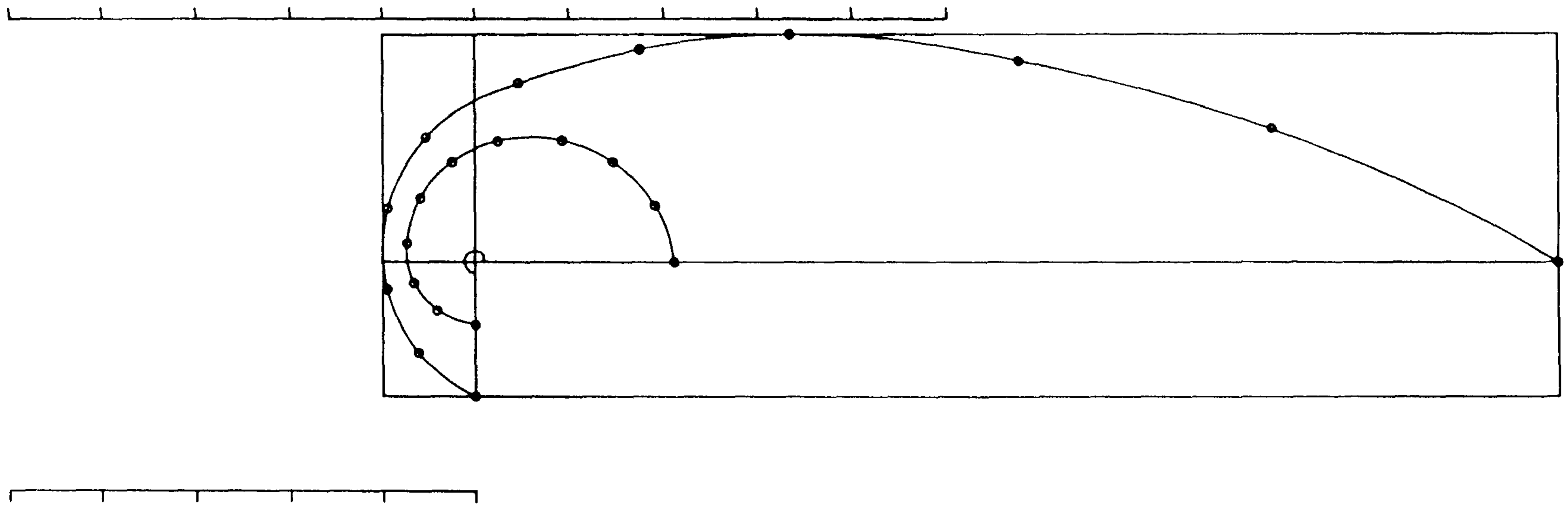
3) Complete dipole cross section



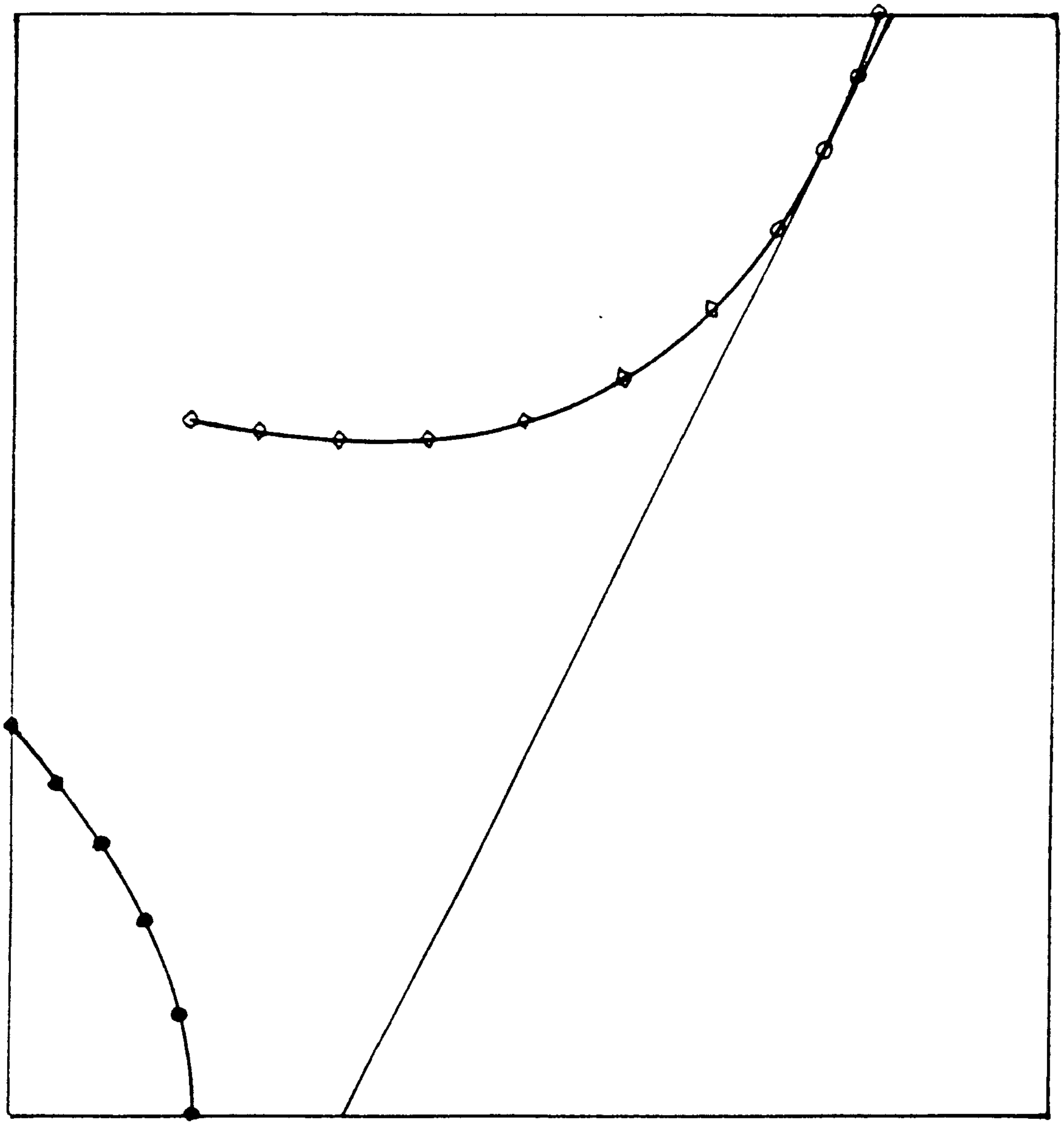
4) Partial dipole cross section



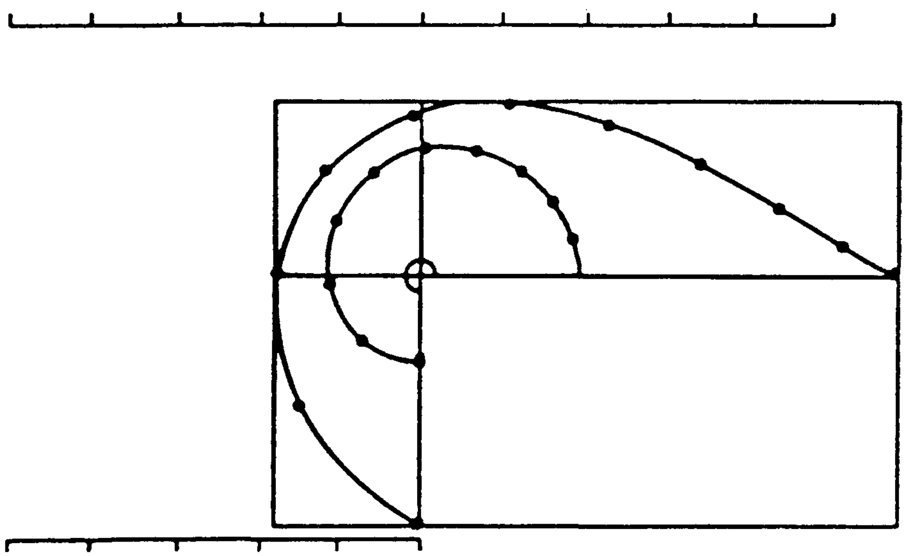
