

## NLO evolution of color dipoles

Ian Balitsky and Giovanni A. Chirilli  
*Physics Dept., ODU, Norfolk VA 23529,*  
*and*  
*Theory Group, Jlab, 12000 Jefferson Ave,*  
*Newport News, VA 23606 \**  
 (Dated: October 17, 2007)

The small- $x$  deep inelastic scattering in the saturation region is governed by the non-linear evolution of Wilson-lines operators. In the leading logarithmic approximation it is given by the BK equation for the evolution of color dipoles. In the next-to-leading order the BK equation gets contributions from quark and gluon loops as well as from the tree gluon diagrams with quadratic and cubic nonlinearities. We calculate the gluon contribution to small- $x$  evolution of Wilson lines (the quark part was obtained earlier).

PACS numbers: 12.38.Bx, 11.15.Kc, 12.38.Cy

### I. INTRODUCTION

A general feature of high-energy scattering is that a fast particle moves along its straight-line classical trajectory and the only quantum effect is the eikonal phase factor acquired along this propagation path. In QCD, for the fast quark or gluon scattering off some target, this eikonal phase factor is a Wilson line - the infinite gauge link ordered along the straight line collinear to particle's velocity  $n^\mu$ :

$$U^\eta(x_\perp) = \text{Pexp}\left\{ig \int_{-\infty}^{\infty} du n_\mu A^\mu(un + x_\perp)\right\}, \quad (1)$$

Here  $A_\mu$  is the gluon field of the target,  $x_\perp$  is the transverse position of the particle which remains unchanged throughout the collision, and the index  $\eta$  labels the rapidity of the particle. Repeating the above argument for the target (moving fast in the spectator's frame) we see that particles with very different rapidities perceive each other as Wilson lines and therefore these Wilson-line operators form the convenient effective degrees of freedom in high-energy QCD (for a review, see ref. [1]).

Let us consider the deep inelastic scattering from a hadron at small  $x_B = Q^2/(2p \cdot q)$ . The virtual photon decomposes into a pair of fast quarks moving along straight lines separated by some transverse distance. The propagation of this quark-antiquark pair reduces to the "propagator of the color dipole"  $U(x_\perp)U^\dagger(y_\perp)$  - two Wilson lines ordered along the direction collinear to quarks' velocity. The structure function of a hadron is proportional to a matrix element of this color dipole operator

$$\hat{U}^\eta(x_\perp, y_\perp) = 1 - \frac{1}{N_c} \text{Tr}\{\hat{U}^\eta(x_\perp)\hat{U}^{\dagger\eta}(y_\perp)\} \quad (2)$$

switched between the target's states ( $N_c = 3$  for QCD). The gluon parton density is approximately

$$x_B G(x_B, \mu^2 = Q^2) \simeq \langle p | \hat{U}^\eta(x_\perp, 0) | p \rangle \Big|_{x_\perp^2 = Q^{-2}} \quad (3)$$

where  $\eta = \ln \frac{1}{x_B}$ . (As usual, we denote operators by "hat"). The energy dependence of the structure function is translated then into the dependence of the color dipole on the slope of the Wilson lines determined by the rapidity  $\eta$ .

Thus, the small- $x$  behavior of the structure functions is governed by the rapidity evolution of color dipoles [2, 3]. At relatively high energies and for sufficiently small dipoles we can use the leading logarithmic approximation (LLA) where  $\alpha_s \ll 1$ ,  $\alpha_s \ln x_B \sim 1$  and get the non-linear BK evolution equation for the color dipoles [4, 5]:

$$\frac{d}{d\eta} \hat{U}(x, y) = \frac{\alpha_s N_c}{2\pi^2} \int d^2 z \frac{(x-y)^2}{(x-z)^2(z-y)^2} [\hat{U}(x, z) + \hat{U}(y, z) - \hat{U}(x, y) - \hat{U}(x, z)\hat{U}(z, y)] \quad (4)$$

---

\*Electronic address: balitsky@jlab.org, chirilli@jlab.org

The first three terms correspond to the linear BFKL evolution [6] and describe the parton emission while the last term is responsible for the parton annihilation. For sufficiently high  $x_B$  the parton emission balances the parton annihilation so the partons reach the state of saturation[7] with the characteristic transverse momentum  $Q_s$  growing with energy  $1/x_B$  (for a review, see [8])

As usual, to get the region of application of the leading-order evolution equation one needs to find the next-to-leading order (NLO) corrections. In the case of the small- $x$  evolution equation (4) there is another reason why NLO corrections are important. Unlike the DGLAP evolution, the argument of the coupling constant in Eq. (4) is left undetermined in the LLA, and usually it is set by hand to be  $Q_s$ . Careful analysis of this argument is very important from both theoretical and experimental points of view. From the theoretical viewpoint, we need to know whether the coupling constant is determined by the size of the original dipole  $|x - y|$  or of the size of the produced dipoles  $|x - z|$  and/or  $|z - y|$  since we may get a very different behavior of the solutions of the equation (4). On the experimental side, the cross section is proportional to some power of the coupling constant so the argument determines how big (or how small) is the cross section. The typical argument of  $\alpha_s$  is the characteristic transverse momenta of the process. For high enough energies, they are of order of the saturation scale  $Q_s$  which is  $\sim 2 \div 3$  GeV for the LHC collider, so even the difference between  $\alpha(Q_s)$  and  $\alpha(2Q_s)$  can make a substantial impact on the cross section. The precise form of the argument of  $\alpha_s$  should come from the solution of the BK equation with the running coupling constant, and the starting point of the analysis of the argument of  $\alpha_s$  in Eq. (4) is the calculation of the NLO evolution.

Let us present our result for the NLO evolution of the color dipole (hereafter we use notations  $X \equiv x - z$ ,  $X' \equiv x - z'$ ,  $Y \equiv y - z$ , and  $Y' \equiv y - z'$ )

$$\begin{aligned}
& \frac{d}{d\eta} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \\
&= \frac{\alpha_s}{2\pi^2} \int d^2z \frac{(x-y)^2}{X^2 Y^2} \left\{ 1 + \frac{\alpha_s}{4\pi} \left[ b \ln(x-y)^2 \mu^2 - b \frac{X^2 - Y^2}{(x-y)^2} \ln \frac{X^2}{Y^2} + \left(\frac{67}{9} - \frac{\pi^2}{3}\right) N_c - \frac{10}{9} n_f \right. \right. \\
&\quad \left. \left. - 2N_c \ln \frac{X^2}{(x-y)^2} \ln \frac{Y^2}{(x-y)^2} \right] \right\} \left[ \text{Tr}\{\hat{U}_x \hat{U}_z^\dagger\} \text{Tr}\{\hat{U}_z \hat{U}_y^\dagger\} - N_c \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \right] \\
&+ \frac{\alpha_s^2}{16\pi^4} \int d^2z d^2z' \left[ \left( -\frac{4}{(z-z')^4} + \left\{ 2 \frac{X^2 Y'^2 + X'^2 Y^2 - 4(x-y)^2 (z-z')^2}{(z-z')^4 [X^2 Y'^2 - X'^2 Y^2]} \right. \right. \right. \\
&+ \left. \frac{(x-y)^4}{X^2 Y'^2 - X'^2 Y^2} \left[ \frac{1}{X^2 Y'^2} + \frac{1}{Y^2 X'^2} \right] + \frac{(x-y)^2}{(z-z')^2} \left[ \frac{1}{X^2 Y'^2} - \frac{1}{X'^2 Y^2} \right] \right\} \ln \frac{X^2 Y'^2}{X'^2 Y^2} \right. \\
&\quad \left. \times \left[ \text{Tr}\{\hat{U}_x \hat{U}_z^\dagger\} \text{Tr}\{\hat{U}_z \hat{U}_{z'}^\dagger\} \text{Tr}\{\hat{U}_{z'} \hat{U}_y^\dagger\} - \text{Tr}\{\hat{U}_x \hat{U}_z^\dagger \hat{U}_z \hat{U}_{z'}^\dagger \hat{U}_{z'} \hat{U}_y^\dagger\} - (z' \rightarrow z) \right] \right. \\
&+ \left\{ \frac{(x-y)^2}{(z-z')^2} \left[ \frac{1}{X^2 Y'^2} + \frac{1}{Y^2 X'^2} \right] - \frac{(x-y)^4}{X^2 Y'^2 X'^2 Y^2} \right\} \ln \frac{X^2 Y'^2}{X'^2 Y^2} \text{Tr}\{\hat{U}_x \hat{U}_z^\dagger\} \text{Tr}\{\hat{U}_z \hat{U}_{z'}^\dagger\} \text{Tr}\{\hat{U}_{z'} \hat{U}_y^\dagger\} \\
&+ \left. 4n_f \left\{ \frac{4}{(z-z')^4} - 2 \frac{X'^2 Y^2 + Y'^2 X^2 - (x-y)^2 (z-z')^2}{(z-z')^4 (X^2 Y'^2 - X'^2 Y^2)} \ln \frac{X^2 Y'^2}{X'^2 Y^2} \right\} \text{Tr}\{t^a \hat{U}_x t^b \hat{U}_y^\dagger\} \left[ \text{Tr}\{t^a \hat{U}_z t^b \hat{U}_{z'}^\dagger\} - (z' \rightarrow z) \right] \right]
\end{aligned} \tag{5}$$

Here  $\mu$  is the normalization point in the  $\overline{MS}$  scheme and  $b = \frac{11}{3} N_c - \frac{2}{3} n_f$  is the first coefficient of the  $\beta$ -function. The result of this paper is the gluon part of the evolution, the quark part of Eq. (5) proportional to  $n_f$  was found earlier [9, 10]. The NLO kernel is a sum of the running-coupling part (proportional to  $b$ ), the non-conformal double-log term  $\sim \ln \frac{(x-y)^2}{(x-z)^2} \ln \frac{(x-y)^2}{(x-z')^2}$  and three conformal terms which depend on the two four-point conformal ratios  $\frac{X^2 Y'^2}{X'^2 Y^2}$  and  $\frac{(x-y)^2 (z-z')^2}{X^2 Y'^2}$ . Note that the logarithm of the second conformal ratio  $\ln \frac{(x-y)^2 (z-z')^2}{X^2 Y'^2}$  is absent.

It should be emphasized that the NLO result itself does not lead automatically to the argument of coupling constant  $\alpha_s$  in Eq. 4. In order to get this argument one can use the renormalon-based approach[11]: first get the quark part of the running coupling constant coming from the bubble chain of quark loops and then make a conjecture that the gluon part of the  $\beta$ -function will follow that pattern. The Eq. (5) proves this conjecture in the first nontrivial order: the quark part of the  $\beta$  - function  $\frac{2}{3} n_f$  calculated earlier gets promoted to full  $b$ . The analysis of the argument of the coupling constant was performed in Refs. [9, 10] and we briefly review it in Sect. 7 for completeness. Roughly speaking, the argument of  $\alpha_s$  is determined by the size of the smallest dipole  $\min(|x - y|, |x - z|, |y - z|)$ .

The paper is organized as follows. In Sect. II we remind the derivation of the BK equation in the leading order in  $\alpha_s$ . In Sect. III and IV, which are central to the paper, we calculate the gluon contribution to the NLO kernel of the small- $x$  evolution of color dipoles: in Sect. III we calculate the part of the NLO kernel corresponding to one-to-three dipoles transition and in Sect. IV the one-to-two dipoles part. In Sect. V we assemble the NLO BK kernel and in Sect. VI compare the forward NLO BK kernel to the NLO BFKL results[12]. The results of the analysis of the

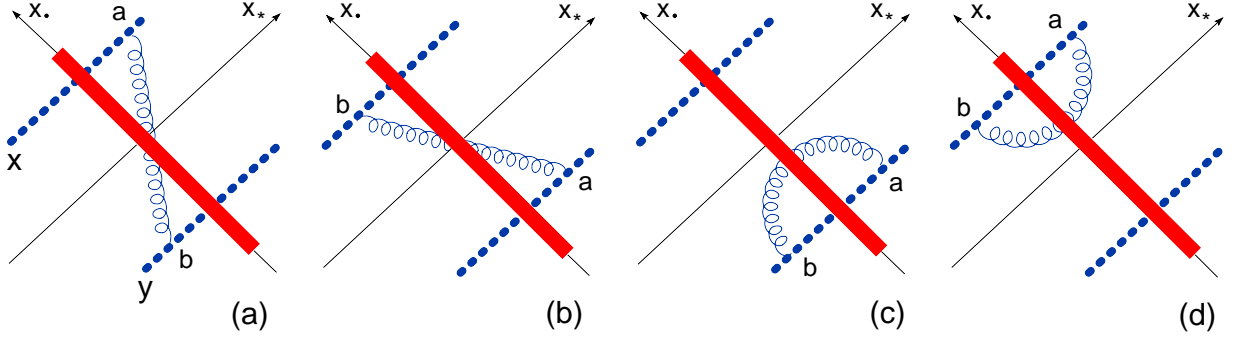


FIG. 1: Leading-order diagrams for the small- $x$  evolution of color dipole. Gauge links are denoted by dotted lines.

argument of coupling constant are briefly reviewed in Sect. VII. Appendix A is devoted to the calculation of the UV-divergent part of the one-to-three dipole kernel and in Appendix B we discuss the dependence of the NLO kernel on the cutoff in the longitudinal momenta.

## II. DERIVATION OF THE BK EQUATION

Before discussing of the small- $x$  evolution of color dipole in the next-to-leading approximation it is instructive to recall the derivation of the leading-order (BK) evolution equation. As discussed in the Introduction, the dependence of the structure functions on  $x_B$  comes from the dependence of Wilson-line operators

$$\hat{U}^\eta(x_\perp) = \text{Pexp}\left\{ig \int_{-\infty}^{\infty} du n_\mu \hat{A}^\mu(un + x_\perp)\right\}, \quad n \equiv p_1 + e^{-2\eta}p_2 \quad (6)$$

on the slope of the supporting line. The momenta  $p_1$  and  $p_2$  are the light-like vectors such that  $q = p_1 - x_B p_2$  and  $p = p_2 + \frac{m^2}{s}p_1$  where  $p$  is the momentum of the target and  $m$  is the mass. Throughout the paper, we use the Sudakov variables  $p = \alpha p_1 + \beta p_2 + p_\perp$  and the notations  $x_\bullet \equiv x_\mu p_1^\mu$  and  $x_* \equiv x_\mu p_2^\mu$  related to the light-cone coordinates:  $x_* = x^+ \sqrt{s/2}$ ,  $x_\bullet = x^- \sqrt{s/2}$ .

To find the evolution of the color dipole (2) with respect to the slope of the Wilson lines in the leading log approximation we consider the matrix element of the color dipole between (arbitrary) target states and integrate over the gluons with rapidities  $\eta_1 > \eta > \eta_2 = \eta_1 - \Delta\eta$  leaving the gluons with  $\eta < \eta_2$  as a background field (to be integrated over later). In the frame of gluons with  $\eta \sim \eta_1$  the fields with  $\eta < \eta_2$  shrink to a pancake and we obtain the four diagrams shown in Fig. 1. Technically, to find the kernel in the leading-order approximation we write down the general form of the operator equation for the evolution of the color dipole

$$\frac{\partial}{\partial\eta} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} = K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} + \dots \quad (7)$$

(where dots stand for the higher orders of the expansion) and calculate the l.h.s. of Eq. (7) in the shock-wave background

$$\frac{\partial}{\partial\eta} \langle \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle_{\text{shockwave}} = \langle K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle_{\text{shockwave}} \quad (8)$$

In what follows we replace  $\langle \dots \rangle_{\text{shockwave}}$  by  $\langle \dots \rangle$  for brevity.

With future NLO computation in view, we will perform the leading-order calculation in the lightcone gauge  $p_2^\mu A_\mu = 0$ . The gluon propagator in a shock-wave external field has the form[13, 14]

$$\begin{aligned} \langle \hat{A}_\mu^a(x) \hat{A}_\nu^b(y) \rangle &= \theta(x_* y_*) \delta^{ab} \frac{s}{2} \int \bar{d}\alpha \bar{d}\beta (x_\perp | \frac{d_{\mu\nu}}{i(\alpha\beta s - p_\perp^2 + i\epsilon)} | y_\perp) \quad (9) \\ &- \theta(x_*) \theta(-y_*) \int_0^\infty \bar{d}\alpha \frac{e^{-i\alpha(x-y)_\bullet}}{2\alpha} (x_\perp | e^{-i\frac{p_\perp^2}{\alpha s} x_*} \left[ g_{\mu\xi}^\perp - \frac{2}{\alpha s} (p_\mu^\perp p_{2\xi} + p_{2\mu} p_\xi^\perp) \right] U^{ab} \left[ g_{\xi\nu}^\perp - \frac{2}{\alpha s} (p_{2\xi} p_\nu^\perp + p_{2\nu} p_\xi^\perp) \right] e^{i\frac{p_\perp^2}{\alpha s} y_*} | y_\perp) \\ &- \theta(-x_*) \theta(y_*) \int_0^\infty \bar{d}\alpha \frac{e^{i\alpha(x-y)_\bullet}}{2\alpha} (x_\perp | e^{i\frac{p_\perp^2}{\alpha s} x_*} \left[ g_{\mu\xi}^\perp - \frac{2}{\alpha s} (p_\mu^\perp p_{2\xi} + p_{2\mu} p_\xi^\perp) \right] U^{\dagger ab} \left[ g_{\xi\nu}^\perp - \frac{2}{\alpha s} (p_{2\xi} p_\nu^\perp + p_{2\nu} p_\xi^\perp) \right] e^{-i\frac{p_\perp^2}{\alpha s} y_*} | y_\perp) \end{aligned}$$

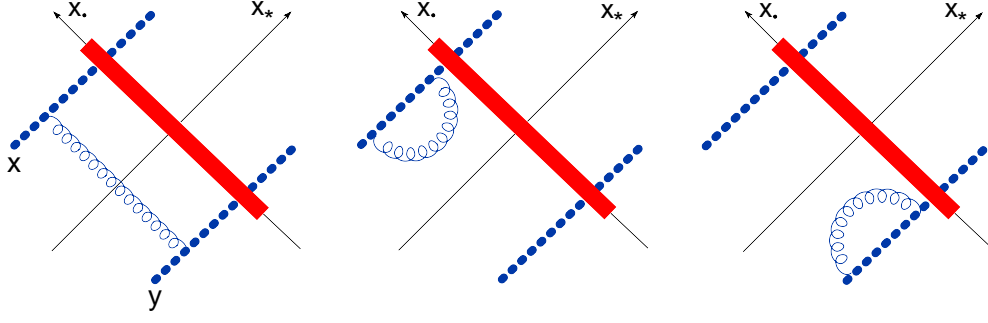


FIG. 2: Leading-order diagrams proportional to the original dipole.

where

$$d_{\mu\nu}(k) \equiv g_{\mu\nu}^{\perp} - \frac{2}{s\alpha}(k_{\mu}^{\perp}p_{2\nu} + k_{\nu}^{\perp}p_{2\mu}) - \frac{4\beta}{s\alpha}p_{2\mu}p_{2\nu} \quad (10)$$

Hereafter use Schwinger's notations  $(x_{\perp}|F(p_{\perp})|y_{\perp}) \equiv \int \tilde{d}p e^{i(p \cdot x - y)_{\perp}} F(p_{\perp})$  (the scalar product of the four-dimensional vectors in our notations is  $x \cdot y = \frac{2}{s}(x_{*}y_{\bullet} + x_{\bullet}y_{*}) - (x, y)_{\perp}$ ). Note that the interaction with the shock wave does not change the  $\alpha$ -component of the gluon momentum.