5) Vacuum chamber and good field region cross sections



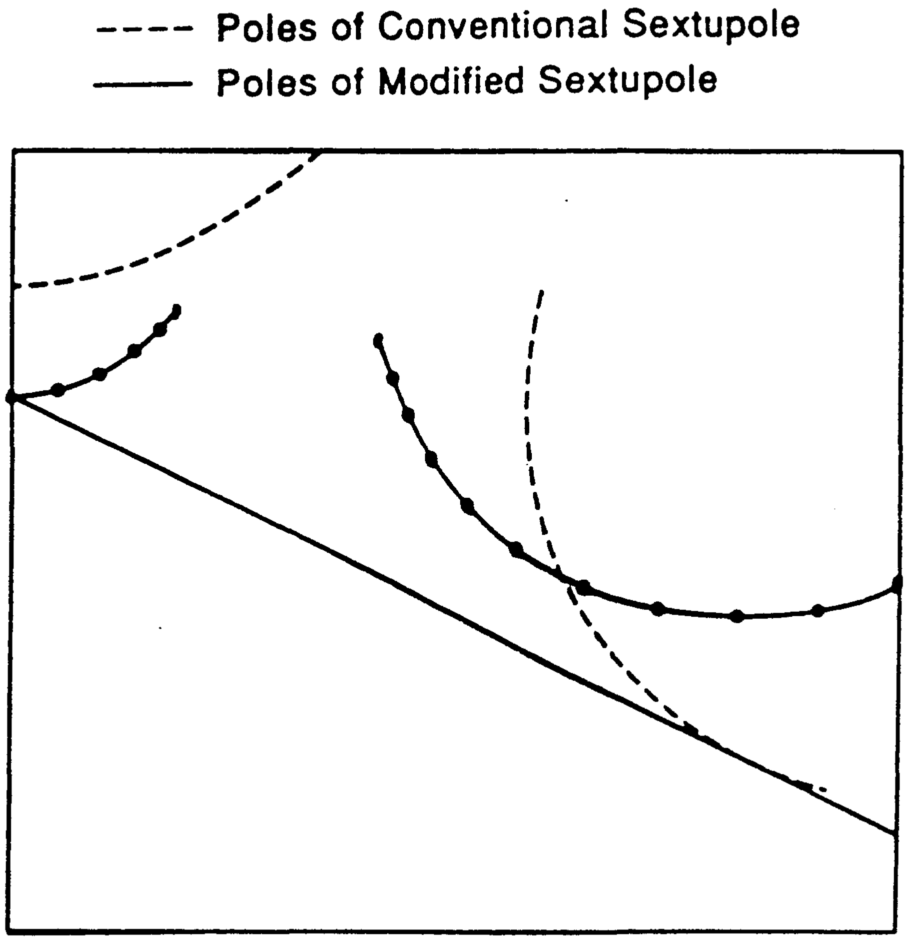
6a) Map of vacuum chamber and good field regions for sextupole



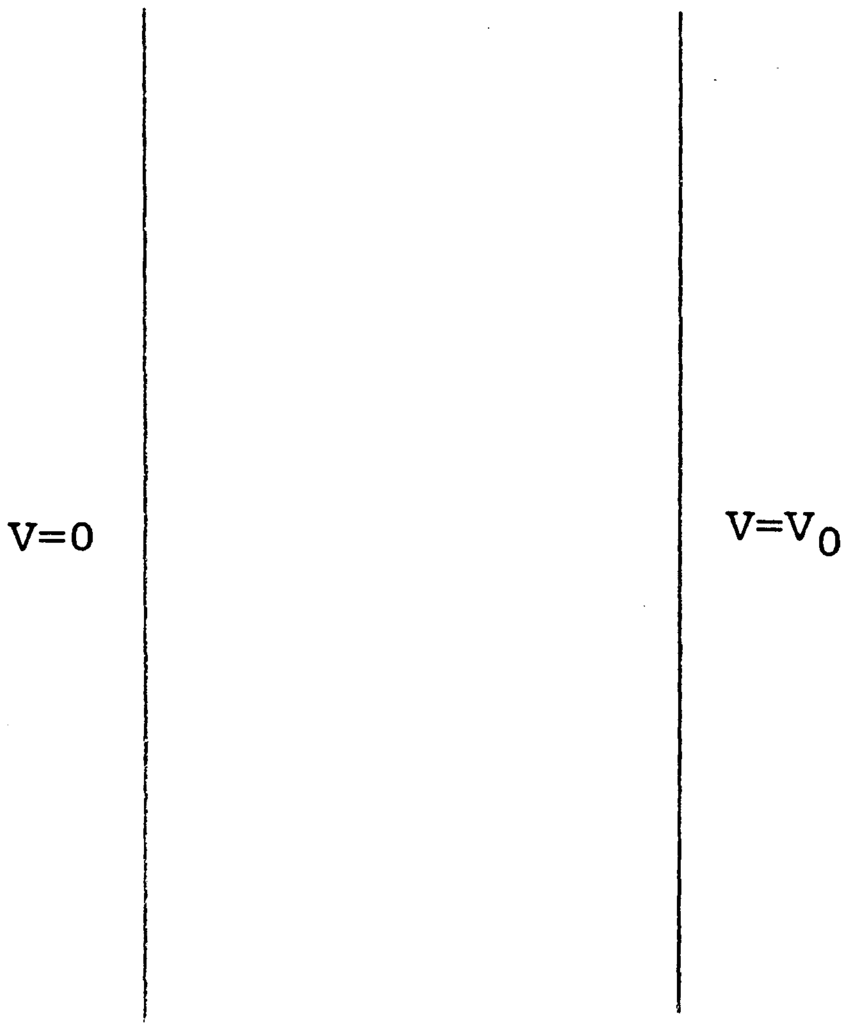
6b) Poles of sextupole



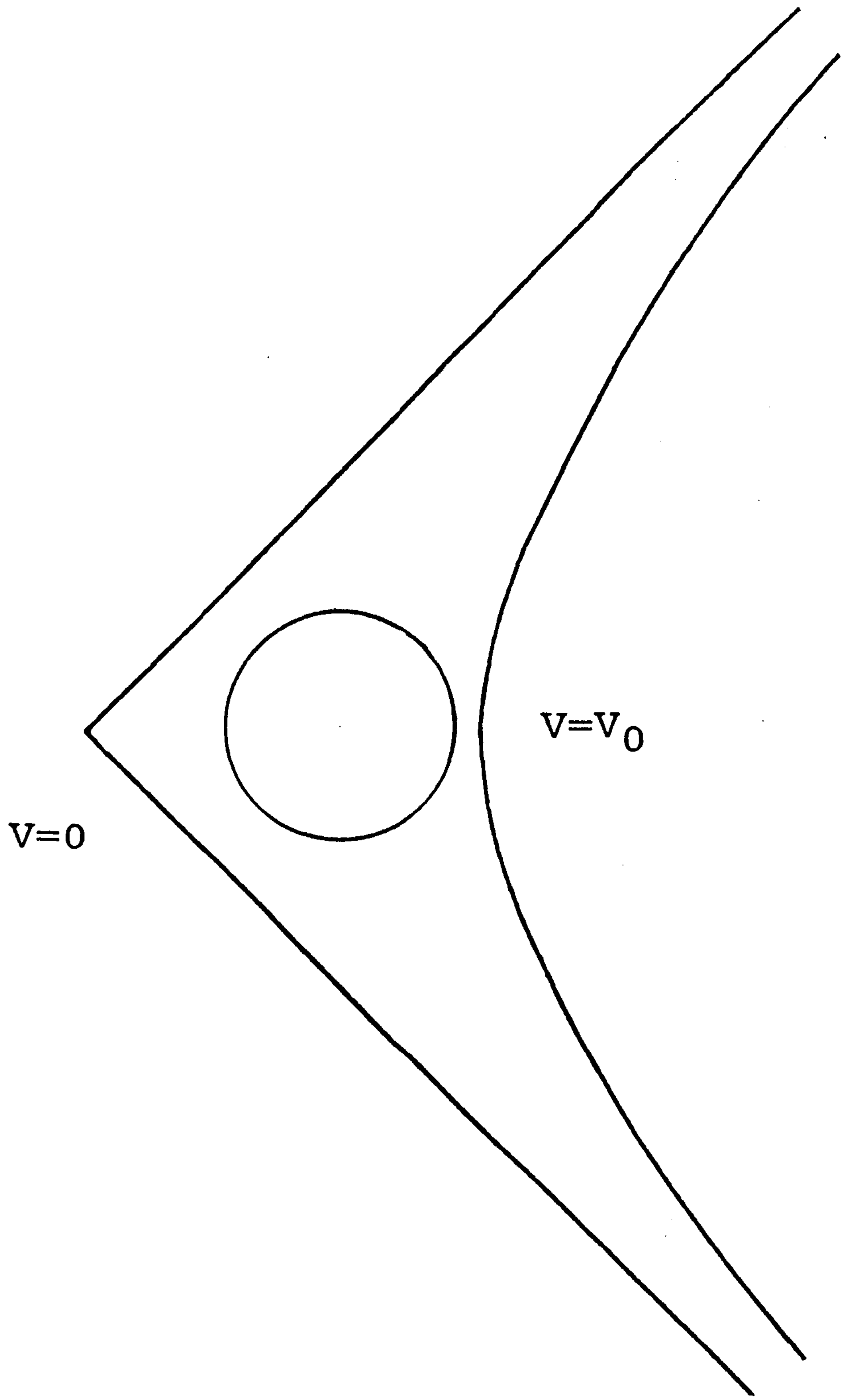
7a) Map of vacuum chamber and good field regions for modified sextupole