We obtain

$$\int_0^{\infty} du \int_{-\infty}^0 dv \langle \hat{A}_{\bullet}^a(un + x_{\perp}) \hat{A}_{\bullet}^b(vn + y_{\perp}) \rangle_{\text{Fig.1a}} = -4\alpha_s \int_{e^{-\eta_2}}^{\infty} \frac{d\alpha}{\alpha} (x_{\perp} | \frac{p_i}{p_{\perp}^2 + \alpha^2 e^{-2\eta_1} s} U^{ab} \frac{p_i}{p_{\perp}^2 + \alpha^2 e^{-2\eta_1} s} | y_{\perp}) \quad (11)$$

Formally, the integral over  $\alpha$  diverges at the lower limit, but since we integrate over the rapidities  $\eta > \eta_2$  we get in the LLA

$$\int_0^{\infty} du \int_{-\infty}^0 dv \langle \hat{A}_{\bullet}^a(un + x_{\perp}) \hat{A}_{\bullet}^b(vn + y_{\perp}) \rangle_{\text{Fig.1a}} = -4\alpha_s \Delta\eta (x_{\perp} | \frac{p_i}{p_{\perp}^2} U^{ab} \frac{p_i}{p_{\perp}^2} | y_{\perp}) \quad (12)$$

and therefore

$$\langle \hat{U}_x \otimes \hat{U}_y^{\dagger} \rangle_{\text{Fig.1a}}^{\eta_1} = -\frac{\alpha_s \Delta\eta}{\pi^2} \{t^a U_x \otimes t^b U_y^{\dagger}\}^{\eta_2} \int d^2 z_{\perp} \frac{(x-z, y-z)_{\perp}}{(x-z)_{\perp}^2 (y-z)_{\perp}^2} (U_z^{\eta_2})^{ab} \quad (13)$$

The contribution of the diagram in Fig. 1b is obtained from Eq. (13) by the replacement  $t^a U_x \otimes t^b U_y^{\dagger} \rightarrow U_x t^b \otimes U_y^{\dagger} t^a$ ,  $x \leftrightarrow y$  and the two remaining diagrams are obtained from Eq. 12 by taking  $y = x$  (Fig. 1c) and  $x = y$  (Fig. 1d). Finally, one obtains

$$\begin{aligned} \langle \hat{U}_x \otimes \hat{U}_y^{\dagger} \rangle_{\text{Fig.1}}^{\eta_1} &= -\frac{\alpha_s \Delta\eta}{\pi^2} \{t^a U_x \otimes t^b U_y^{\dagger} + U_x t^b \otimes U_y^{\dagger} t^a\}^{\eta_2} \int d^2 z_{\perp} \frac{(x-z, y-z)_{\perp}}{(x-z)_{\perp}^2 (y-z)_{\perp}^2} (U_z^{\eta_2})^{ab} \\ &+ \frac{\alpha_s \Delta\eta}{\pi^2} \{t^a U_x t^b \otimes U_y^{\dagger}\}^{\eta_2} \int \frac{d^2 z_{\perp}}{(x-z)_{\perp}^2} (U_z^{\eta_2})^{ab} + \frac{\alpha_s \Delta\eta}{\pi^2} \{U_x \otimes t^b U_y^{\dagger} t^a\}^{\eta_2} \int \frac{d^2 z_{\perp}}{(y-z)_{\perp}^2} (U_z^{\eta_2})^{ab} \end{aligned} \quad (14)$$

so

$$\langle \text{Tr}\{\hat{U}_x \hat{U}_y^{\dagger}\} \rangle_{\text{Fig.1}}^{\eta_1} = \frac{\alpha_s \Delta\eta}{2\pi^2} \int d^2 z_{\perp} \frac{(x-y)_{\perp}^2}{(x-z)_{\perp}^2 (y-z)_{\perp}^2} [\text{Tr}\{U_x U_z^{\dagger}\} \text{Tr}\{U_z U_y^{\dagger}\} - \frac{1}{N_c} \text{Tr}\{U_x U_y^{\dagger}\}]^{\eta_2} \quad (15)$$

There are also contributions coming from the diagrams shown in Fig. 2 (plus graphs obtained by reflection with respect to the shock wave). These diagrams are proportional to the original dipole  $\text{Tr}\{U_x U_y^{\dagger}\}$  and therefore the corresponding term can be derived from the contribution of Fig. 1 graphs using the requirement that the r.h.s. of the evolution equation should vanish for  $x = y$  since  $\lim_{x \rightarrow y} \frac{\partial}{\partial \eta} \text{Tr}\{U_x U_y^{\dagger}\} = 0$ . It is easy to see that this requirement leads to

$$\langle \text{Tr}\{\hat{U}_x \hat{U}_y^{\dagger}\} \rangle^{\eta_1} = \frac{\alpha_s \Delta\eta}{2\pi^2} \int d^2 z_{\perp} \frac{(x-y)_{\perp}^2}{(x-z)_{\perp}^2 (y-z)_{\perp}^2} [\text{Tr}\{U_x U_z^{\dagger}\} \text{Tr}\{U_z U_y^{\dagger}\} - N_c \text{Tr}\{U_x U_y^{\dagger}\}]^{\eta_2} \quad (16)$$

which is equivalent to the BK equation for the evolution of the color dipole (4).

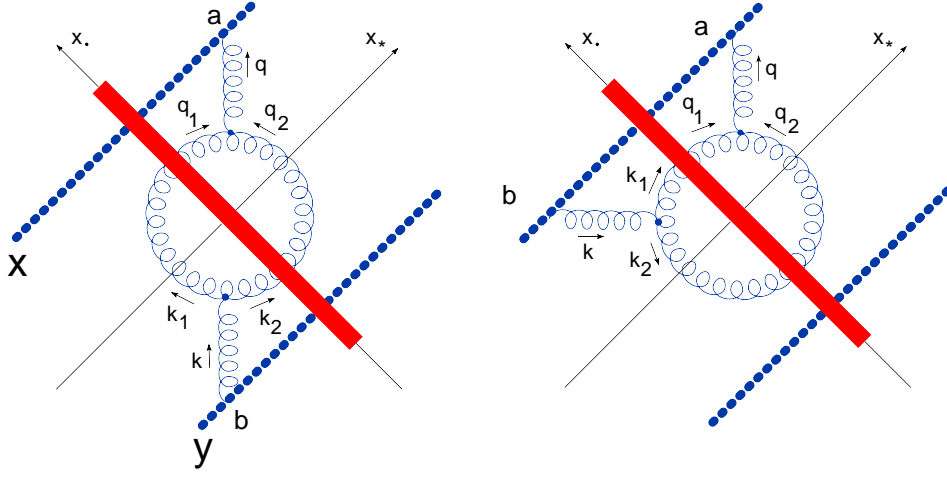


FIG. 3: “Cut self-energy” diagram.

### III. DIAGRAMS WITH TWO GLUON-SHOCKWAVE INTERSECTIONS

#### A. “Cut self-energy” diagrams

In the next-to-leading order there are three types of diagrams. Diagrams of the first type have two intersections of the emitted gluons with the shock wave, diagram of the second type have one intersection, and finally there are diagrams of the third type without intersections. In principle, there could have been contributions coming from the gluon loop which lies entirely in the shock wave, but we will demonstrate below that such terms are absent (see the discussion at the end of Sect. VI).

For the NLO calculation we use the lightcone gauge  $p_2^\mu A_\mu = 0$ . Also, we find it convenient to change the prescription for the cutoff in the longitudinal direction. We consider the light-like dipoles (in the  $p_1$  direction) and impose the cutoff on the maximal  $\alpha$  emitted by any gluon from the Wilson lines so

$$U_x^\eta = \text{Pexp} \left[ ig \int_{-\infty}^{\infty} du p_1^\mu A_\mu^\eta(u p_1 + x_\perp) \right]$$

$$A_\mu^\eta(x) = \int \tilde{d}^4 k \theta(e^\eta - |\alpha_k|) e^{-ik \cdot x} A_\mu(k) \quad (17)$$

As we will see below, the (almost) conformal result (5) comes from the regularization (17). In Appendix B we will present the NLO kernel for the cutoff with the slope (6).

We start with the calculation of the Fig. 3a diagram. Multiplying two propagators (9), two 3-gluon vertices and two bare propagators we obtain (AGREES)

$$g^2 \int_0^\infty du \int_{-\infty}^0 dv \langle \hat{A}_\bullet^a(u p_1 + x_\perp) \hat{A}_\bullet^b(v p_1 + y_\perp) \rangle \quad (18)$$

$$= \frac{1}{2} g^4 \frac{s^2}{4} f^{anl} f^{bn'l'} \int \tilde{d}\alpha \tilde{d}\beta \tilde{d}\beta' \tilde{d}\beta_1 \tilde{d}\beta_1' \tilde{d}\beta_2 \tilde{d}\beta_2' \int d^2 z d^2 z' \int \tilde{d}^2 q_1 \tilde{d}^2 q_2 \tilde{d}^2 k_1 \tilde{d}^2 k_2 e^{i(q_1+q_2, x)_\perp - i(k_1+k_2, y)_\perp}$$

$$\frac{4\alpha_1(\alpha - \alpha_1) U_z^{nn'} U_{z'}^{l'l'} e^{-i(q_1-k_1, z)_\perp - i(q_2-k_2, z')_\perp}}{(\beta - \beta_1 - \beta_2 + i\epsilon)(\beta' - \beta_1' - \beta_2' + i\epsilon)(\beta - i\epsilon)(\beta' - i\epsilon)} \frac{d_{\bullet\lambda}(\alpha p_1 + \beta p_2 + q_{1\perp} + k_{1\perp})}{\alpha\beta s - (q_1 + q_2)_\perp^2 + i\epsilon} \frac{d_{\lambda'\bullet}(\alpha p_1 + \beta' p_2 + q_{2\perp} + k_{2\perp})}{\alpha\beta' s - (k_1 + k_2)_\perp^2 + i\epsilon}$$

$$\frac{d_{\mu\xi}(\alpha_1 p_1 + \beta_1 p_2 + q_{1\perp})}{\alpha_1 \beta_1 s - q_{1\perp}^2 + i\epsilon} \frac{d_{\mu\xi}^{\xi}(\alpha_1 p_1 + \beta_1' p_2 + k_{1\perp})}{\alpha_1 \beta_1' s - k_{1\perp}^2 + i\epsilon} \frac{d_{\nu\eta}((\alpha - \alpha_1) p_1 + \beta_2 p_2 + q_{2\perp})}{(\alpha - \alpha_1) \beta_2 s - q_{2\perp}^2 + i\epsilon} \frac{d_{\nu\eta}^{\eta}((\alpha - \alpha_1) p_1 + \beta_2' p_2 + k_{2\perp})}{(\alpha - \alpha_1) \beta_2' s - k_{2\perp}^2 + i\epsilon}$$

$$\Gamma^{\mu\nu\lambda}(\alpha p_1 + q_{1\perp}, (\alpha - \alpha_1) p_1 + q_{2\perp}, -\alpha p_1 - q_{1\perp} - q_{2\perp}) \Gamma^{\mu'\nu'\lambda'}(\alpha p_1 + k_{1\perp}, (\alpha - \alpha_1) p_1 + k_{2\perp}, -\alpha p_1 - k_{1\perp} - k_{2\perp})$$

where

$$\Gamma_{\mu\nu\lambda}(p, k, -p - k) = (p - k)_\lambda g_{\mu\nu} + (2k + p)_\mu g_{\nu\lambda} + (-2p - k)_\nu g_{\lambda\mu} \quad (19)$$

In this formula  $\frac{1}{\beta-i\epsilon}$  comes from the integration over  $u$  parameter in the l.h.s. and  $\frac{1}{\beta-\beta_1-\beta_2+i\epsilon}$  from the integration of the left three-gluon vertex over the half-space  $x_* > 0$ . Similarly, we get  $\frac{1}{\beta'-i\epsilon}$  from the integration over  $v$  parameter and  $\frac{1}{\beta'-\beta'_1-\beta'_2+i\epsilon}$  from the integration of the right three-gluon vertex over the half-space  $x_* < 0$ . The factor  $\frac{1}{2}$  in the r.h.s. is combinatorial. Note that in the light-cone gauge one can always neglect the  $\beta p_{2\xi}$  components of the momenta in the three-gluon vertex since they are always multiplied by the some  $d_{\xi\eta}$ .

Taking residues at  $\beta = \beta' = 0$  and  $\beta_2 = -\beta_1, \beta'_2 = -\beta'_1$  we obtain (AGREES)

$$\begin{aligned}
& g^2 \int_0^\infty du \int_{-\infty}^0 dv \langle \hat{A}_\bullet^a(up_1 + x_\perp) \hat{A}_\bullet^b(vp_1 + y_\perp) \rangle \tag{20} \\
&= \frac{1}{2} g^4 \frac{s^2}{4} f^{anl} f^{bn'l'} \int d\alpha d\alpha_1 d\beta_1 d\beta'_1 \int d^2 z d^2 z' \int d^2 q_1 d^2 q_2 d^2 k_1 d^2 k_2 e^{i(q_1+q_2,x)_\perp - i(k_1+k_2,y)_\perp} \\
& 4 \frac{\alpha_1(\alpha - \alpha_1)}{\alpha^2} U_z^{nn'} U_{z'}^{ll'} e^{-i(q_1-k_1)z - i(q_2-k_2)z'} \frac{(q_{1\perp} + q_{2\perp})_\lambda (k_{1\perp} + k_{2\perp})_{\lambda'}}{(q_1 + q_2)_\perp^2 (k_1 + k_2)_\perp^2} \\
& \frac{d_\mu^\xi(\alpha p_1 + q_{1\perp}) d_{\xi\mu'}(\alpha p_1 + k_{1\perp}) d_\eta^n((\alpha - \alpha_1)p_1 + q_{2\perp}) d_{\eta\nu'}((\alpha - \alpha_1)p_1 + k_{2\perp})}{\alpha_1 \beta_1 s - q_{1\perp}^2 + i\epsilon \alpha_1 \beta'_1 s - k_{1\perp}^2 + i\epsilon - (\alpha - \alpha_1)\beta_1 s - q_{2\perp}^2 + i\epsilon - (\alpha - \alpha_1)\beta'_1 s - k_{2\perp}^2 + i\epsilon} \\
& \Gamma^{\mu\nu\lambda}(\alpha p_1 + q_{1\perp}, (\alpha - \alpha_1)p_1 + q_{2\perp}, -\alpha p_1 - q_{1\perp} - q_{2\perp}) \Gamma^{\mu'\nu'\lambda'}(\alpha p_1 + k_{1\perp}, (\alpha - \alpha_1)p_1 + k_{2\perp}, -\alpha p_1 - k_{1\perp} - k_{2\perp})
\end{aligned}$$

We have omitted terms  $\sim \beta p_2$  in the arguments of  $d_{\xi\eta}$  since they do not contribute to  $d_{\mu\xi} d^{\xi\mu'}$ , see Eq. (10). Introducing the variable  $u = \alpha_1/\alpha$  and taking residues at  $\beta_1 = \frac{q_1^2}{\alpha_1 s}$  and  $\beta'_1 = \frac{k_1^2}{\alpha_1 s}$  we obtain

$$\begin{aligned}
& - \frac{g^4}{8\pi^2} f^{anl} f^{bn'l'} \int_0^\sigma \frac{d\alpha}{\alpha} \int_0^1 du \bar{u} u \int d^2 z d^2 z' \int d^2 q_1 d^2 q_2 d^2 k_1 d^2 k_2 e^{i(q_1+q_2,x)_\perp - i(k_1+k_2,y)_\perp - i(q_1-k_1,z) - i(q_2-k_2,z')} \\
& \times U_z^{nn'} U_{z'}^{ll'} \frac{(q_{1\perp} + q_{2\perp})_\lambda (k_{1\perp} + k_{2\perp})_{\lambda'}}{(q_1 + q_2)_\perp^2 (k_1 + k_2)_\perp^2} \frac{d_{\mu\xi}(u\alpha p_1 + q_{1\perp}) d_{\mu'\nu'}^\xi(u\alpha p_1 + k_{1\perp}) d_{\nu\eta}(\bar{u}\alpha p_1 + q_{2\perp}) d_{\nu'\lambda'}^\eta(\bar{u}\alpha p_1 + k_{2\perp})}{q_{1\perp}^2 \bar{u} + q_{2\perp}^2 u} \frac{d_{\nu\eta}(\bar{u}\alpha p_1 + q_{2\perp}) d_{\nu'\lambda'}^\eta(\bar{u}\alpha p_1 + k_{2\perp})}{k_{1\perp}^2 \bar{u} + k_{2\perp}^2 u} \\
& \times \Gamma^{\mu\nu\lambda}(u\alpha p_1 + q_{1\perp}, \bar{u}\alpha p_1 + q_{2\perp}, -\alpha p_1 - q_{1\perp} - q_{2\perp}) \Gamma^{\mu'\nu'\lambda'}(u\alpha p_1 + k_{1\perp}, \bar{u}\alpha p_1 + k_{2\perp}, -\alpha p_1 - k_{1\perp} - k_{2\perp}) \tag{21}
\end{aligned}$$

where we have imposed a cutoff  $\alpha < \sigma$  in accordance with Eq. (17).