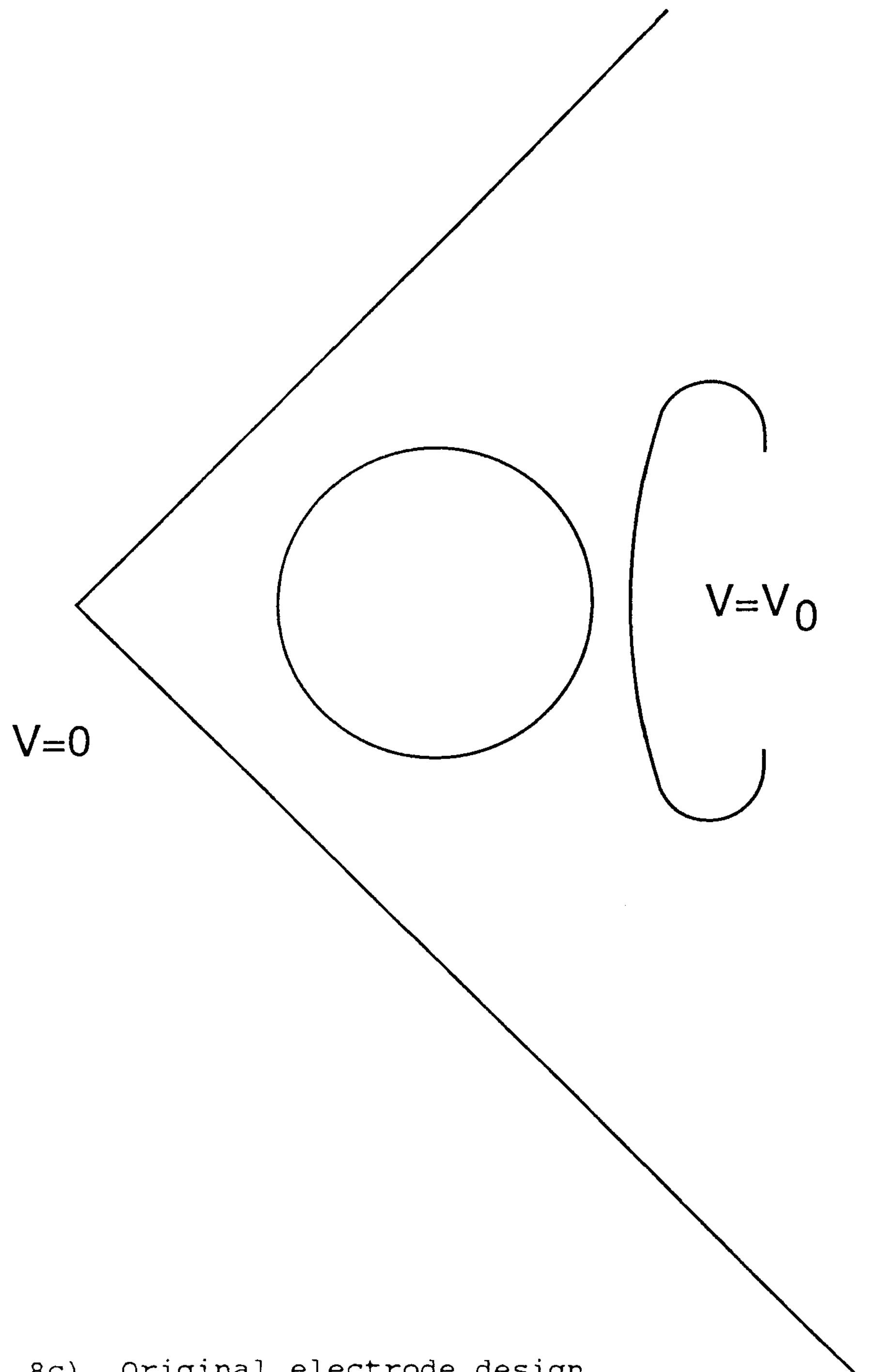
7b) Poles of modified sextupole



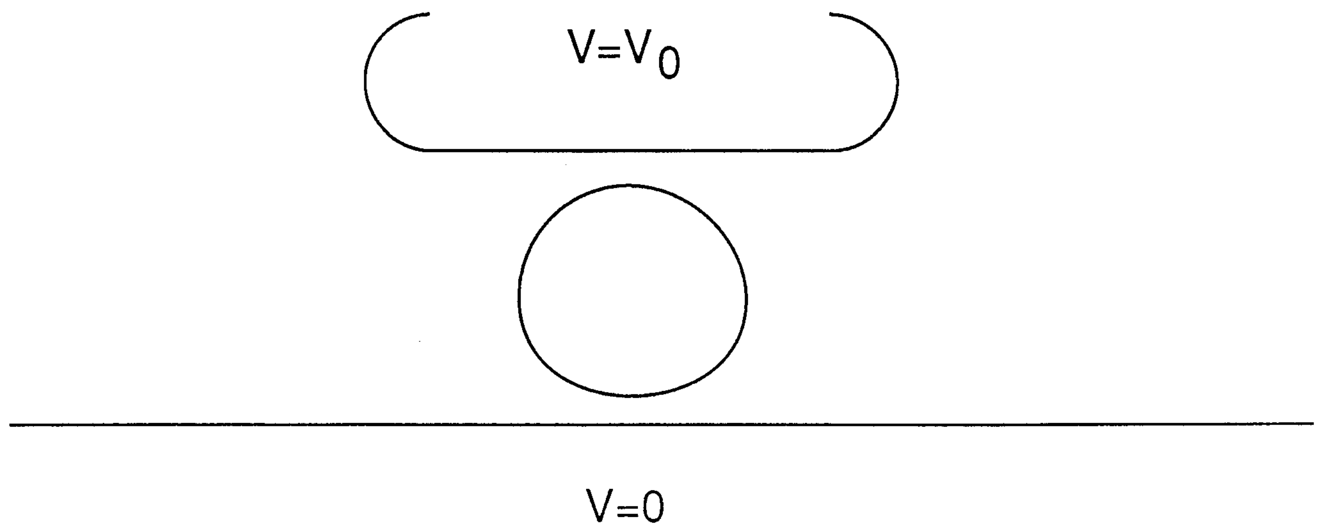
8a) Electrodes for uniform field extraction



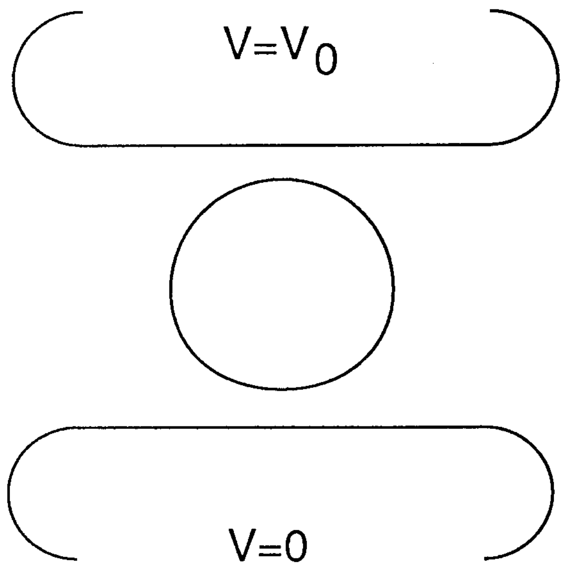
8b) Idealized electrodes for extraction with focusing