Using formulas (AGREES)

$$\begin{aligned}
& d_{\mu\xi}(u\alpha p_1 + q_\perp) d^{\xi\mu'}(u\alpha p_1 + k_\perp) = \left( g_{\mu\xi}^\perp - \frac{2}{s\alpha u} p_{2\mu} q_\xi^\perp \right) \left( g_{\perp}^{\xi\mu'} - \frac{2}{s\alpha u} k_\perp^\xi \right), \\
& -(q_1 + q_2)_\perp^\lambda \Gamma_{\mu\nu\lambda}(\alpha u p_1 + q_{1\perp}, \alpha \bar{u} p_1 + q_{2\perp}, -\alpha p_1 - (q_1 + q_2)_\perp) \left( g_{\perp}^{\mu i} - \frac{2}{s\alpha u} p_2^\mu q_1^i \right) \left( g_{\perp}^{\nu j} - \frac{2}{s\alpha u} p_2^\nu q_2^j \right) \\
& = (q_{1\perp}^2 - q_{2\perp}^2) g_\perp^{ij} + \frac{2}{u} q_1^i (q_1 + q_2)^j - \frac{2}{\bar{u}} (q_1 + q_2)^i q_2^j \tag{22}
\end{aligned}$$

we can represent the contribution of diagram in Fig. 3a in the form

$$\begin{aligned}
\langle \text{Tr} \{ \hat{U}_x \hat{U}_y^\dagger \} \rangle_{\text{Fig.3a}} &= - \frac{g^4}{8\pi^2} \text{Tr} \{ t^a U_x t^b U_y^\dagger \} f^{anl} f^{bn'l'} \int d^2 z d^2 z' U_z^{nn'} U_{z'}^{ll'} \tag{23} \\
& \times \int_0^\sigma \frac{d\alpha}{\alpha} \int_0^1 du \bar{u} u \int d^2 q_1 d^2 q_2 d^2 k_1 d^2 k_2 \frac{e^{i(q_1,x-z)_\perp + i(q_2,x-z')_\perp - i(k_1,y-z)_\perp - i(k_2,y-z')_\perp}}{(q_1 + q_2)^2 (k_1 + k_2)^2 (q_1^2 \bar{u} + q_2^2 u) (k_1^2 \bar{u} + k_2^2 u)} \\
& \times \left[ (q_1^2 - q_2^2) \delta_{ij} - \frac{2}{u} q_{1i} (q_1 + q_2)_j + \frac{2}{\bar{u}} (q_1 + q_2)_i q_{2j} \right] \left[ (k_1^2 - k_2^2) \delta_{ij} - \frac{2}{u} k_{1i} (k_1 + k_2)_j + \frac{2}{\bar{u}} (k_1 + k_2)_i k_{2j} \right]
\end{aligned}$$

The diagram shown in Fig. 3b is obtained by the substitution  $e^{-i(k_1+k_2,y)_\perp} \rightarrow -e^{-i(k_1+k_2,x)_\perp}$  (the different sign comes from the replacement  $[-\infty p_1, 0]_y$  by  $[0, -\infty p_1]_x$ ). We get

$$\begin{aligned}
\langle \text{Tr} \{ \hat{U}_x \hat{U}_y^\dagger \} \rangle_{\text{Fig.3a+b}} &= \frac{g^4}{8\pi^2} \text{Tr} \{ t^a U_x t^b U_y^\dagger \} f^{anl} f^{bn'l'} \int d^2 z d^2 z' U_z^{nn'} U_{z'}^{ll'} \tag{24} \\
& \times \int_0^\sigma \frac{d\alpha}{\alpha} \int_0^1 du \bar{u} u \int d^2 q_1 d^2 q_2 d^2 k_1 d^2 k_2 \frac{e^{i(q_1+q_2,x)_\perp - i(q_1-k_1,z) - i(q_2-k_2,z')} [e^{-i(k_1+k_2,x)_\perp} - e^{-i(k_1+k_2,y)_\perp}]}{(q_1 + q_2)^2 (k_1 + k_2)^2 (q_1^2 \bar{u} + q_2^2 u) (k_1^2 \bar{u} + k_2^2 u)} \\
& \times \left[ (q_1^2 - q_2^2) \delta_{ij} - \frac{2}{u} q_{1i} (q_1 + q_2)_j + \frac{2}{\bar{u}} (q_1 + q_2)_i q_{2j} \right] \left[ (k_1^2 - k_2^2) \delta_{ij} - \frac{2}{u} k_{1i} (k_1 + k_2)_j + \frac{2}{\bar{u}} (k_1 + k_2)_i k_{2j} \right]
\end{aligned}$$

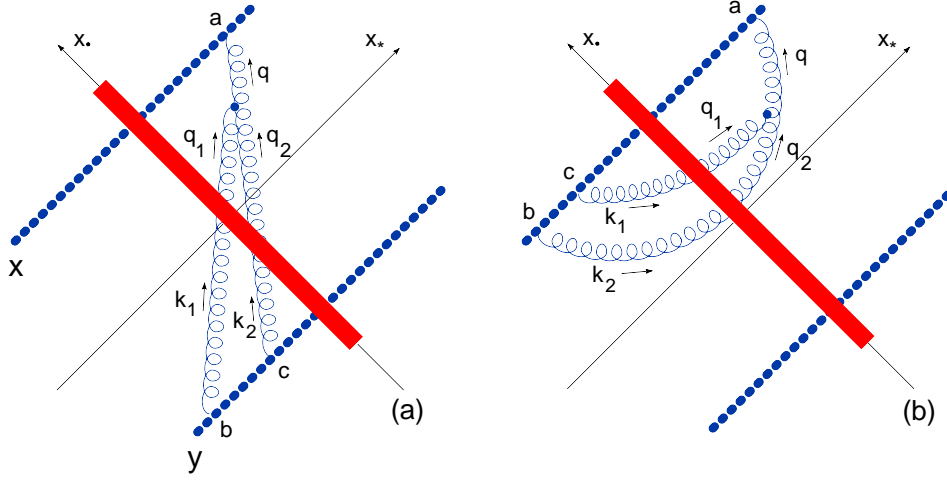


FIG. 4: “Cut vertex” diagram.

### B. “Cut vertex” diagrams

Next, consider the “cut vertex” diagram in Fig. 4a. The analog of Eq. (20) has the form: (AGREES)

$$\begin{aligned}
& g^3 \int_0^\infty dt \int_{-\infty}^0 du \int_u dv \langle \hat{A}_\bullet^a(tp_1 + x_\perp) \hat{A}_\bullet^b(up_1 + y_\perp) \hat{A}_\bullet^c(vp_1 + y_\perp) \rangle \\
&= 2g^4 s f^{mna} \int \dot{d}\alpha_1 \dot{d}\beta_1 \dot{d}\alpha_2 \dot{d}\beta_2 \int \dot{d}^2 q_1 \dot{d}^2 q_2 \dot{d}^2 k_1 \dot{d}^2 k_2 \Gamma^{\mu\nu\lambda}(\alpha_1 p_1 + q_{1\perp}, \alpha_2 p_1 + q_{2\perp}, -(\alpha_1 + \alpha_2)p_1 - (q_1 + q_2)_\perp) \\
&\quad \times \frac{\alpha_1 \theta(\alpha_1) \theta(\alpha_2) e^{i(q_1 + q_2, x)_\perp} \left( (q_1 + q_2)_\perp + 2(\beta_1 + \beta_2)p_2 \right)_\lambda}{(\alpha_1 + \alpha_2)(\beta_1 + \beta_2 - i\epsilon)} \frac{\left( k_1 + \frac{2(k_1, q_1)_\perp}{\alpha_1 s} p_2 \right)_\mu \left( k_2 + \frac{2(k_2, q_2)_\perp}{\alpha_2 s} p_2 \right)_\nu}{[(\alpha_1 + \alpha_2)(\beta_1 + \beta_2)s - (q_1 + q_2)_\perp^2 + i\epsilon]} \\
&\quad \times \int d^2 z d^2 z' \frac{e^{-i(k_1 + k_2, y)_\perp} U_z^{mb} U_{z'}^{nc} e^{-i(q_1 - k_1, z)_\perp - i(q_2 - k_2, z')_\perp}}{k_1^2 (k_1^2 \alpha_2 + k_2^2 \alpha_1) (\alpha_1 \beta_1 s - q_{1\perp}^2 + i\epsilon) (\alpha_2 \beta_2 s - q_{2\perp}^2 + i\epsilon)} \quad (25)
\end{aligned}$$

Going to variables  $\alpha = \alpha_1 + \alpha_2$ ,  $u = \alpha_1/\alpha$  and taking residues at  $\beta_1 + \beta_2 = 0$  and  $\beta_1 = \frac{q_1^2}{\alpha_1}$  we get (AGREES)

$$\begin{aligned}
& g^3 \int_0^\infty dt \int_{-\infty}^0 du \int_u dv \langle \hat{A}_\bullet^a(tp_1 + x_\perp) \hat{A}_\bullet^b(up_1 + y_\perp) \hat{A}_\bullet^c(vp_1 + y_\perp) \rangle \\
&= \frac{g^4}{2\pi^2} f^{mna} \int_0^\sigma \frac{d\alpha}{\alpha^2} \int_0^1 du u \int \dot{d}^2 q_1 \dot{d}^2 q_2 \dot{d}^2 k_1 \dot{d}^2 k_2 \int d^2 z d^2 z' U_z^{mb} U_{z'}^{nc} \frac{e^{i(q_1 + q_2, x)_\perp - i(k_1 + k_2, y)_\perp - i(q_1 - k_1, z)_\perp - i(q_2 - k_2, z')_\perp}}{(q_1 + q_2)_\perp^2 (q_1^2 \bar{u} + q_2^2 u) k_1^2 (k_1^2 \bar{u} + k_2^2 u)} \\
&\quad \times \left( k_1 + \frac{2(k_1, q_1)_\perp}{\alpha u s} p_2 \right)_\mu \left( k_2 + \frac{2(k_2, q_2)_\perp}{\alpha \bar{u} s} p_2 \right)_\nu (q_1 + q_2)_\perp \Gamma^{\mu\nu\lambda}(\alpha u p_1 + q_{1\perp}, \alpha \bar{u} p_1 + q_{2\perp}, -\alpha p_1 - (q_1 + q_2)_\perp) \quad (26)
\end{aligned}$$

and therefore

$$\begin{aligned}
& \langle \text{Tr} \{ \hat{U}_x \hat{U}_y^\dagger \} \rangle_{\text{Fig. 4a}} \quad (27) \\
&= -i \frac{g^4}{2\pi^2} f^{mna} \text{Tr} \{ t^a U_x t^b t^c U_y^\dagger \} \int_0^\sigma \frac{d\alpha}{\alpha} \int_0^1 du \bar{u} u \int \dot{d}^2 k_1 \dot{d}^2 k_2 \dot{d}^2 q_1 \dot{d}^2 q_2 \int d^2 z d^2 z' U_z^{mb} U_{z'}^{nc} \\
&\quad \times \frac{(q_1^2 - q_2^2) \delta_{ij} - \frac{2}{u} q_{1i} (q_1 + q_2)_j + \frac{2}{\bar{u}} (q_1 + q_2)_i q_{2j}}{(q_1 + q_2)_\perp^2 (q_1^2 \bar{u} + q_2^2 u)} \frac{k_{1i} k_{2j}}{\bar{u} k_1^2 (k_1^2 \bar{u} + k_2^2 u)} e^{i(q_1 + q_2, x)_\perp - i(k_1 + k_2, y)_\perp - i(q_1 - k_1, z)_\perp - i(q_2 - k_2, z')_\perp}
\end{aligned}$$

where we have used the formula

$$\begin{aligned}
& \left( k_1 + \frac{2(k_1, q_1)_\perp}{\alpha u s} p_2 \right)_\mu \left( k_2 + \frac{2(k_2, q_2)_\perp}{\alpha \bar{u} s} p_2 \right)_\nu (q_1 + q_2)_\perp \Gamma^{\mu\nu\lambda}(\alpha u p_1 + q_{1\perp}, \alpha \bar{u} p_1 + q_{2\perp}, -\alpha p_1 - (q_1 + q_2)_\perp) \\
&= k_{1i} k_{2j} \left[ (q_1^2 - q_2^2) \delta_{ij} - \frac{2}{u} q_{1i} (q_1 + q_2)_j + \frac{2}{\bar{u}} (q_1 + q_2)_i q_{2j} \right] \quad (28)
\end{aligned}$$

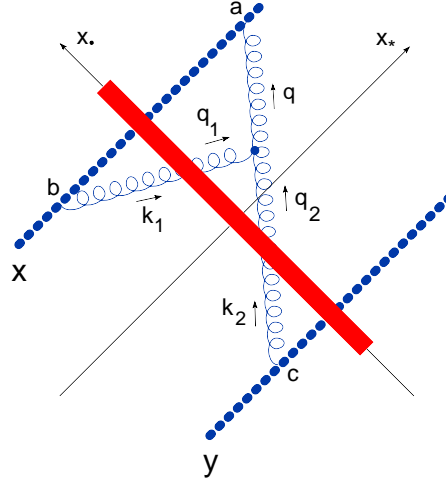


FIG. 5: “Cut vertex” diagram.

following from Eq. (22).

The contribution of the diagram shown in Fig. 4b differs by the substitution  $e^{-i(k_1+k_2,y)_\perp} \rightarrow e^{-i(k_1+k_2,x)_\perp}$  and changing the order of  $t^b, t^c$  matrices. (Similarly to the case of the Fig. (3)b diagram, this prescription follows from the replacement  $[-\infty p_1, 0]_y$  by  $[0, \infty p_1]_x$  but now we consider the second term of the expansion in the gauge field). We get

$$\begin{aligned} & \langle \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle_{\text{Fig.4a+b}} \quad (29) \\ &= -i \frac{g^4}{2\pi^2} \int_0^\sigma \frac{d\alpha}{\alpha} \int_0^1 du \bar{u}u \int \bar{d}^2 k_1 \bar{d}^2 k_2 \bar{d}^2 q_1 \bar{d}^2 q_2 \int d^2 z d^2 z' U_z^{mb} U_{z'}^{nc} \frac{(q_1^2 - q_2^2) \delta_{ij} - \frac{2}{u} q_{1i} (q_1 + q_2)_j + \frac{2}{\bar{u}} (q_1 + q_2)_i q_{2j}}{(q_1 + q_2)^2 (q_1^2 \bar{u} + q_2^2 u)} \\ & \times \frac{k_{1i} k_{2j}}{\bar{u} k_1^2 (k_1^2 \bar{u} + k_2^2 u)} e^{i(q_1+q_2,x)_\perp - i(q_1-k_1,z)_\perp - i(q_2-k_2,z')_\perp} f^{mna} \text{Tr}\{t^a U_x [t^c t^b e^{-i(k_1+k_2,x)} + t^b t^c e^{-i(k_1+k_2,y)}] U_y^\dagger\} \end{aligned}$$

There is another type of “cut vertex” diagram shown in Fig. 5 (AGREES)

$$\begin{aligned} & g^3 \int_0^\infty dt \int_{-\infty}^0 du \int_{-\infty}^0 dv \langle \hat{A}_\bullet^a(tp_1 + x_\perp) \hat{A}_\bullet^b(up_1 + x_\perp) \hat{A}_\bullet^c(vp_1 + y_\perp) \rangle \quad (30) \\ &= 2g^4 s \int \bar{d}^2 \alpha_1 \bar{d}^2 \alpha_2 \bar{d}^2 \beta_1 \bar{d}^2 \beta_2 \int d^2 z d^2 z' U_z^{mb} U_{z'}^{nc} \int \frac{\bar{d}^2 k_1 \bar{d}^2 k_2}{k_1^2 k_2^2} \int \bar{d}^2 q_1 \bar{d}^2 q_2 \theta(\alpha_1) \theta(\alpha_2) \\ & \times f^{mna} \Gamma^{\mu\nu\lambda}(\alpha_1 p_1 + q_1, \alpha_2 p_1 + q_2, -(\alpha_1 + \alpha_2) p_1 - (q_1 + q_2)_\perp) e^{i(q_1, x-z)_\perp + i(q_2, x-z')_\perp - i(k_1, x-z)_\perp - i(k_2, y-z')_\perp} \\ & \times \left. \frac{[(q_1 + q_2)_\perp + 2(\beta_1 + \beta_2) p_2]_\lambda}{(\alpha_1 + \alpha_2)(\beta_1 + \beta_2 - i\epsilon)[(\alpha_1 + \alpha_2)(\beta_1 + \beta_2)s - (q_1 + q_2)_\perp^2 + i\epsilon]} \frac{k_{1\mu} + \frac{2(k_1, q_1)_\perp}{\alpha_1 s} p_{2\mu}}{\alpha_1 \beta_1 s - q_{1\perp}^2 + i\epsilon} \frac{k_{2\nu} + \frac{2(k_2, q_2)_\perp}{\alpha_2 s} p_{2\nu}}{\alpha_2 \beta_2 s - q_{2\perp}^2 + i\epsilon} \right\} \end{aligned}$$