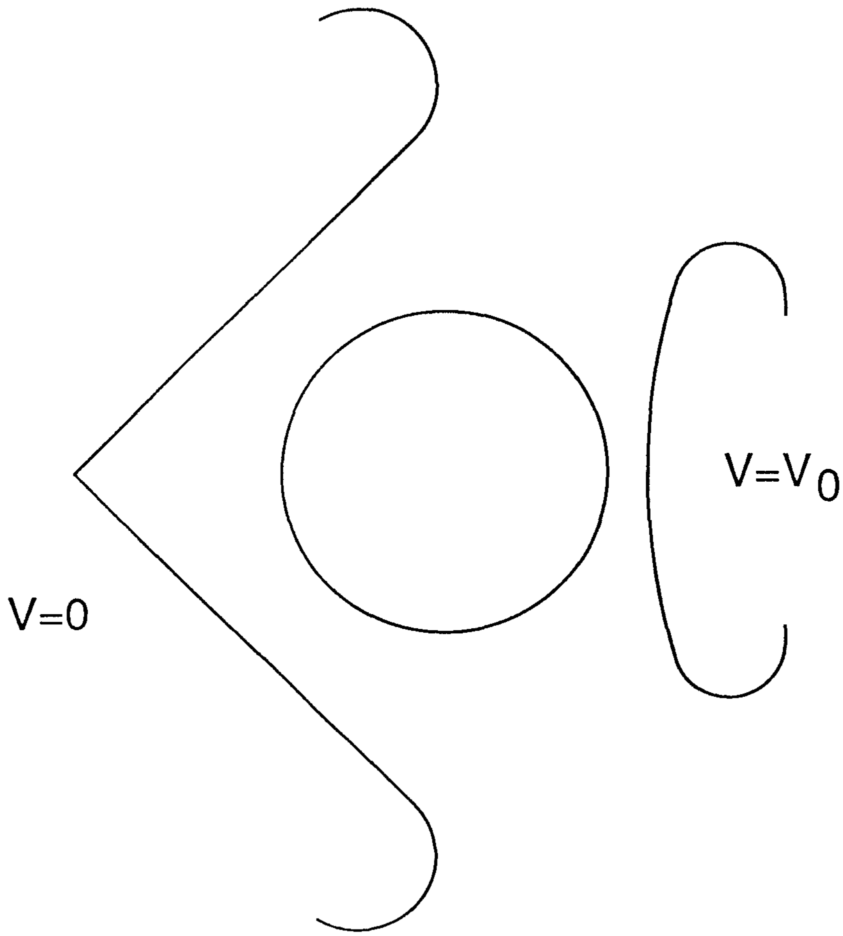
8c) Original electrode design



8d) Map of original electrodes

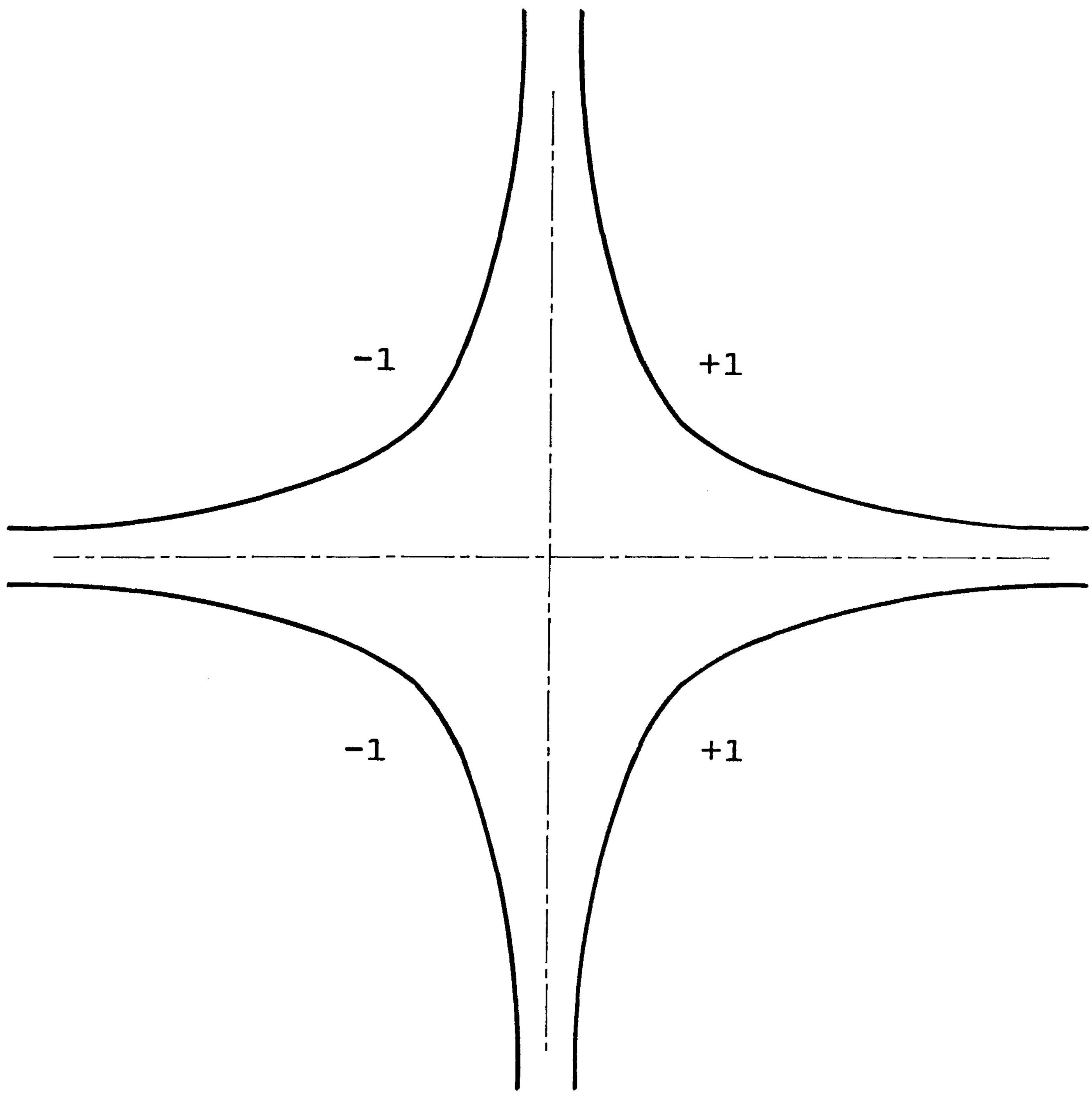


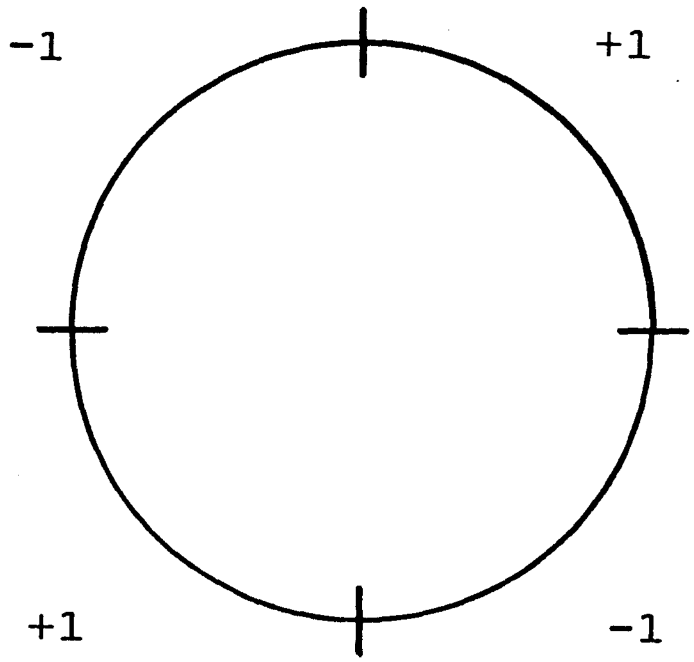
8e) Map of original electrodes with modification



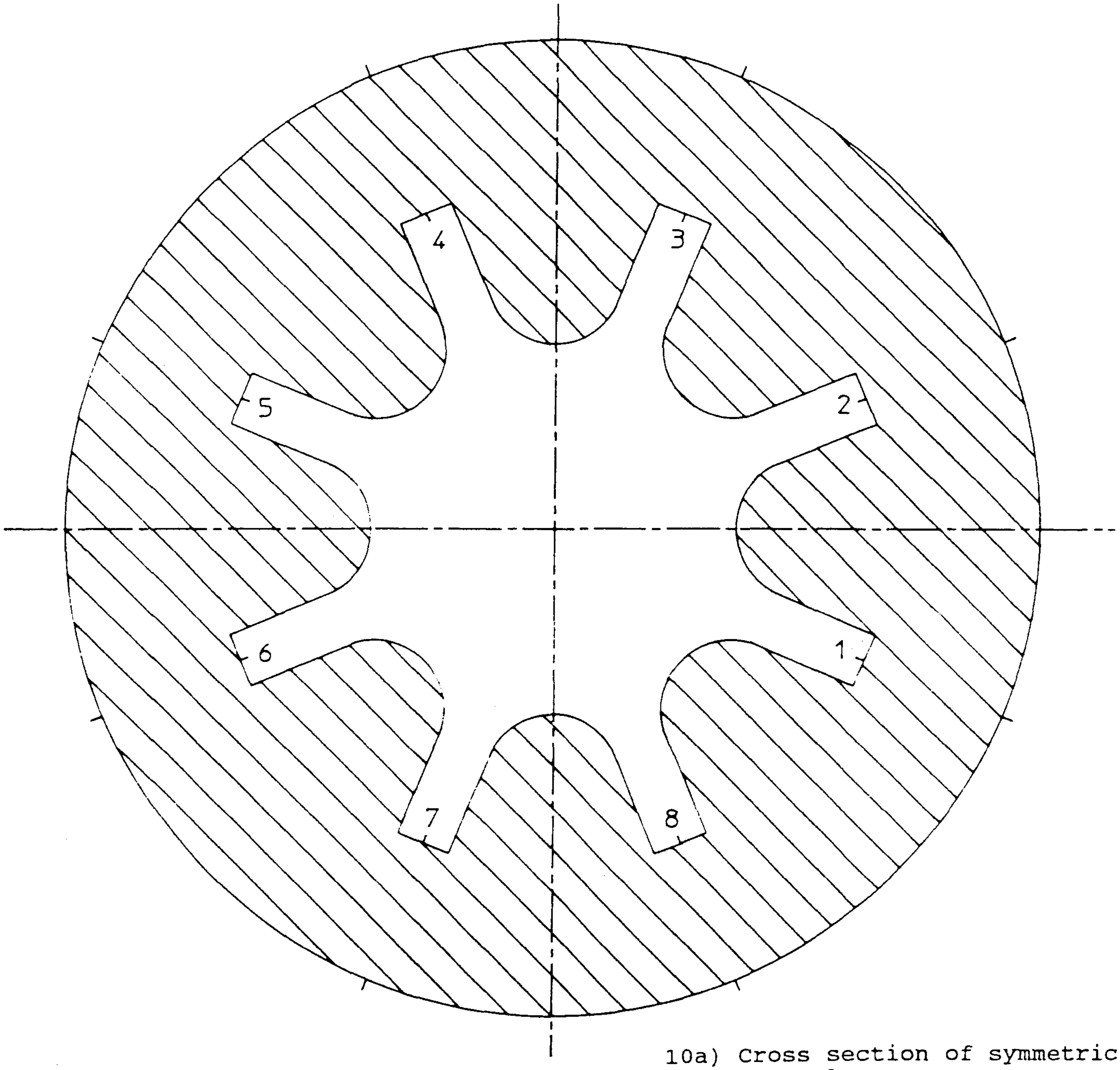
8f) Improved electrode design

9a) Cross section of perfect symmetrical quadrupole

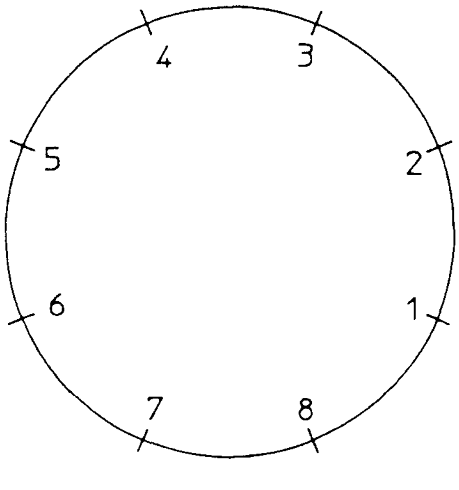


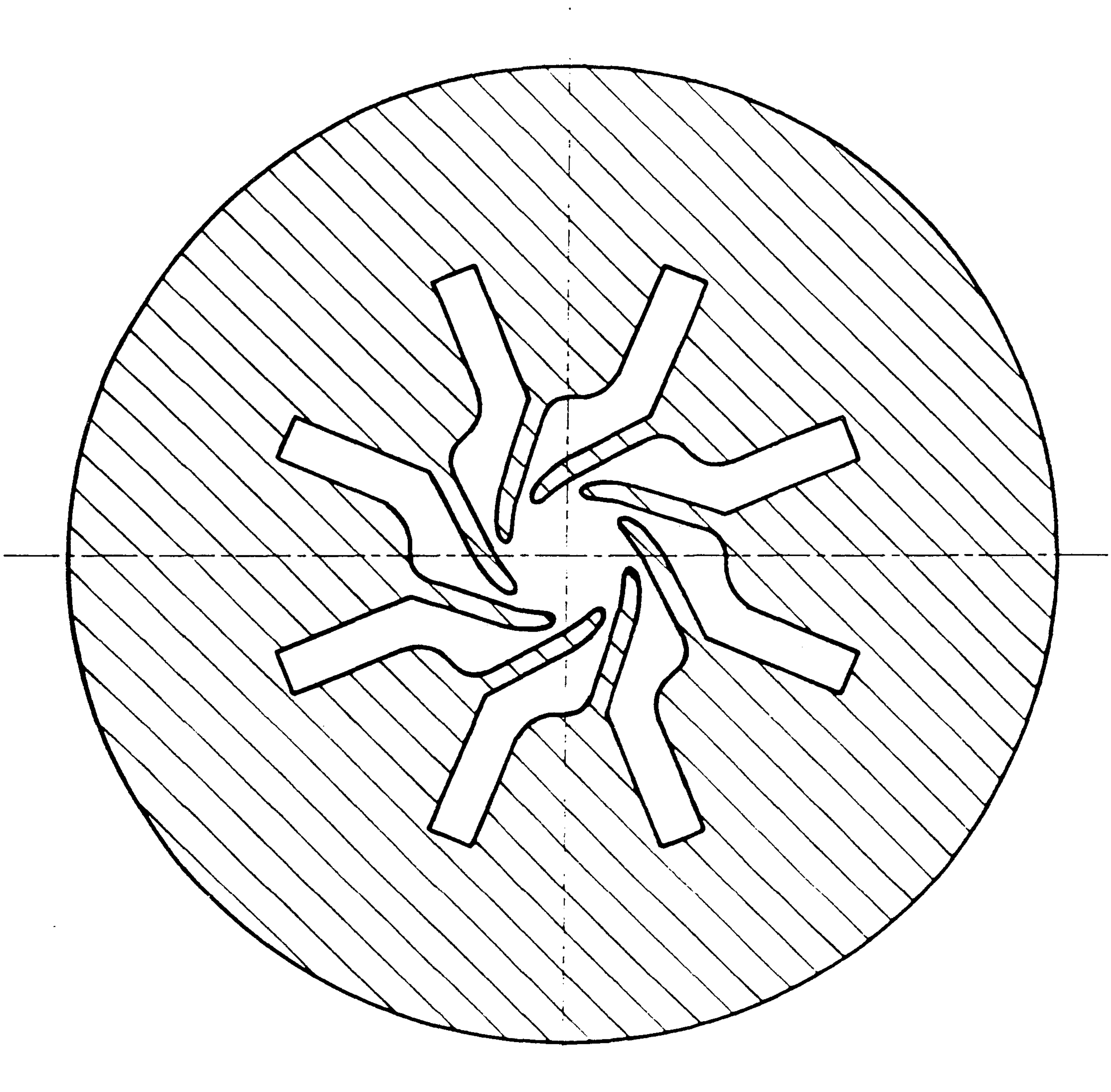


9b) Map of interior of perfect quadrupole onto circular disk