Taking residues at  $\beta_1 + \beta_2 = 0$  and at  $\beta_1 = \frac{q_1^2}{\alpha_1 s}$  and going to variables  $\alpha = \alpha_1 + \alpha_2$  and  $u = \alpha_1/\alpha$  we get (AGREES)

$$\begin{aligned} & g^3 \int_0^\infty dt \int_{-\infty}^0 du \int_{-\infty}^0 dv \langle \hat{A}_\bullet^a(tp_1 + x_\perp) \hat{A}_\bullet^b(up_1 + x_\perp) \hat{A}_\bullet^c(vp_1 + y_\perp) \rangle \quad (31) \\ &= \frac{g^4}{2\pi^2} f^{amn} \int_0^\sigma \frac{d\alpha}{\alpha^2} \int_0^1 du \int \bar{d}^2 q_1 \bar{d}^2 q_2 \bar{d}^2 k_1 \bar{d}^2 k_2 U_z^{mb} U_{z'}^{nc} \frac{e^{i(q_1, x-z)_\perp + i(q_2, x-z')_\perp - i(k_1, x-z)_\perp - i(k_2, y-z')_\perp}}{k_1^2 k_2^2 (q_1 + q_2)^2 (\bar{u} q_1^2 + u q_2^2)} \\ & \left( k_1 + \frac{2(q_1, k_1)_\perp p_2}{s \alpha u} \right)^\mu \left( k_2 + \frac{2(q_2, k_2)_\perp p_2}{s \alpha \bar{u}} \right)^\nu (q_1 + q_2)^\lambda \Gamma_{\mu\nu\lambda}(\alpha u p_1 + q_1, \alpha \bar{u} p_1 + q_2, -\alpha p_1 - (q_1 + q_2)_\perp) \\ &= \frac{g^4}{2\pi^2} f^{amn} \int_0^\sigma \frac{d\alpha}{\alpha^2} \int_0^1 du \int \bar{d}^2 q_1 \bar{d}^2 q_2 \bar{d}^2 k_1 \bar{d}^2 k_2 U_z^{mb} U_{z'}^{nc} \frac{e^{i(q_1, x-z)_\perp + i(q_2, x-z')_\perp - i(k_1, x-z)_\perp - i(k_2, y-z')_\perp}}{k_1^2 k_2^2 (q_1 + q_2)^2 (\bar{u} q_1^2 + u q_2^2)} \\ & \times \left[ (q_{1\perp}^2 - q_{2\perp}^2) \delta^{ij} - \frac{2}{u} q_1^i (q_1 + q_2)^j + \frac{2}{\bar{u}} (q_1 + q_2)^i q_2^j \right] k_{1i} k_{2j} \end{aligned}$$



and therefore (AGREES)

$$\begin{aligned}
& \langle \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle_{\text{Fig.5}} \\
&= i \frac{g^4}{2\pi^2} f^{amn} \int_0^\sigma \frac{d\alpha}{\alpha} \int_0^1 du \int d^2 z d^2 z' \int \bar{d}^2 q_1 \bar{d}^2 q_2 \int \bar{d}^2 k_1 \bar{d}^2 k_2 e^{-i(q_1-k_1, z)_\perp - i(q_2-k_2, z')_\perp} U_z^{mm'} U_{z'}^{nn'} \\
&\times \frac{e^{i(q_1+q_2, x)_\perp - i(k_1, x)_\perp - i(k_2, y)_\perp}}{(q_1+q_2)^2 (q_1^2 \bar{u} + q_2^2 u)} \left[ (q_1^2 - q_2^2) \delta_{ij} - \frac{2}{u} q_{1i} (q_1+q_2)_j + \frac{2}{\bar{u}} (q_1+q_2)_i q_{2j} \right] \frac{k_{1i} k_{2j}}{k_1^2 k_2^2} \text{Tr}\{t^a U_x t^{m'} t^{n'} U_y^\dagger\}
\end{aligned} \tag{32}$$

where again we have used formula (28).

The sum of the contributions (24), (29) and (32) can be represented as follows

$$\begin{aligned}
& \langle \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle_{\text{Fig.3+Fig.4+Fig.5}} \equiv \langle \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle_{\text{Fig.6 I+III+V+VII+IX}} \\
&= \frac{g^2}{8\pi^2} \int_0^\infty \frac{d\alpha}{\alpha} \int_0^1 du \bar{u} u \int d^2 z d^2 z' \int \bar{d}^2 q_1 \bar{d}^2 q_2 \int \bar{d}^2 k_1 \bar{d}^2 k_2 e^{-i(q_1-k_1, z)_\perp - i(q_2-k_2, z')_\perp} U_z^{mm'} U_{z'}^{nn'} \\
&\times f^{amn} \text{Tr}\left\{ t^a \frac{e^{i(q_1+q_2, x)_\perp}}{(q_1+q_2)^2 (q_1^2 \bar{u} + q_2^2 u)} \left[ (q_1^2 - q_2^2) \delta_{ij} - \frac{2}{u} q_{1i} (q_1+q_2)_j + \frac{2}{\bar{u}} (q_1+q_2)_i q_{2j} \right] U_x \right. \\
&\times \left[ t^b f^{bm'n'} \frac{(e^{-i(k_1+k_2, x)_\perp} - e^{-i(k_1+k_2, y)_\perp})}{(k_1+k_2)^2 (k_1^2 \bar{u} + k_2^2 u)} \left[ (k_1^2 - k_2^2) \delta_{ij} - \frac{2}{u} k_{1i} (k_1+k_2)_j + \frac{2}{\bar{u}} (k_1+k_2)_i k_{2j} \right] \right. \\
&\left. \left. - 4i k_{1i} k_{2j} \left( \frac{t^{n'} t^{m'} e^{-i(k_1+k_2, x)_\perp}}{\bar{u} k_1^2 (k_1^2 \bar{u} + k_2^2 u)} - \frac{e^{-i(k_1, x)_\perp - i(k_2, y)_\perp}}{\bar{u} u k_1^2 k_2^2} t^{m'} t^{n'} + \frac{t^{m'} t^{n'} e^{-i(k_1+k_2, y)_\perp}}{\bar{u} k_1^2 (k_1^2 \bar{u} + k_2^2 u)} \right) \right] U_y^\dagger \right\}
\end{aligned} \tag{33}$$

If we add contribution of the diagrams with the gluon on the right side of the shock wave attached to the Wilson line at the point  $y$  instead of  $x$  (which differs by the substitution  $e^{i(q_1+q_2, x)_\perp} \rightarrow -e^{i(q_1+q_2, y)_\perp}$ ) we obtain

$$\begin{aligned}
& \langle \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle_{\text{Fig.6 I+II+...+X}} \\
&= \frac{g^2}{8\pi^2} \int_0^\infty \frac{d\alpha}{\alpha} \int_0^1 du \bar{u} u \int d^2 z d^2 z' \int \bar{d}^2 q_1 \bar{d}^2 q_2 \int \bar{d}^2 k_1 \bar{d}^2 k_2 e^{-i(q_1-k_1, z)_\perp - i(q_2-k_2, z')_\perp} U_z^{mm'} U_{z'}^{nn'} \\
&\times f^{amn} \text{Tr}\left\{ t^a \frac{(e^{i(q_1+q_2, x)_\perp} - e^{i(q_1+q_2, y)_\perp})}{(q_1+q_2)^2 (q_1^2 \bar{u} + q_2^2 u)} \left[ (q_1^2 - q_2^2) \delta_{ij} - \frac{2}{u} q_{1i} (q_1+q_2)_j + \frac{2}{\bar{u}} (q_1+q_2)_i q_{2j} \right] U_x \right. \\
&\times \left[ t^b f^{bm'n'} \frac{(e^{-i(k_1+k_2, x)_\perp} - e^{-i(k_1+k_2, y)_\perp})}{(k_1+k_2)^2 (k_1^2 \bar{u} + k_2^2 u)} \left[ (k_1^2 - k_2^2) \delta_{ij} - \frac{2}{u} k_{1i} (k_1+k_2)_j + \frac{2}{\bar{u}} (k_1+k_2)_i k_{2j} \right] \right. \\
&\left. \left. - 4i k_{1i} k_{2j} \left( \frac{t^{n'} t^{m'} e^{-i(k_1+k_2, x)_\perp}}{\bar{u} k_1^2 (k_1^2 \bar{u} + k_2^2 u)} - \frac{e^{-i(k_1, x)_\perp - i(k_2, y)_\perp}}{\bar{u} u k_1^2 k_2^2} t^{m'} t^{n'} + \frac{t^{m'} t^{n'} e^{-i(k_1+k_2, y)_\perp}}{\bar{u} k_1^2 (k_1^2 \bar{u} + k_2^2 u)} \right) \right] U_y^\dagger \right\}
\end{aligned} \tag{34}$$

The result (34) can be obtained from the self-energy contribution (24) by the replacement of the term corresponding to the emission of the two gluons via the three-gluon vertex

$$t^b f^{bm'n'} \frac{(e^{-i(k_1+k_2, x)_\perp} - e^{-i(k_1+k_2, y)_\perp})}{(k_1+k_2)^2 (k_1^2 \bar{u} + k_2^2 u)} \left[ (k_1^2 - k_2^2) \delta_{ij} - \frac{2}{u} k_{1i} (k_1+k_2)_j + \frac{2}{\bar{u}} (k_1+k_2)_i k_{2j} \right]$$

with similar contribution containing the ‘‘effective vertex’’

$$\begin{aligned}
& S^{m'n'}(k_1, k_2; x, y) \equiv \\
& t^b f^{bm'n'} \frac{(e^{-i(k_1+k_2, x)_\perp} - e^{-i(k_1+k_2, y)_\perp})}{(k_1+k_2)^2 (k_1^2 \bar{u} + k_2^2 u)} \left[ (k_1^2 - k_2^2) \delta_{ij} - \frac{2}{u} k_{1i} (k_1+k_2)_j + \frac{2}{\bar{u}} (k_1+k_2)_i k_{2j} \right] \\
& - 4i k_{1i} k_{2j} \left( \frac{t^{n'} t^{m'} e^{-i(k_1+k_2, x)_\perp}}{\bar{u} k_1^2 (k_1^2 \bar{u} + k_2^2 u)} - \frac{e^{-i(k_1, x)_\perp - i(k_2, y)_\perp}}{\bar{u} u k_1^2 k_2^2} t^{m'} t^{n'} + \frac{t^{m'} t^{n'} e^{-i(k_1+k_2, y)_\perp}}{\bar{u} k_1^2 (k_1^2 \bar{u} + k_2^2 u)} \right)
\end{aligned} \tag{35}$$

It can be demonstrated that the sum of the contributions of the diagrams shown in Fig. 6 I, ..., IV, XI, ..., XVI can be obtained from the self-energy contribution (24) by replacing the gluon vertex

$$t^a f^{amn} \frac{(e^{i(q_1+q_2, x)_\perp} - e^{i(q_1+q_2, y)_\perp})}{(q_1+q_2)^2 (q_1^2 \bar{u} + q_2^2 u)} \left[ (q_1^2 - q_2^2) \delta_{ij} - \frac{2}{u} q_{1i} (q_1+q_2)_j + \frac{2}{\bar{u}} (q_1+q_2)_i q_{2j} \right] \tag{36}$$

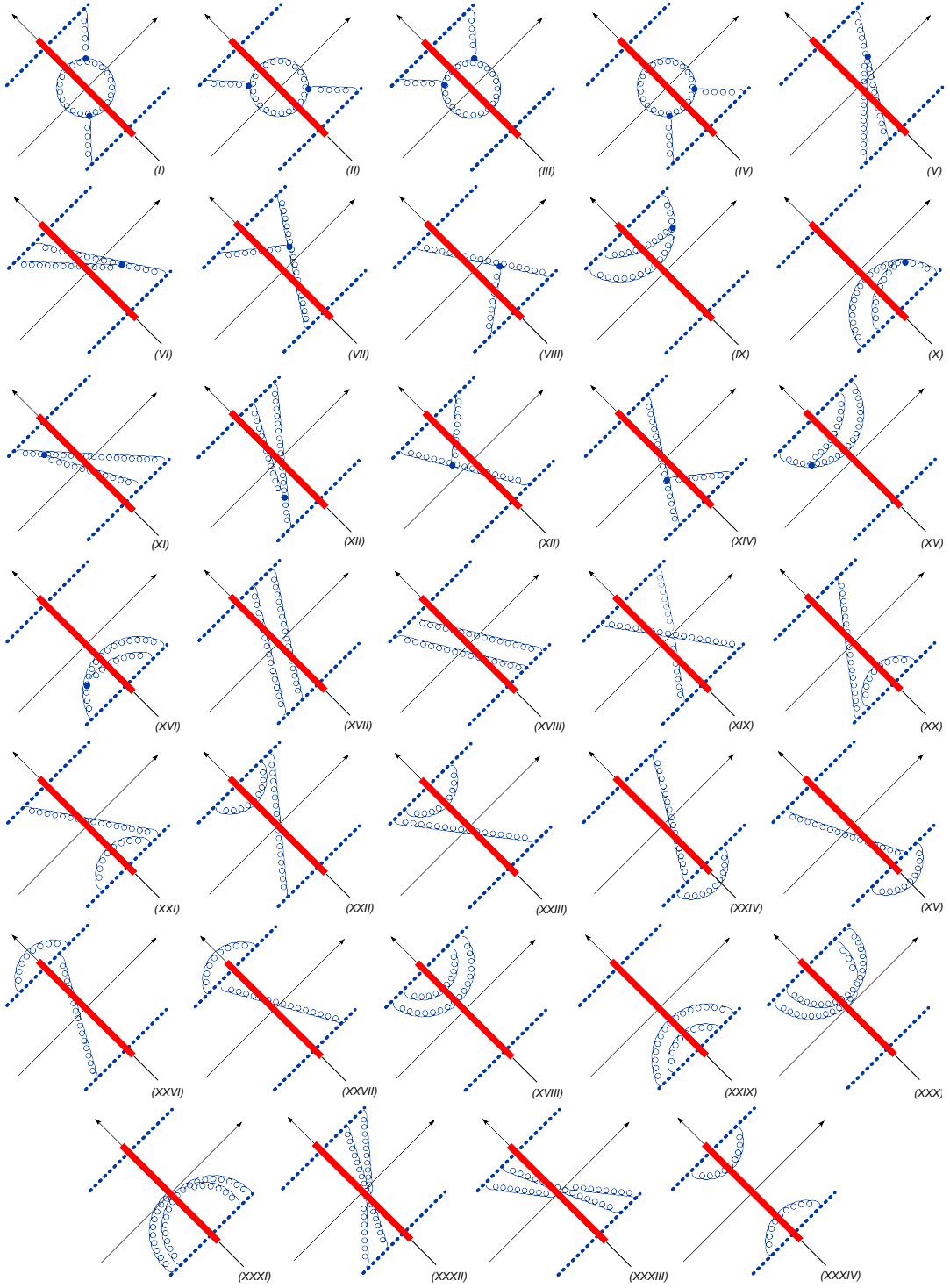


FIG. 6: Diagrams with two cuts.

with similar “effective vertex”

$$\begin{aligned}
& t^a f^{amn} \frac{(e^{i(q_1+q_2, x)_\perp} - e^{i(q_1+q_2, y)_\perp})}{(q_1 + q_2)^2 (q_1^2 \bar{u} + q_2^2 u)} \left[ (q_1^2 - q_2^2) \delta_{ij} - \frac{2}{u} q_{1i} (q_1 + q_2)_j + \frac{2}{\bar{u}} (q_1 + q_2)_i q_{2j} \right] \\
& + 4i q_{1i} q_{2j} \left( \frac{t^m t^n e^{i(q_1+q_2, x)_\perp}}{\bar{u} q_1^2 (q_1^2 \bar{u} + q_2^2 u)} - \frac{e^{i(q_1, x)_\perp + i(q_2, y)_\perp}}{\bar{u} u q_1^2 q_2^2} t^n t^m + \frac{t^n t^m e^{i(q_1+q_2, y)_\perp}}{\bar{u} q_1^2 (q_1^2 \bar{u} + q_2^2 u)} \right)
\end{aligned} \tag{37}$$

Note that (37) is equal to  $S^{\dagger mn}(q_1, q_2; x, y)$ . Let us consider now the box diagrams topology shown in fig. 6 XVII-XXXIV. The calculation of these diagrams is similar to the above calculation of “cut self-energy” and “cut vertex” diagrams so we present here only the final result

$$\begin{aligned}
& \langle \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle_{\text{Fig.6 XVII}+\dots+\text{XXXIV}} \tag{38} \\
&= \frac{g^2}{2\pi^2} \int_0^\sigma \frac{d\alpha}{\alpha} \int_0^1 du \bar{u}u \int d^2 z d^2 z' \int \bar{d}^2 q_1 \bar{d}^2 q_2 \int \bar{d}^2 k_1 \bar{d}^2 k_2 e^{-i(q_1-k_1, z)_\perp - i(q_2-k_2, z')_\perp} U_z^{mm'} U_{z'}^{nn'} \\
&\times \text{Tr}\left\{ \left[ q_{1i} q_{2j} \left( \frac{t^n t^n e^{i(q_1+q_2, x)_\perp}}{\bar{u}q_1^2(q_1^2 \bar{u} + q_2^2 u)} + \frac{t^n t^n e^{i(q_1+q_2, x)_\perp}}{uq_2^2(q_1^2 \bar{u} + q_2^2 u)} - \frac{e^{i(q_1, x)_\perp + i(q_2, y)_\perp}}{\bar{u}uq_1^2 q_2^2} t^n t^m - \frac{e^{i(q_1, y)_\perp + i(q_2, x)_\perp}}{\bar{u}uq_1^2 q_2^2} t^m t^n \right. \right. \right. \\
&\quad \left. \left. \left. + \frac{t^n t^m e^{i(q_1+q_2, y)_\perp}}{\bar{u}q_1^2(q_1^2 \bar{u} + q_2^2 u)} + \frac{t^m t^n e^{i(q_1+q_2, y)_\perp}}{uq_2^2(q_1^2 \bar{u} + q_2^2 u)} \right) \right] U_x \right. \\
&\times \left[ k_{1i} k_{2j} \left( \frac{t^{n'} t^{m'} e^{-i(k_1+k_2, x)_\perp}}{\bar{u}k_1^2(k_1^2 \bar{u} + k_2^2 u)} + \frac{t^{m'} t^{n'} e^{-i(k_1+k_2, x)_\perp}}{uk_2^2(k_1^2 \bar{u} + k_2^2 u)} - \frac{e^{-i(k_1, x)_\perp - i(k_2, y)_\perp}}{\bar{u}uk_1^2 k_2^2} t^{m'} t^{n'} - \frac{e^{-i(k_2, x)_\perp - i(k_1, y)_\perp}}{\bar{u}uk_1^2 k_2^2} t^{n'} t^{m'} \right. \right. \\
&\quad \left. \left. \left. + \frac{t^{m'} t^{n'} e^{-i(k_1+k_2, y)_\perp}}{\bar{u}k_1^2(k_1^2 \bar{u} + k_2^2 u)} + \frac{t^{n'} t^{m'} e^{-i(k_1+k_2, y)_\perp}}{uk_2^2(k_1^2 \bar{u} + k_2^2 u)} \right) \right] U_y^\dagger \left. \right\}
\end{aligned}$$

This expression agrees with the sum of “box topology” diagrams in Ref. [14].