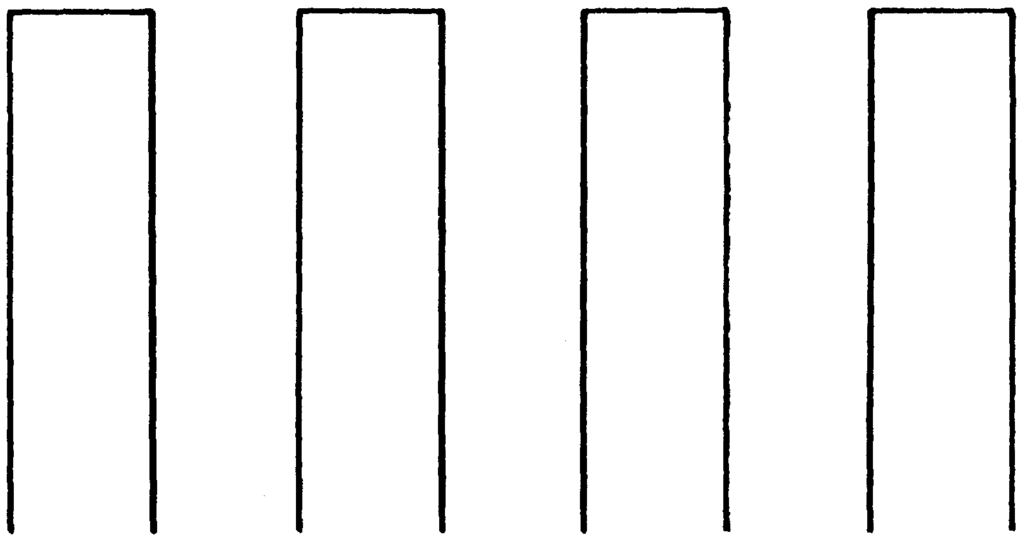


10a) Cross section of symmetric octupole

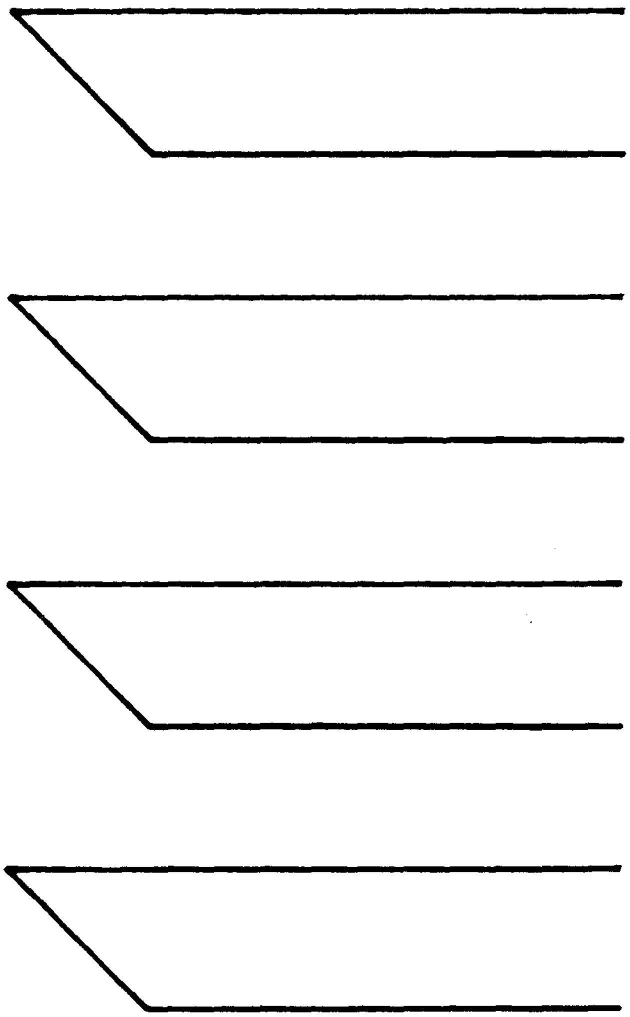




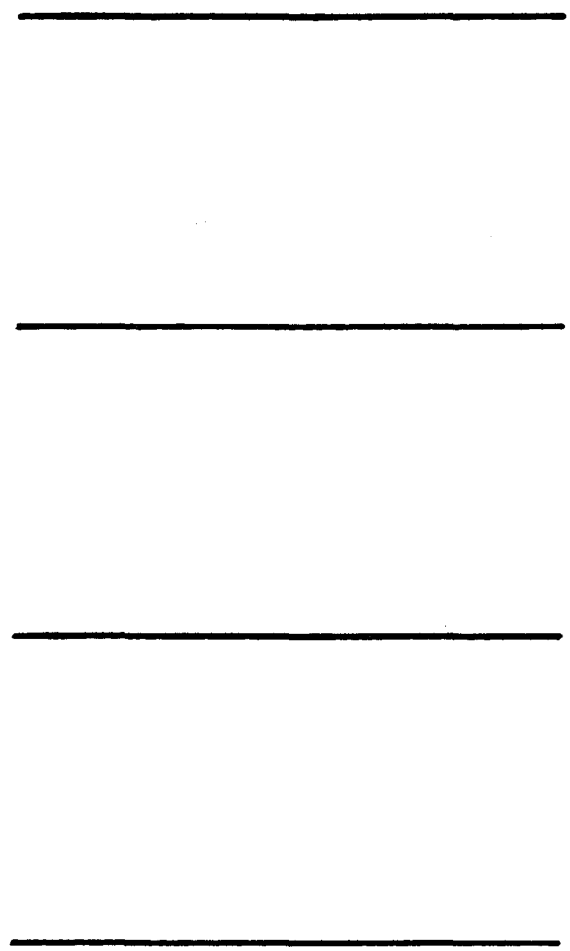
10b) Cross section of octupole with different geometry