Now we observe that each three-gluon vertex diagram is equal to its own cross diagram (the same cannot be said for box diagrams). Thus we may redefine the “effective vertex” (35) in the following way

$$\begin{aligned}
& \mathcal{S}^{m'n'}(k_1, k_2; x, y) = \\
& t^b f^{bm'n'} \frac{(e^{-i(k_1+k_2, x)_\perp} - e^{-i(k_1+k_2, y)_\perp})}{(k_1+k_2)^2(k_1^2 \bar{u} + k_2^2 u)} \left[ (k_1^2 - k_2^2) \delta_{ij} - \frac{2}{u} k_{1i}(k_1+k_2)_j + \frac{2}{\bar{u}}(k_1+k_2)_i k_{2j} \right] \\
& - i2k_{1i} k_{2j} \left( \frac{t^n t^{m'} e^{-i(k_1+k_2, x)_\perp}}{\bar{u}k_1^2(k_1^2 \bar{u} + k_2^2 u)} + \frac{t^{m'} t^{n'} e^{-i(k_1+k_2, x)_\perp}}{uk_2^2(k_1^2 \bar{u} + k_2^2 u)} - \frac{e^{-i(k_1, x)_\perp - i(k_2, y)_\perp}}{\bar{u}uk_1^2 k_2^2} t^{m'} t^{n'} - \frac{e^{-i(k_2, x)_\perp - i(k_1, y)_\perp}}{\bar{u}uk_1^2 k_2^2} t^{n'} t^{m'} \right. \\
&\quad \left. + \frac{t^{m'} t^{n'} e^{-i(k_1+k_2, y)_\perp}}{\bar{u}k_1^2(k_1^2 \bar{u} + k_2^2 u)} + \frac{t^{n'} t^{m'} e^{-i(k_1+k_2, y)_\perp}}{uk_2^2(k_1^2 \bar{u} + k_2^2 u)} \right) \tag{39}
\end{aligned}$$

which corresponds to writing each contribution of the three-gluon vertex diagrams as a sum of two equal terms.

A similar expression can be written for the “effective vertex” (37) and therefore the sum of all diagrams with two gluon-shockwave intersections can be written as

$$\begin{aligned}
& \langle \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle_{\text{Fig.6}} = \frac{g^2}{8\pi^2} \int_0^\sigma \frac{d\alpha}{\alpha} \int_0^1 du \bar{u}u \int d^2 z d^2 z' \int \bar{d}^2 q_1 \bar{d}^2 q_2 \int \bar{d}^2 k_1 \bar{d}^2 k_2 e^{-i(q_1-k_1, z)_\perp - i(q_2-k_2, z')_\perp} \\
& \times U_z^{mm'} U_{z'}^{nn'} \text{Tr}\left\{ [S^{\dagger mn}(q_1, q_2; x, y) U_x] [\mathcal{S}^{m'n'}(k_1, k_2; x, y) U_y^\dagger] \right\} \tag{40}
\end{aligned}$$

Separating the contributions of different color structures one obtains

$$\begin{aligned}
& \langle \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle_{\text{Fig.6}} \tag{41} \\
&= \frac{g^2}{8\pi^2} \int_0^\sigma \frac{d\alpha}{\alpha} \int_0^1 \frac{du}{\bar{u}u} \int d^2 z d^2 z' \int \bar{d}^2 q_1 \bar{d}^2 q_2 \int \bar{d}^2 k_1 \bar{d}^2 k_2 U_z^{mm'} U_{z'}^{nn'} \\
&\times \text{Tr}\left\{ t^a f^{amn} \frac{(e^{i(q_1, X)+i(q_2, X')} - x \leftrightarrow y)}{(q_1^2 \bar{u} + q_2^2 u)} \left[ \frac{(q_1^2 - q_2^2) \bar{u}u \delta_{ij} - 2\bar{u}q_{1i}(q_1+q_2)_j + 2u(q_1+q_2)_i q_{2j}}{(q_1+q_2)^2} - u \frac{q_{1i} q_{2j}}{q_1^2} + \bar{u} \frac{q_{1i} q_{2j}}{q_2^2} \right] \right. \\
&\quad \left. - t^a f^{amn} \frac{q_{1i} q_{2j}}{q_1^2 q_2^2} (e^{i(q_1, X)+i(q_2, Y')} - x \leftrightarrow y) + i\{t^m, t^n\} \frac{q_{1i} q_{2j}}{q_1^2 q_2^2} (e^{i(q_1, X)} - e^{i(q_1, Y)})(e^{i(q_2, X')} - e^{i(q_2, Y')}) \right\} U_x \\
&\times \left\{ t^b f^{bm'n'} \frac{(e^{-i(k_1, X)-i(k_2, X')} - x \leftrightarrow y)}{(k_1^2 \bar{u} + k_2^2 u)} \left[ \frac{(k_1^2 - k_2^2) \bar{u}u \delta_{ij} - 2\bar{u}k_{1i}(k_1+k_2)_j + 2u(k_1+k_2)_i k_{2j}}{(k_1+k_2)^2} - u \frac{k_{1i} k_{2j}}{k_1^2} + \bar{u} \frac{k_{1i} k_{2j}}{k_2^2} + \right] \right. \\
&\quad \left. - t^b f^{bm'n'} \frac{k_{1i} k_{2j}}{k_1^2 k_2^2} (e^{-i(k_1, X)-i(k_2, Y')} - x \leftrightarrow y) - i\{t^{m'}, t^{n'}\} \frac{k_{1i} k_{2j}}{k_1^2 k_2^2} (e^{-i(k_1, X)} - e^{-i(k_1, Y)})(e^{-i(k_2, X')} - e^{-i(k_2, Y')}) \right\} U_y
\end{aligned}$$

This result agrees with Ref. 14.

Performing the Fourier transformation (AGREES)

$$\begin{aligned}
& \int \bar{d}^2 q_1 \bar{d}^2 q_2 e^{i(q_1, x_1) + i(q_2, x_2)} \frac{q_{1i} q_{2j}}{q_1^2 (q_1^2 \bar{u} + q_2^2 u)} = - \frac{x_{1i} x_{2j}}{4\pi^2 x_2^2 (u x_1^2 + \bar{u} x_2^2)} \\
& \int \bar{d}^2 q_1 \bar{d}^2 q_2 e^{i(q_1, x_1) + i(q_2, x_2)} \frac{\delta_{ij} (q_1^2 - q_2^2) - \frac{2}{u} q_{1i} (q_1 + q_2)_j + \frac{2}{\bar{u}} (q_1 + q_2)_i q_{2j}}{(q_1 + q_2)^2 (q_1^2 \bar{u} + q_2^2 u)} \\
& = \frac{-(x_1^2 - x_2^2) \delta_{ij} + \frac{2}{u} (x_1 - x_2)_i x_{2j} + \frac{2}{\bar{u}} x_{1i} (x_1 - x_2)_j}{4\pi^2 (x_1 - x_2)^2 (u x_1^2 + \bar{u} x_2^2)} \tag{42}
\end{aligned}$$

we get

$$\begin{aligned}
& \frac{d}{d \ln \sigma} \langle \text{Tr} \{ \hat{U}_x \hat{U}_y^\dagger \} \rangle_{\text{Fig.6}} = \frac{\alpha_s^2}{8\pi^4} \int_0^1 du u \bar{u} \int d^2 z d^2 z' U_z^{bb'} U_{z'}^{cc'} \\
& \times \text{Tr} \left\{ t^a f^{abc} \left[ \frac{\frac{X_{ij}}{(z-z')^2} + \frac{X_i X'_j}{u X^2} - \frac{X_i X'_j}{\bar{u} X'^2}}{u X^2 + \bar{u} X'^2} - \frac{X_i Y'_j}{\bar{u} u X^2 Y'^2} - (x \leftrightarrow y) \right] + i \frac{\{t^b, t^c\}}{\bar{u} u} \left( \frac{X_i}{X^2} - \frac{Y_i}{Y^2} \right) \left( \frac{X'_j}{X'^2} - \frac{Y'_j}{Y'^2} \right) \right\} U_x \\
& \times \left\{ t^{a'} f^{a'b'c'} \left[ \frac{\frac{X_{ij}}{(z-z')^2} + \frac{X_i X'_j}{u X^2} - \frac{X_i X'_j}{\bar{u} X'^2}}{u X^2 + \bar{u} X'^2} - \frac{X_i Y'_j}{\bar{u} u X^2 Y'^2} - (x \leftrightarrow y) \right] - i \frac{\{t^{b'}, t^{c'}\}}{\bar{u} u} \left( \frac{X_i}{X^2} - \frac{Y_i}{Y^2} \right) \left( \frac{X'_j}{X'^2} - \frac{Y'_j}{Y'^2} \right) \right\} U_y^\dagger \tag{43}
\end{aligned}$$

where we introduced the notations

$$\begin{aligned}
X_{ij} & \equiv (X^2 - X'^2) \delta_{ij} + \frac{2}{u} (z - z')_i X'_j + \frac{2}{\bar{u}} X_i (z - z')_j \\
Y_{ij} & \equiv (Y^2 - Y'^2) \delta_{ij} + \frac{2}{u} (z - z')_i Y'_j + \frac{2}{\bar{u}} Y_i (z - z')_j \tag{44}
\end{aligned}$$

### C. Subtraction of the $(LO)^2$ contribution

It is easy to see that our result for the sum of diagrams in Fig. 6 (43) diverges as  $u \rightarrow 0$  and  $u \rightarrow 1$ . If we put a lower cutoff  $\alpha > \sigma'$  on the  $\alpha$  integrals we would get a contribution  $\sim \ln^2 \frac{\sigma}{\sigma'}$  coming from the region  $\alpha_2 \gg \alpha_1 > \sigma'$  (or  $\alpha_1 \gg \alpha_2 > \sigma'$ ) which corresponds to the the square of the leading-order BK kernel rather than to the NLO kernel. To get the NLO kernel we need to subtract this  $(LO)^2$  contribution. Indeed, the operator form of the evolution equation for the color dipole up to the next-to-leading order looks like

$$\frac{\partial}{\partial \eta} \text{Tr} \{ \hat{U}_x \hat{U}_y^\dagger \} = K_{\text{LO}} \text{Tr} \{ \hat{U}_x \hat{U}_y^\dagger \} + K_{\text{NLO}} \text{Tr} \{ \hat{U}_x \hat{U}_y^\dagger \} \tag{45}$$

where  $\eta = \ln \sigma$ . Our goal is to find  $K_{\text{NLO}}$  by considering the l.h.s. of this equation in the external shock-wave background so

$$\langle K_{\text{NLO}} \text{Tr} \{ \hat{U}_x \hat{U}_y^\dagger \} \rangle_{\text{shockwave}} = \frac{\partial}{\partial \eta} \langle \text{Tr} \{ \hat{U}_x \hat{U}_y^\dagger \} \rangle_{\text{shockwave}} - \langle K_{\text{LO}} \text{Tr} \{ \hat{U}_x \hat{U}_y^\dagger \} \rangle_{\text{shockwave}} \tag{46}$$

The subtraction (46) leads to the  $\left[ \frac{1}{u} \right]_+$  prescription for the terms divergent as  $\frac{1}{u}$  (and similarly  $\frac{1}{\bar{u}} \rightarrow \left[ \frac{1}{\bar{u}} \right]_+$ ) for the contribution divergent as  $u \rightarrow 1$ ). Here we define  $\left[ \frac{1}{u} \right]_+$  in the usual way

$$\int_0^1 du f(u) \left[ \frac{1}{u} \right]_+ \equiv \int_0^1 du \frac{f(u) - f(0)}{u}, \quad \int_0^1 du f(u) \left[ \frac{1}{\bar{u}} \right]_+ \equiv \int_0^1 du \frac{f(u) - f(1)}{\bar{u}} \tag{47}$$

To illustrate this prescription, consider the divergent terms in Eq. (40) proportional to  $(X, Y)(Y', z - z')$  or

$(X', Y')(Y, z - z')$

$$\begin{aligned}
\frac{d}{d\eta} \langle \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle &= \frac{\alpha_s^2}{4\pi^4} \int_0^1 \frac{du}{\bar{u}} \int d^2 z d^2 z' U_z^{bb'} U_{z'}^{cc'} \frac{(X, Y)(Y', z - z')}{(z - z')^2 Y'^2} \left[ \frac{1}{uX^2 + \bar{u}X'^2} \right. \\
&\times \left[ f^{abc} \text{Tr}\{t^a U_x \left( \frac{u f^{a'b'c'} t^{a'}}{uY^2 + \bar{u}Y'^2} - \frac{i}{Y^2} \{t^{b'}, t^{c'}\} \right) U_y^\dagger\} + f^{a'b'c'} \text{Tr}\left\{ \left( t^a \frac{u f^{abc}}{uY^2 + \bar{u}Y'^2} + \frac{i}{Y^2} \{t^b, t^c\} \right) U_x t^{a'} U_y^\dagger \right\} \right] \\
&+ \frac{1}{X^2(uY^2 + \bar{u}Y'^2)} \text{Tr}\{(f^{abc} t^a + i\{t^b, t^c\}) U_x f^{a'b'c'} t^{a'} U_y^\dagger + t^a f^{abc} U_x (f^{a'b'c'} t^{a'} - i\{t^b, t^c\}) U_y^\dagger\} \\
&+ \frac{\alpha_s^2}{4\pi^4} \int_0^1 \frac{du}{u} \int d^2 z d^2 z' U_z^{bb'} U_{z'}^{cc'} \frac{(X', Y')(Y, z - z')}{(z - z')^2 Y^2} \left[ \frac{1}{uX^2 + \bar{u}X'^2} \right. \\
&\times \left[ f^{abc} \text{Tr}\{-t^a U_x \left( \frac{\bar{u} f^{a'b'c'} t^{a'}}{(uY^2 + \bar{u}Y'^2)} - \frac{i}{Y'^2} \{t^{b'}, t^{c'}\} \right) U_y^\dagger\} + f^{a'b'c'} \text{Tr}\left\{ \left( -t^a \frac{\bar{u} Y_i Y'_j f^{abc}}{uY^2 + \bar{u}Y'^2} + \frac{i}{Y'^2} \{t^b, t^c\} \right) U_x t^{a'} U_y^\dagger \right\} \right] \\
&+ \frac{1}{X'^2(uY^2 + \bar{u}Y'^2)} \text{Tr}\{(-f^{abc} t^a + i\{t^b, t^c\}) U_x f^{a'b'c'} t^{a'} U_y^\dagger - t^a f^{abc} U_x (f^{a'b'c'} t^{a'} + i\{t^b, t^c\}) U_y^\dagger\} \left. \right] \quad (48)
\end{aligned}$$

Note that the second term is equal to the first one after the replacement  $u \leftrightarrow \bar{u}$ ,  $z \leftrightarrow z'$  and  $b \leftrightarrow c, b' \leftrightarrow c'$ .

It is convenient to return back to the notation  $\alpha_1$  and  $\alpha_2 = \sigma - \alpha_1$  (after  $\frac{d}{d \ln \sigma}$  the value of  $\alpha$  is set equal to  $\sigma$ ).

$$\begin{aligned}
\frac{d}{d\eta} \langle \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle &= \frac{\alpha_s^2}{2\pi^4} \int_0^\sigma \frac{d\alpha_2}{\alpha_2} \int d^2 z d^2 z' U_z^{bb'} U_{z'}^{cc'} \frac{(X, Y)(Y', z - z')}{(z - z')^2 Y'^2} \quad (49) \\
&\times \left[ \frac{\sigma}{\alpha_1 X^2 + \alpha_2 X'^2} \left[ f^{abc} \text{Tr}\{t^a U_x \left( \frac{\alpha_1 f^{a'b'c'} t^{a'}}{\alpha_1 Y^2 + \alpha_2 Y'^2} - \frac{i}{Y^2} \{t^{b'}, t^{c'}\} \right) U_y^\dagger\} + f^{a'b'c'} \text{Tr}\left\{ \left( \frac{\alpha_1 f^{abc} t^a}{\alpha_1 Y^2 + \alpha_2 Y'^2} \right. \right. \right. \\
&\quad \left. \left. \left. + \frac{i}{Y^2} \{t^b, t^c\} \right) U_x t^{a'} U_y^\dagger \right\} \right] + \frac{2i\sigma}{X^2(\alpha_1 Y^2 + \alpha_2 Y'^2)} \text{Tr}\{t^c t^b U_x f^{a'b'c'} t^{a'} U_y^\dagger - t^a f^{abc} U_x t^b t^c U_y^\dagger\} \right]
\end{aligned}$$

The corresponding term in  $K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\}$  is (see Eq. (4))

$$K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} = -\frac{4\alpha_s}{\pi^2} \int d^2 z \frac{(x - z, y - z)}{(x - z)^2 (y - z)^2} \text{Tr}\{t^a \hat{U}_x t^b \hat{U}_y^\dagger\} \text{Tr}\{t^a \hat{U}_z t^b \hat{U}_z^\dagger\} \quad (50)$$

The relevant term in the ‘‘matrix element’’  $\langle K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle$  in the external shock-wave background comes from  $\hat{U}_x, \hat{U}_z$  taken in the leading order in  $\alpha_s$  (so that  $\hat{U}_x \rightarrow U_x, \hat{U}_z \rightarrow U_z$ ) and  $\hat{U}_z \otimes \hat{U}_y^\dagger$  taken in the first order in  $\alpha_s$

$$\langle \{\hat{U}_z \otimes \hat{U}_y^\dagger\} \rangle \sim -\frac{\alpha_s}{\pi^2} \int_0^\sigma \frac{d\alpha_2}{\alpha_2} \int d^2 z' \frac{(z - z', y - z')}{(z - z')^2 (y - z')^2} (t^c U_z \otimes t^c U_y^\dagger + U_z t^c \otimes U_y^\dagger t^c) U_{z'}^{cc'}, \quad (51)$$

or vice versa:  $\hat{U}_x \rightarrow U_x, \hat{U}_{z'} \rightarrow U_{z'}$  and

$$\langle \{\hat{U}_z^\dagger \otimes \hat{U}_y^\dagger\} \rangle \sim \frac{\alpha_s}{\pi^2} \int_0^\sigma \frac{d\alpha_2}{\alpha_2} \int d^2 z' \frac{(z - z', y - z')}{(z - z')^2 (y - z')^2} (t^c U_z^\dagger \otimes U_y^\dagger t^c + U_z^\dagger t^c \otimes t^c U_y^\dagger) U_{z'}^{cc'} \quad (52)$$

Here we have used the leading-order equations for Wilson lines with arbitrary color indices [4, 15]. Substituting eqs. (51) and (52) in Eq. (50) we obtain

$$\langle K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle = \frac{2\alpha_s^2}{\pi^4} \int_0^\sigma \frac{d\alpha_2}{\alpha_2} \int d^2 z \frac{(X, Y)(Y', z - z')}{X^2 Y^2 Y'^2 (z - z')^2} \text{Tr}\{i f^{a'b'c'} t^c t^b U_x t^b U_y^\dagger - i f^{abc} t^a U_x t^b t^c U_y^\dagger\} U_z^{bb'} U_{z'}^{cc'} \quad (53)$$

From Eq. (46) we get

$$\begin{aligned}
\langle K_{\text{NLO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle &= \frac{\alpha_s^2}{2\pi^4} \int_0^\sigma \frac{d\alpha_2}{\alpha_2} \int d^2 z d^2 z' U_z^{bb'} U_{z'}^{cc'} \frac{(X, Y)(Y', z - z')}{(z - z')^2 Y'^2} \left[ \frac{\sigma}{\alpha_1 X^2 + \alpha_2 X'^2} \right. \\
&\times \left[ f^{abc} \text{Tr}\{t^a U_x \left( \frac{\alpha_1 f^{a'b'c'} t^{a'}}{\alpha_1 Y^2 + \alpha_2 Y'^2} - \frac{i}{Y^2} \{t^{b'}, t^{c'}\} \right) U_y^\dagger\} + f^{a'b'c'} \text{Tr}\left\{ \left( \frac{\alpha_1 f^{abc} t^a}{\alpha_1 Y^2 + \alpha_2 Y'^2} + \frac{i}{Y^2} \{t^b, t^c\} \right) U_x t^a U_y^\dagger \right\} \right] \\
&\quad \left. + \frac{2i\sigma}{X^2(\alpha_1 Y^2 + \alpha_2 Y'^2)} \text{Tr}\{t^c t^b U_x f^{a'b'c'} t^a U_y^\dagger - t^a f^{abc} U_x t^b t^c U_y^\dagger\} \right] \\
&- \frac{2\alpha_s^2}{\pi^4} \int_0^\sigma \frac{d\alpha_2}{\alpha_2} \int d^2 z \frac{(X, Y)(Y', z - z')}{X^2 Y^2 Y'^2 (z - z')^2} \text{Tr}\{i f^{a'b'c'} t^c t^b U_x t^b U_y^\dagger - i f^{abc} t^a U_x t^b t^c U_y^\dagger\} U_z^{bb'} U_{z'}^{cc'} \\
&= \frac{\alpha_s^2}{2\pi^4} \int_0^1 \frac{du}{\bar{u}} \int d^2 z d^2 z' U_z^{bb'} U_{z'}^{cc'} \frac{(X, Y)(Y', z - z')}{(z - z')^2 Y'^2} \left[ f^{abc} f^{a'b'c'} \text{Tr}\{t^a U_x t^a U_y^\dagger\} \left[ \frac{2u}{(uX^2 + \bar{u}X'^2)(uY^2 + \bar{u}Y'^2)} - \frac{2}{X^2 Y^2} \right] \right. \\
&- \frac{i}{Y^2} \left[ \frac{1}{uX^2 + \bar{u}X'^2} - \frac{1}{X^2} \right] f^{abc} \text{Tr}\{t^a U_x \{t^{b'}, t^{c'}\} U_y^\dagger\} + \frac{i}{Y^2} \left[ \frac{1}{uX^2 + \bar{u}X'^2} - \frac{1}{X^2} \right] f^{a'b'c'} \text{Tr}\{\{t^b, t^c\} U_x t^a U_y^\dagger\} \\
&\quad \left. + \frac{2i}{X^2} \left[ \frac{1}{uY^2 + \bar{u}Y'^2} - \frac{1}{Y^2} \right] \text{Tr}\{t^c t^b U_x f^{a'b'c'} t^a U_y^\dagger - t^a f^{abc} U_x t^b t^c U_y^\dagger\} \right]
\end{aligned} \tag{54}$$

which corresponds to the  $\left[\frac{1}{\bar{u}}\right]$  prescription (47) (the same prescription was used in Ref. [14]). Note that the ‘‘plus’’ prescription (47) is a consequence of the ‘‘rigid’’ cutoff  $|\alpha| < \sigma$  (17); with the ‘‘smooth’’ cutoff (6) we would get different results - see Appendix B.

#### D. Assembling the result for 1→3 dipoles transition

There are four color structures in the r.h.s. of Eq. (43). Three of them reduce to

$$\begin{aligned}
f^{abc} f^{a'b'c'} U_z^{bb'} U_{z'}^{cc'} \text{Tr}\{t^a U_x t^a U_y^\dagger\} &= \frac{1}{4} \text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_{z'}^\dagger\} \text{Tr}\{U_{z'} U_y^\dagger\} - \frac{1}{4} \text{Tr}\{U_x U_z^\dagger U_{z'} U_y^\dagger U_z U_{z'}^\dagger\} + (z \leftrightarrow z') \\
i f^{abc} U_z^{bb'} U_{z'}^{cc'} \text{Tr}(t^a U_x \{t^{b'}, t^{c'}\} U_y^\dagger) &= -\frac{1}{4} \text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_{z'}^\dagger\} \text{Tr}\{U_{z'} U_y^\dagger\} + \frac{1}{4} \text{Tr}\{U_x U_z^\dagger U_{z'} U_y^\dagger U_z U_{z'}^\dagger\} - (z \leftrightarrow z') \\
i f^{a'b'c'} U_z^{bb'} U_{z'}^{cc'} \text{Tr}(\{t^b, t^c\} U_x t^a U_y^\dagger) &= \frac{1}{4} \text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_{z'}^\dagger\} \text{Tr}\{U_{z'} U_y^\dagger\} + \frac{1}{4} \text{Tr}\{U_x U_z^\dagger U_{z'} U_y^\dagger U_z U_{z'}^\dagger\} - (z \leftrightarrow z') \tag{55}
\end{aligned}$$

We will not need the explicit form of the fourth color structure  $U_z^{aa'} U_{z'}^{bb'} \text{Tr}(\{t^a, t^b\} U_x \{t^{a'}, t^{b'}\} U_y^\dagger)$  since it is multiplied by pure LO<sup>2</sup> integral  $\int \frac{du}{\bar{u}} = \int \frac{du}{u} + \int \frac{du}{\bar{u}}$  and does not contribute to the NLO kernel.

Performing integration over  $u$  using the prescription (47) after some algebra we get

$$\begin{aligned}
&\int_0^1 du u \bar{u} \left[ \frac{1}{uX^2 + \bar{u}X'^2} \left( \frac{X_{ij}}{(z - z')^2} + \frac{X_i X'_j}{uX^2} - \frac{X_i X'_j}{\bar{u}X'^2} \right) - \frac{X_i Y'_j}{\bar{u}u X^2 Y'^2} - (x \leftrightarrow y) \right]^2 \\
&= \frac{1}{(z - z')^4} \left[ -4 + 2 \frac{X^2 Y'^2 + X'^2 Y^2 - 4\Delta^2 (z - z')^2}{X^2 Y'^2 - X'^2 Y^2} \ln \frac{X^2 Y'^2}{X'^2 Y^2} \right] \\
&+ \left( \frac{(x - y)^4}{X^2 Y'^2 - X'^2 Y^2} \left[ \frac{1}{X^2 Y'^2} + \frac{1}{Y^2 X'^2} \right] + \frac{(x - y)^2}{(z - z')^2} \left[ \frac{1}{X^2 Y'^2} - \frac{1}{X'^2 Y^2} \right] \right) \ln \frac{X^2 Y'^2}{X'^2 Y^2} \tag{56}
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 du \left[ \frac{1}{uX^2 + \bar{u}X'^2} \left( \frac{X_{ij}}{(z - z')^2} + \frac{X_i X'_j}{uX^2} - \frac{X_i X'_j}{\bar{u}X'^2} \right) - \frac{X_i Y'_j}{\bar{u}u X^2 Y'^2} - (x \leftrightarrow y) \right] \left( \frac{X_i}{X^2} - \frac{Y_i}{Y^2} \right) \left( \frac{X'_j}{X'^2} - \frac{Y'_j}{Y'^2} \right) \\
&= -\frac{(x - y)^4}{2X^2 Y^2 X'^2 Y'^2} \ln \frac{X^2 Y'^2}{X'^2 Y^2} + \frac{(x - y)^2}{2(z - z')^2} \left[ \frac{1}{X^2 Y'^2} + \frac{1}{X'^2 Y^2} \right] \ln \frac{X^2 Y'^2}{X'^2 Y^2} \tag{57}
\end{aligned}$$

so the two-cut contribution (43) reduces to

$$\begin{aligned}
\frac{d}{d\eta} \text{Tr}\{U_x U_y^\dagger\}_{\text{Fig.6}} &= \frac{\alpha_s^2}{16\pi^4} \int d^2 z d^2 z' \left[ \left\{ -\frac{4}{(z-z')^4} + \left( 2 \frac{X^2 Y'^2 + X'^2 Y^2 - 4(x-y)^2(z-z')^2}{(z-z')^4 [X^2 Y'^2 - X'^2 Y^2]} \right. \right. \right. \\
&+ \left. \frac{(x-y)^4}{X^2 Y'^2 - X'^2 Y^2} \left[ \frac{1}{X^2 Y'^2} + \frac{1}{Y^2 X'^2} \right] + \frac{(x-y)^2}{(z-z')^2} \left[ \frac{1}{X^2 Y'^2} - \frac{1}{X'^2 Y^2} \right] \right\} \ln \frac{X^2 Y'^2}{X'^2 Y^2} \Big\} \\
&\quad \times [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_{z'}^\dagger\} \text{Tr}\{U_{z'} U_y^\dagger\} - \text{Tr}\{U_x U_z^\dagger U_{z'} U_y^\dagger U_z U_{z'}^\dagger\}] \\
&+ \left. \left\{ -\frac{(x-y)^4}{X^2 Y'^2 X'^2 Y^2} + \frac{(x-y)^2}{(z-z')^2} \left( \frac{1}{X^2 Y'^2} + \frac{1}{Y^2 X'^2} \right) \right\} \ln \frac{X^2 Y'^2}{X'^2 Y^2} \text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_{z'}^\dagger\} \text{Tr}\{U_{z'} U_y^\dagger\} \right] \quad (58)
\end{aligned}$$

This result agrees with the 1→3 dipoles kernel calculated in Ref. [14].

### E. Subtraction of the UV part

The integral in the r.h.s. of Eq. (58) diverges as  $z \rightarrow z'$ . It is convenient to separate the divergent term by subtracting and adding the contribution at  $z = z'$ :

$$\begin{aligned}
&\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_{z'}^\dagger\} \text{Tr}\{U_{z'} U_y^\dagger\} - \text{Tr}\{U_x U_z^\dagger U_{z'} U_y^\dagger U_z U_{z'}^\dagger\} \quad (59) \\
&= [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_{z'}^\dagger\} \text{Tr}\{U_{z'} U_y^\dagger\} - \text{Tr}\{U_x U_z^\dagger U_{z'} U_y^\dagger U_z U_{z'}^\dagger\} - (z' \rightarrow z)] + [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_{z'}^\dagger\} - \text{Tr}\{U_x U_{z'}^\dagger\}]
\end{aligned}$$

For the last line in the r.h.s. of Eq. (58) the subtraction is redundant since

$$\int d^2 z' \left\{ -\frac{(x-y)^4}{X^2 Y'^2 X'^2 Y^2} + \frac{(x-y)^2}{(z-z')^2} \left( \frac{1}{X^2 Y'^2} + \frac{1}{Y^2 X'^2} \right) \right\} \ln \frac{X^2 Y'^2}{X'^2 Y^2} = 0 \quad (60)$$

The easiest way to prove this is to set  $y = 0$  and make an inversion  $x \rightarrow 1/\tilde{x}$  so the integral (60) reduces to

$$\int d^2 \tilde{z}' \frac{(\tilde{x} - \tilde{z}, \tilde{x} - \tilde{z}')}{(\tilde{x} - \tilde{z}')^2 (\tilde{z} - \tilde{z}')^2} \ln \frac{(\tilde{x} - \tilde{z})^2}{(\tilde{x} - \tilde{z}')^2} = 0 \quad (61)$$

We obtain

$$\begin{aligned}
\frac{d}{d\eta} \text{Tr}\{U_x U_y^\dagger\}_{\text{Fig.6}} &= \frac{\alpha_s^2}{16\pi^4} \int d^2 z d^2 z' \left[ \left( -\frac{4}{(z-z')^4} + \left\{ 2 \frac{X^2 Y'^2 + X'^2 Y^2 - 4(x-y)^2(z-z')^2}{(z-z')^4 [X^2 Y'^2 - X'^2 Y^2]} \right. \right. \right. \\
&+ \left. \frac{(x-y)^4}{X^2 Y'^2 - X'^2 Y^2} \left[ \frac{1}{X^2 Y'^2} + \frac{1}{Y^2 X'^2} \right] + \frac{(x-y)^2}{(z-z')^2} \left[ \frac{1}{X^2 Y'^2} - \frac{1}{X'^2 Y^2} \right] \right\} \ln \frac{X^2 Y'^2}{X'^2 Y^2} \Big) \\
&\quad \times [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_{z'}^\dagger\} \text{Tr}\{U_{z'} U_y^\dagger\} - \text{Tr}\{U_x U_z^\dagger U_{z'} U_y^\dagger U_z U_{z'}^\dagger\} - (z' \rightarrow z)] \\
&+ \left. \left\{ -\frac{(x-y)^4}{X^2 Y'^2 X'^2 Y^2} + \frac{(x-y)^2}{(z-z')^2} \left( \frac{1}{X^2 Y'^2} + \frac{1}{Y^2 X'^2} \right) \right\} \ln \frac{X^2 Y'^2}{X'^2 Y^2} \text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_{z'}^\dagger\} \text{Tr}\{U_{z'} U_y^\dagger\} \right] \\
&+ \frac{\alpha_s^2}{16\pi^4} \int d^2 z [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - \text{Tr}\{U_x U_y^\dagger\}] \int d^2 z' \left[ -\frac{4}{(z-z')^4} + \left\{ 2 \frac{X^2 Y'^2 + X'^2 Y^2 - 4(x-y)^2(z-z')^2}{(z-z')^4 [X^2 Y'^2 - X'^2 Y^2]} \right. \right. \\
&+ \left. \left. \frac{(x-y)^4}{X^2 Y'^2 - X'^2 Y^2} \left[ \frac{1}{X^2 Y'^2} + \frac{1}{Y^2 X'^2} \right] + \frac{(x-y)^2}{(z-z')^2} \left[ \frac{1}{X^2 Y'^2} - \frac{1}{X'^2 Y^2} \right] \right\} \ln \frac{X^2 Y'^2}{X'^2 Y^2} \right] \quad (62)
\end{aligned}$$