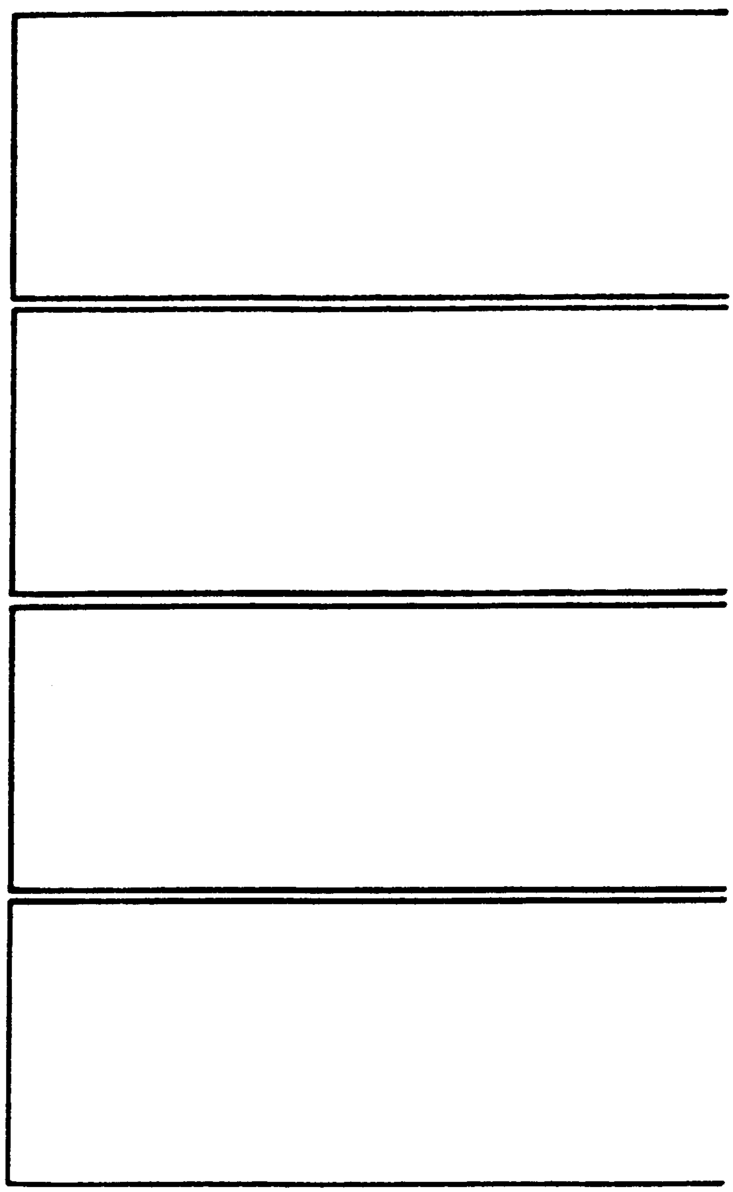
10c) Cross section #1 of linear array of poles



10d) Cross section #2 of linear array of poles



10e) Cross section #3 of linear array of poles



10f) Cross section #4 of linear array of poles