The first term is now finite while the second term contains the UV divergent contribution which reflects the usual UV divergency of the one-loop diagrams. To find the second term we use the dimensional regularization in the transverse space and set  $d_\perp = 2 - \epsilon$ . Because the Fourier transforms (42) are more complicated at  $d_\perp \neq 2$  it is convenient to return back to Eq. (40) and calculate the subtracted term in the momentum representation. The calculation is

performed in Appendix A and here we only quote the final result (164)

$$\begin{aligned}
\frac{d}{d\eta} \langle \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle_{\text{Fig.6}} &= \frac{\alpha_s^2}{16\pi^4} \int d^2 z d^2 z' \left[ \left( -\frac{4}{(z-z')^4} + \left\{ 2 \frac{X^2 Y'^2 + X'^2 Y^2 - 4(x-y)^2(z-z')^2}{(z-z')^4 [X^2 Y'^2 - X'^2 Y^2]} \right. \right. \right. \\
&+ \frac{(x-y)^4}{X^2 Y'^2 - X'^2 Y^2} \left[ \frac{1}{X^2 Y'^2} + \frac{1}{Y^2 X'^2} \right] + \frac{(x-y)^2}{(z-z')^2} \left[ \frac{1}{X^2 Y'^2} - \frac{1}{X'^2 Y^2} \right] \left. \right\} \ln \frac{X^2 Y'^2}{X'^2 Y^2} \\
&\quad \times [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_{z'}^\dagger\} \text{Tr}\{U_{z'} U_y^\dagger\} - \text{Tr}\{U_x U_z^\dagger U_{z'} U_y^\dagger U_z U_{z'}^\dagger\} - (z' \rightarrow z)] \\
&+ \left. \left\{ -\frac{(x-y)^4}{X^2 Y'^2 X'^2 Y^2} + \frac{(x-y)^2}{(z-z')^2} \left( \frac{1}{X^2 Y'^2} + \frac{1}{Y^2 X'^2} \right) \right\} \ln \frac{X^2 Y'^2}{X'^2 Y^2} \text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_{z'}^\dagger\} \text{Tr}\{U_{z'} U_y^\dagger\} \right] \\
&- \frac{\alpha_s^2 N_c}{8\pi^3} \int d^2 z \frac{(x-y)^2}{X^2 Y^2} \left[ \frac{11}{3} \ln \frac{X^2 Y^2}{(x-y)^2} \mu^2 + \frac{67}{9} - \frac{\pi^2}{3} \right] [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - \frac{1}{N_c} \text{Tr}\{U_x U_y^\dagger\}]
\end{aligned} \tag{63}$$

where  $\mu$  is the normalization scale in the  $\overline{MS}$  scheme.

#### IV. DIAGRAMS WITH ONE GLUON-SHOCKWAVE INTERSECTION

##### A. “Running coupling” diagrams

The relevant diagrams are shown in Fig. 7 (plus permutations). Let us start from the sum of diagrams Fig. 7 a and b. It has the form: (AGREES)

$$\begin{aligned}
&\int_0^\infty du \int_{-\infty}^0 dv \langle \hat{A}_\bullet^a(u p_1 + x_\perp) \hat{A}_\bullet^b(v p_1 + y_\perp) \rangle_{\text{Fig.7a+b}} \\
&= g^2 N_c \frac{s}{2} \int \bar{d}^2 k_\perp \bar{d}^2 k'_\perp \frac{\bar{d}^2 q_\perp}{q_\perp^2} \int d^2 z U_z^{ab} e^{i(q, x-z)_\perp - i(k, y-z)_\perp} \int_0^\infty \frac{\bar{d}\alpha}{\alpha} \int \bar{d}\alpha' \int \frac{\bar{d}\beta \bar{d}\beta'}{\beta - i\epsilon} \\
&\times \left[ \frac{(q + \frac{2(k, q)_\perp}{\alpha s} p_2)_\lambda}{(k^2 + i\epsilon)^2} \frac{d_{\mu\mu'}(k - k')}{(k - k')^2 + i\epsilon} \Gamma^{\mu\nu\lambda}((\alpha - \alpha') p_1 + (k - k')_\perp, \alpha' p_1 + k'_\perp, -\alpha p_1 - k_\perp) \frac{d_{\mu\mu'}(k')}{k'^2 + i\epsilon} (k + 2\beta p_2)_\lambda \right. \\
&\times \left. \Gamma^{\mu'\nu'\lambda'}((\alpha - \alpha') p_1 + (k - k')_\perp, \alpha' p_1 + k'_\perp, -\alpha p_1 - k_\perp) - 2 \frac{(k + 2\beta p_2)_\nu (q_\mu + \frac{2(k, q)_\perp}{\alpha s} p_{2\mu})}{(k^2 + i\epsilon)^2} \frac{g^{\mu\nu} d_\xi^\xi(k') - d^{\mu\nu}(k')}{k'^2 + i\epsilon} \right]
\end{aligned} \tag{64}$$

where the first term in the square brackets comes from Fig. 7a and the second from Fig. 7b. We use the principal-value prescription for the  $1/\alpha'$  terms in  $d_{\mu\nu}(k')$  in loop integrals.

To regularize the UV divergence we change the dimension of the transverse space to  $2-\varepsilon$ . After some algebra one



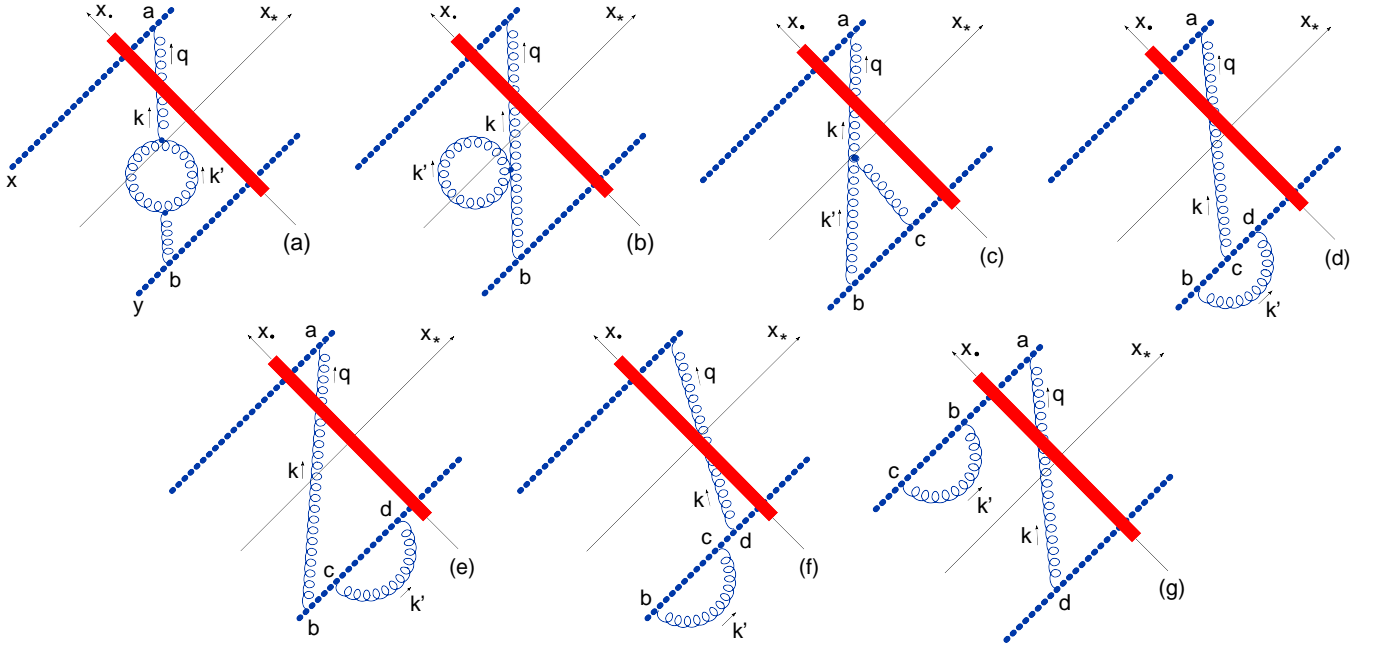


FIG. 7: “Running coupling” diagrams.

obtains (AGREES)

$$\begin{aligned}
& \int_0^\infty du \int_{-\infty}^0 dv \langle \hat{A}_\bullet^a(up_\perp + x_\perp) \hat{A}_\bullet^b(vp_\perp + y_\perp) \rangle_{\text{Fig. 7a+b}} \\
&= g^2 N_c \mu^{2\varepsilon} \int \tilde{d}^{2-\varepsilon} k \tilde{d}^{2-\varepsilon} k' \tilde{d}^{2-\varepsilon} q \int d^{2-\varepsilon} z U_z^{ab} \int_0^\infty \tilde{d}\alpha \int \tilde{d}\beta \tilde{d}\beta' \frac{e^{i(q,X)_\perp - i(k,Y)_\perp}}{\alpha(\beta - i\varepsilon)(\alpha\beta s - k_\perp^2 + i\varepsilon)^2 q^2} \\
&\times \frac{s}{2} \int \tilde{d}\alpha' \tilde{d}\beta' \frac{1}{(\alpha'\beta' s - k_\perp^2 + i\varepsilon)[(\alpha - \alpha')(\beta - \beta')s - (k - k')_\perp^2 + i\varepsilon]} \\
&\times \left\{ -\varepsilon[(\alpha - 2\alpha')\beta s + k_\perp^2 - (k - k')_\perp^2] \left[ \frac{\alpha - 2\alpha'}{\alpha} (k, q)_\perp + (2k' - k, q)_\perp \right] + 2 \frac{(\alpha - 2\alpha')^2}{\alpha} (k, q)_\perp \beta s \right. \\
&+ \frac{(\alpha - 2\alpha')}{\alpha} (k, q)_\perp (2k' - k, k)_\perp + (\alpha - 2\alpha')\beta s (2k' - k, q)_\perp + (q, k)_\perp (k - 2k', k - k')_\perp + (q, k')_\perp (k, 2k' - k)_\perp \\
&+ \frac{\alpha + \alpha'}{\alpha - \alpha'} \left[ \frac{\alpha - 2\alpha'}{\alpha} (q, k)_\perp (k, k - k')_\perp + (q, 2k' - k)_\perp (k, k - k')_\perp + (q, k)_\perp (k_\perp^2 - k_\perp'^2) + (q, k - k')_\perp (k, 2k - k')_\perp \right. \\
&+ (\alpha - 2\alpha')(q, k - k')_\perp \beta s + (q, k - k')_\perp (k_\perp'^2 - (k - k')_\perp^2) - (q, k')_\perp (k', k - k')_\perp + (q, k - k')_\perp (k - k')_\perp^2 \left. \right] \\
&+ \frac{\alpha' - 2\alpha}{\alpha'} \left[ \frac{\alpha - 2\alpha'}{\alpha} (q, k)_\perp (k, k')_\perp + (q, 2k' - k)_\perp (k, k')_\perp - (q, k')_\perp (k, k + k')_\perp + (q, k)_\perp (k', 2k - k')_\perp \right. \\
&+ (\alpha - 2\alpha')(q, k')_\perp \beta s + (q, k')_\perp (k, 2k' - k)_\perp - (q, k')_\perp k_\perp'^2 + (q, k - k')_\perp (k', k - k')_\perp \left. \right] \\
&+ \frac{\alpha + \alpha'}{\alpha - \alpha'} \frac{\alpha' - 2\alpha}{\alpha'} \left[ (k, k')_\perp (q, k - k')_\perp + (q, k')_\perp (k, k - k')_\perp \right] + [(k - k')_\perp^2 + k_\perp'^2] (q, k)_\perp + \\
&+ 4\alpha (q, k)_\perp \left[ \frac{\alpha - \alpha'}{\alpha'} + \frac{\alpha}{\alpha - \alpha'} \right] \beta s - 4\alpha (q, k)_\perp \left[ \frac{k_\perp'^2}{\alpha - \alpha'} + \frac{(k - k')_\perp^2}{\alpha'} \right] \left. \right\}
\end{aligned} \tag{65}$$

where we have omitted the contribution

$$\int \tilde{d}\alpha' \tilde{d}\beta' \tilde{d}^2 k' \frac{1}{\alpha'(\alpha'\beta's - k'^2_{\perp} + i\epsilon)} = \int d^4 k' \frac{1}{k'^2 + i\epsilon} V.p. \frac{1}{(k', p_2)} = 0 \quad (66)$$

Taking residues at  $\beta = 0$  and  $\beta' = \frac{k'^2}{\alpha's}$  and changing to variable  $u = \frac{\alpha'}{\alpha}$  we obtain (AGREES)

$$\begin{aligned} & \int_0^{\infty} du \int_{-\infty}^0 dv \langle \hat{A}_{\bullet}^a(up_1 + x_{\perp}) \hat{A}_{\bullet}^b(vp_1 + y_{\perp}) \rangle_{\text{Fig. 7a+b}} \\ &= -\frac{g^2 N_c \mu^{2\epsilon}}{8\pi^2} \int \tilde{d}^{2-\epsilon} k \tilde{d}^{2-\epsilon} k' \tilde{d}^{2-\epsilon} q \int d^{2-\epsilon} z U_z^{ab} \int_0^{\sigma} \frac{d\alpha}{\alpha} \int_0^1 du \frac{e^{i(q, x-z)_{\perp} - i(k, y-z)_{\perp}}}{k^4 q^2 [k'^2 \bar{u} + (k-k')^2 u]} \\ &\times \left\{ -2\epsilon(2k' - k, k)(q, k' - ku) + (1-2u)(k, q)(2k' - k, k) + (q, k)(k - 2k', k - k') + (q, k')(k, 2k' - k) \right. \\ &+ \frac{1+u}{\bar{u}} \left[ (1-2u)(q, k)(k, k - k') + 2(q, k)k'^2 - 2(k, k')_{\perp}(q, k') \right] \\ &- \frac{2-u}{u} \left[ (1-2u)(q, k)(k, k') + 2(q, k)(k - k', k') - 2(q, k')(k - k', k) \right] \\ &\left. - \frac{(1+u)(2-u)}{u\bar{u}} \left[ (k, k')(q, k - k') + (q, k')(k, k - k') \right] + (q, k) \left[ (k - k')^2 + k'^2 - 4\frac{k'^2}{\bar{u}} - 4\frac{(k - k')^2}{u} \right] \right\} \end{aligned} \quad (67)$$

Using the “plus” - prescription (47) to subtract the (LO)<sup>2</sup> contribution we get

$$\begin{aligned} & \frac{d}{d\eta} \langle \text{Tr} \{ \hat{U}_x \hat{U}_y^{\dagger} \} \rangle_{\text{Fig. 7a+b}} \\ &= -2\alpha_s^2 N_c \mu^{2\epsilon} \int d^{2-\epsilon} z [\text{Tr} \{ U_x U_z^{\dagger} \} \text{Tr} \{ U_z U_y^{\dagger} \} - \frac{1}{N_c} \text{Tr} \{ U_x U_y^{\dagger} \}] \int \tilde{d}^{2-\epsilon} k \tilde{d}^{2-\epsilon} k' \tilde{d}^{2-\epsilon} q \frac{e^{i(q, X) - i(k, Y)}}{k^2 q^2} \\ &\times \left[ \left( \frac{(q, 2k - k')}{k'^2} - \frac{(q, k + k')}{(k - k')^2} \right) \ln \frac{(k - k')^2}{k'^2} + \int_0^1 du \frac{(2 - \epsilon)(q, ku - k')(k, k - 2k') + 2(q, k)k^2}{k^2 [k'^2 \bar{u} + (k - k')^2 u]} \right] \end{aligned} \quad (68)$$

Next we calculate diagram shown in Fig. 7c.

$$\begin{aligned} & g^3 \int_0^{\infty} dt \int_{-\infty}^0 du \int_u dv \langle \hat{A}_{\bullet}^c(tp_1 + x) \hat{A}_{\bullet}^a(up_1 + y) \hat{A}_{\bullet}^b(vp_1 + y) \rangle_{\text{Fig. 7c}} = -2g^4 f^{lbc} \text{Tr} \{ t^a U_x t^b t^c U_y^{\dagger} \} \\ &\times \int \tilde{d}\alpha \tilde{d}\beta \tilde{d}\alpha' \tilde{d}\beta' \tilde{d}^2 k \tilde{d}^2 k' \tilde{d}^2 q \tilde{d}^{2-\epsilon} k \tilde{d}^{2-\epsilon} k' e^{i(q, x-z)_{\perp} - i(k, y-z)_{\perp}} \frac{\theta(\alpha) U_z^{al}}{q_{\perp}^2 (\alpha\beta s - k^2 + i\epsilon)} \frac{(k'_{\perp} + 2\beta' p_2)_{\mu}}{\alpha'\beta's - k'^2 + i\epsilon} \\ &\times \frac{(q_{\perp} + \frac{2}{\alpha s}(q, k)_{\perp} p_2)_{\lambda}}{\alpha'(\alpha - \alpha')} \frac{((k - k')_{\perp} + 2\beta'' p_2)_{\nu}}{(\alpha - \alpha')\beta''s - (k - k')^2_{\perp} + i\epsilon} \frac{\Gamma^{\mu\nu\lambda}(\alpha' p_1 + k'_{\perp}, (\alpha - \alpha') p_1 + (k - k')_{\perp}, -\alpha p_1 - k_{\perp})}{(\beta' - i\epsilon)(\beta'' + \beta' - i\epsilon)(\beta - \beta' - \beta'' - i\epsilon)} \end{aligned} \quad (69)$$

There are 2 regions of integration over  $\alpha$ 's:  $\alpha > |\alpha'|$  and  $\alpha < |\alpha'|$ . Taking relevant residues, we obtain (AGREES)

$$\begin{aligned} & -\frac{g^4}{2\pi^2} f^{abl} \mu^{2\epsilon} \int_0^{\infty} \frac{d\alpha}{\alpha} \int \tilde{d}^{2-\epsilon} k \tilde{d}^{2-\epsilon} k' \tilde{d}^{2-\epsilon} q \int d^{2-\epsilon} z_{\perp} U_z^{cl} \frac{e^{i(q, X) - i(k, Y)}}{k'^2 q^2 k^2} \int_0^1 du \left\{ \frac{1}{k'^2 \bar{u} + (k - k')^2 u} \left[ \frac{(q, k')}{\bar{u}} [k'^2 + k^2] \right. \right. \\ &+ \left. \frac{k'^2}{u} (q, 2k - k') - 2(q, k)(k', k - k') \right] + \frac{1}{(k - k')^2 \bar{u} + k^2 u} \left[ \frac{1}{u} [(q, k)k'^2 + (q, k')k^2] - (k', 2k - k')(q, k) \right] \right\} \end{aligned} \quad (70)$$

where we have introduced the variable  $u = |\alpha'|/\alpha$ . As usual. After integration over  $u$  with help of Eq. (47) this reduces to (AGREES)

$$\begin{aligned} & -\frac{g^4}{2\pi^2} f^{abl} \mu^{2\epsilon} \int_0^{\infty} \frac{d\alpha}{\alpha} \int \tilde{d}^{2-\epsilon} k \tilde{d}^{2-\epsilon} k' \tilde{d}^{2-\epsilon} q \int d^{2-\epsilon} z_{\perp} U_z^{cl} \left\{ \left[ \frac{(q, k')}{(k - k')^2} [k'^2 + k^2] + (q, k' - 2k) \right] \ln \frac{(k - k')^2}{k'^2} \right. \\ &+ \left. \left[ \frac{1}{(k - k')^2} [(q, k)k'^2 + (q, k')k^2] + (q, k) \right] \ln \frac{(k - k')^2}{k^2} - \int_0^1 du \frac{2(q, k)(k', k - k')}{k'^2 \bar{u} + (k - k')^2 u} \right\} \frac{e^{i(q, X) - i(k, Y)}}{k'^2 q^2 k^2} \end{aligned} \quad (71)$$

and therefore (AGREES)

$$\begin{aligned} \frac{d}{d\eta} \langle \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle_{\text{Fig. 7c}} &= -\frac{g^4 N_c}{8\pi^2} \mu^{2\varepsilon} \int d^{2-\varepsilon} z [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - \frac{1}{N_c} \text{Tr}\{U_x U_y^\dagger\}] \\ &\times \int \hat{d}^{2-\varepsilon} k \hat{d}^{2-\varepsilon} k' \hat{d}^{2-\varepsilon} q \frac{e^{i(q,X)-i(k,Y)}}{k^2 q^2} \left\{ \left[ \frac{(q,k')}{(k-k')^2} + \frac{(q,k')k^2}{k'^2(k-k')^2} + \frac{(q,k'-2k)}{k'^2} \right] \ln \frac{(k-k')^2}{k'^2} \right. \\ &\left. - \left[ \frac{(q,k)}{k'^2} + \frac{(q,k)}{(k-k')^2} + \frac{(q,k-k')k^2}{k'^2(k-k')^2} \right] \ln \frac{k^2}{k'^2} - 2 \int_0^1 du \frac{(q,k)(k',k-k')}{k'^2[k'^2 \bar{u} + (k-k')^2 u]} \right\} \end{aligned} \quad (72)$$

Next we calculate the sum of diagrams in Fig. 7 d,e, and f. The contribution of the diagram shown in Fig. 7d is

$$\begin{aligned} &\langle \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle_{\text{Fig. 7d}} \\ &= \int \hat{d}\alpha \hat{d}\alpha' \hat{d}\beta \hat{d}\beta' \int \hat{d}^{2-\varepsilon} k \hat{d}^{2-\varepsilon} k' \hat{d}^{2-\varepsilon} q \int d^{2-\varepsilon} z U_z^{ab} \frac{4g^4 \mu^{2\varepsilon} \theta(\alpha)(q,k)_\perp e^{i(q,x-z)_\perp - i(k,y-z)_\perp} \text{Tr}\{t^a U_x t^c t^b t^c U_y^\dagger\}}{\alpha\alpha'(\beta+\beta'-i\varepsilon)(\beta-i\varepsilon)(\alpha'\beta's - k'^2_\perp + i\varepsilon)(\alpha\beta s - k^2_\perp + i\varepsilon)q^2_\perp} \\ &= -\frac{g^4}{\pi^2} \mu^{2\varepsilon} \int_0^\sigma \frac{d\alpha}{\alpha} \int_0^1 du \int \hat{d}^{2-\varepsilon} q \hat{d}^{2-\varepsilon} k \hat{d}^{2-\varepsilon} k' \int d^{2-\varepsilon} z U_z^{ab} \frac{(q,k)_\perp e^{i(q,x-z)_\perp - i(k,y-z)_\perp}}{uk^2(uk^2 + \bar{u}k'^2)q^2} \text{Tr}\{t^a U_x t^c t^b t^c U_y^\dagger\} \end{aligned} \quad (73)$$

where we took residues at  $\beta = \frac{k^2}{\alpha s}$ ,  $\beta' = \frac{k'^2}{\alpha' s}$  and introduced the variable  $u = \frac{\alpha}{\alpha+\alpha'}$ . It should be noted that the cutoff  $\alpha < \sigma$  in the r.h.s. of this equation translates into  $\int_0^\infty d\alpha d\alpha' \theta(\sigma - \alpha - \alpha')$  while our cutoff (17) corresponds to  $\int_0^\infty d\alpha d\alpha' \theta(\sigma - \alpha)\theta(\sigma - \alpha')$ . Fortunately, the difference

$$\int_0^\infty \frac{d\alpha d\alpha'}{\alpha'(\alpha k'^2 + \alpha' k^2)} [\theta(\sigma - \alpha)\theta(\sigma - \alpha') - \theta(\sigma - \alpha - \alpha')] = \frac{1}{k'^2} \int_0^1 \frac{dv}{v} \ln \frac{k'^2 + k^2 v}{k'^2 \bar{v} + k^2 v} \quad (74)$$

does not contain  $\ln \sigma$  and hence does not contribute to the NLO kernel. Similarly, one can impose the cutoff  $\alpha_1 + \alpha_2 < \sigma$  instead of the cutoff  $\alpha_1, \alpha_2 < \sigma$  in other diagrams whenever convenient.

Before calculating the diagrams in Fig. 7e and Fig. 7f it is convenient to make a replacement

$$\int_{-\infty}^0 du \int_u^0 dv \int_v^0 dt \hat{A}^a(u) \hat{A}^b(v) \hat{A}^c(t) \rightarrow \frac{1}{2} \int_{-\infty}^0 du \int_u^0 dv dt \hat{A}^a(u) \hat{A}^b(v) \hat{A}^c(t) \quad (75)$$

which can be performed since the color indices  $b$  and  $c$  in  $\dots t^b t^c \dots$  are contracted. For the diagram in Fig. 7e we get

$$\begin{aligned} &\langle \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle_{\text{Fig. 7e}} = 2g^4 \mu^{2\varepsilon} c_F \text{Tr}\{t^a U_x t^b U_y^\dagger\} \\ &\times \int \hat{d}\alpha \hat{d}\alpha' \hat{d}\beta \hat{d}\beta' \int \hat{d}^{2-\varepsilon} k \hat{d}^{2-\varepsilon} k' \hat{d}^{2-\varepsilon} q \int d^{2-\varepsilon} z U_z^{ab} \frac{e^{i(q,X)_\perp - i(k,Y)_\perp}}{\alpha'(\beta-i\varepsilon)^2(k'^2 + i\varepsilon)} \left( \frac{\beta'}{\beta+\beta'-i\varepsilon} + \frac{\beta'}{\beta-\beta'-i\varepsilon} \right) \frac{\theta(\alpha)(k,q)_\perp}{\alpha(k^2 + i\varepsilon)q^2_\perp} \\ &= 4g^4 \mu^{2\varepsilon} c_F \text{Tr}\{t^a U_x t^b U_y^\dagger\} \int \hat{d}\alpha \hat{d}\alpha' \hat{d}\beta \hat{d}\beta' \int \hat{d}^{2-\varepsilon} k \hat{d}^{2-\varepsilon} k' \hat{d}^{2-\varepsilon} q \int d^{2-\varepsilon} z U_z^{ab} e^{i(q,X)_\perp - i(k,Y)_\perp} \\ &\quad \times \frac{\beta'}{\alpha\alpha'(\beta-i\varepsilon)(\alpha'\beta's - k'^2_\perp + i\varepsilon)(\beta+\beta'-i\varepsilon)(\beta-\beta'-i\varepsilon)} \frac{\theta(\alpha)(k,q)_\perp}{(\alpha\beta s - k^2_\perp + i\varepsilon)q^2_\perp} \end{aligned} \quad (76)$$

where  $c_F = \frac{N_c^2 - 1}{2N_c}$ . Taking residues at  $\beta' = -\beta$ ,  $\beta = \frac{k^2_\perp}{\alpha s}$  at  $\alpha' > 0$  and  $\beta' = \beta$ ,  $\beta = \frac{k^2_\perp}{\alpha s}$  at  $\alpha' < 0$  we obtain

$$\begin{aligned} &\langle \text{Tr}\{U_x U_y^\dagger\} \rangle_{\text{Fig. 7e}} \\ &= \frac{g^4}{\pi^2} c_F \mu^{2\varepsilon} \int_0^\sigma \frac{d\alpha}{\alpha} \int_0^1 du \int \hat{d}^{2-\varepsilon} k \hat{d}^{2-\varepsilon} k' \frac{e^{i(q,X)_\perp - i(k,Y)_\perp}}{uk^2(uk^2 + \bar{u}k'^2)} \int \hat{d}^{2-\varepsilon} q \int d^{2-\varepsilon} z \frac{(k,q)_\perp}{q^2_\perp} U_z^{db} \text{Tr}\{t^d U_x t^b U_y^\dagger\} \end{aligned} \quad (77)$$

The diagram in Fig. 7f yields

$$\begin{aligned} &\langle \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle_{\text{Fig. 7f}} \\ &= g^4 \mu^{2\varepsilon} \int \hat{d}\alpha \hat{d}\beta \hat{d}^{2-\varepsilon} k \int \hat{d}\alpha' \hat{d}\beta' \hat{d}^{2-\varepsilon} k' \frac{1}{\alpha'(\beta-i\varepsilon)(\beta'-i\varepsilon)(\alpha'\beta's - k'^2_\perp + i\varepsilon)} \\ &\times 2\theta(\alpha) \int \hat{d}^{2-\varepsilon} q \int d^{2-\varepsilon} z e^{i(q,X)_\perp - i(k,Y)_\perp} \frac{(k,q)_\perp}{\alpha(\alpha\beta s - k^2_\perp + i\varepsilon)q^2_\perp} U_z^{ab} \text{Tr}\{t^a U_x t^c t^c t^b U_y^\dagger\} \\ &= -\frac{g^4}{2\pi^2} \mu^{2\varepsilon} \int_0^\sigma \frac{d\alpha}{\alpha} \int_0^1 \frac{du}{\bar{u}u} \int \hat{d}^{2-\varepsilon} k \hat{d}^{2-\varepsilon} k' \int \hat{d}^{2-\varepsilon} q \int d^{2-\varepsilon} z e^{i(q,x-z)_\perp - i(k,y-z)_\perp} \frac{(k,q)_\perp}{k^2 k'^2 q^2_\perp} U_z^{ab} \text{Tr}\{t^a U_x t^c t^c t^b U_y^\dagger\} \end{aligned} \quad (78)$$

Adding Eqs. (73), (77), (78) and integrating over  $u$  using Eq. (47) we get

$$\begin{aligned} & \frac{d}{d\eta} \langle \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle_{\text{Fig. 7d+e+f}} \\ &= -\frac{g^4 N_c}{4\pi^2} \mu^{2\epsilon} \int d^{2-\epsilon} q d^{2-\epsilon} k d^{2-\epsilon} k' \int d^{2-\epsilon} z \frac{(q, k)}{k^2 k'^2 q^2} \ln \frac{k^2}{k'^2} e^{i(q, X) - i(k, Y)} [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - \frac{1}{N_c} \text{Tr}\{U_x U_y^\dagger\}] \end{aligned} \quad (79)$$

Note that the diagram in Fig. 7f does not contribute to the NLO kernel.

The contribution of the last ‘‘running coupling’’ diagram shown in Fig. 7g has the form

$$\begin{aligned} & \langle \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle_{\text{Fig. 7g}} \\ &= -\frac{g^4}{2} \text{Tr}\{t^a U_x t^b t^c t^d U_y^\dagger\} \int_0^\infty du \int_{-\infty}^0 dv \langle \hat{A}_\bullet^a(ups_1 + x_\perp) \hat{A}_\bullet^d(vps_1 + y_\perp) \rangle \int_{-\infty}^0 dt dw \langle \hat{A}^b(tp_1 + x_\perp) \hat{A}^c(wp_1 + x_\perp) \rangle \end{aligned} \quad (80)$$

where we have again replaced  $\int_{-\infty}^0 dt \int_{-\infty}^t dw \hat{A}^b(t) \hat{A}^c(w)$  by  $\frac{1}{2} \int_{-\infty}^0 dt dw \hat{A}^b(t) \hat{A}^c(w)$ . Using the Eq. (11) we get

$$\begin{aligned} & \langle \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle_{\text{Fig. 7g}} \\ &= \frac{ig^4}{\pi} \text{Tr}\{t^a U_x t^b t^c t^d U_y^\dagger\} (x | \frac{p_i}{p_\perp^2} U^{ad} \frac{p_i}{p_\perp^2} | y) \int d\alpha d\beta' \frac{\beta'}{\alpha'(\beta'^2 + \epsilon^2)} (x | \frac{1}{\alpha' \beta' s - p_\perp^2 + i\epsilon} | x) \\ &= \frac{g^4}{2\pi^2} \text{Tr}\{t^a U_x t^b t^c t^d U_y^\dagger\} (x | \frac{p_i}{p_\perp^2} U^{ad} \frac{p_i}{p_\perp^2} | y) (x | \frac{1}{p_\perp^2} | x) \int_0^\sigma \frac{d\alpha}{\alpha} \frac{d\alpha'}{\alpha'} \end{aligned} \quad (81)$$

which is obviously a (LO)<sup>2</sup> term which does not contribute to the NLO kernel.

Combining the expressions (68), (72), and (79) we get

$$\begin{aligned} & \frac{d}{d\eta} \langle \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle_{\text{Fig. 7}} \\ &= -2\alpha_s^2 N_c \mu^{2\epsilon} \int d^{2-\epsilon} z [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - \frac{1}{N_c} \text{Tr}\{U_x U_y^\dagger\}] \int d^{2-\epsilon} k d^{2-\epsilon} k' d^{2-\epsilon} q \frac{e^{i(q, X) - i(k, Y)}}{k^2 q^2} \\ &\times \left\{ \left[ \frac{k^2}{k'^2} (q, 2k - k') - \frac{k^2}{(k - k')^2} (q, k + k') \right] \ln \frac{(k - k')^2}{k'^2} + \int_0^1 du \frac{(2 - \epsilon)(q, ku - k')(k, k - 2k') + 2(q, k)k^2}{k'^2 \bar{u} + (k - k')^2 u} \right. \\ &+ \left[ \frac{(q, k')}{(k - k')^2} + \frac{(q, k')k^2}{k'^2 (k - k')^2} + \frac{(q, k' - 2k)}{k'^2} \right] \ln \frac{(k - k')^2}{k'^2} \\ &\left. - \left[ \frac{(q, k)}{k'^2} + \frac{(q, k)}{(k - k')^2} + \frac{(q, k - k')k^2}{k'^2 (k - k')^2} \right] \ln \frac{k^2}{k'^2} - 2 \int_0^1 du \frac{(q, k)(k', k - k')}{k'^2 [k'^2 \bar{u} + (k - k')^2 u]} + \frac{2(q, k)}{k'^2} \ln \frac{k^2}{k'^2} \right\} \\ &= -2\alpha_s^2 N_c \mu^{2\epsilon} \int d^{2-\epsilon} z [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - \frac{1}{N_c} \text{Tr}\{U_x U_y^\dagger\}] \int d^{2-\epsilon} k d^{2-\epsilon} k' d^{2-\epsilon} q \frac{e^{i(q, X) - i(k, Y)}}{k^2 q^2} \left\{ \ln \frac{k^2}{k'^2} \right. \\ &\times \left. \frac{3(q, k')k^2 - 4(q, k)(k, k')}{k'^2 (k - k')^2} + \int_0^1 du \frac{(2 - \epsilon)(q, ku - k')(k, k - 2k') + 2(q, k)k^2}{k^2 [k'^2 \bar{u} + (k - k')^2 u]} - \int_0^1 du \frac{2(q, k)(k', k - k')}{k'^2 [k'^2 \bar{u} + (k - k')^2 u]} \right\} \end{aligned} \quad (82)$$

Using the integral over  $k'$  (AGREES)

$$\begin{aligned} & \int d^{2-\epsilon} k'_\perp \left\{ \frac{3(q, k')k^2 - 4(q, k)(k, k')}{k'^2 (k - k')^2} \ln \frac{k^2}{k'^2} + \int_0^1 du \frac{(2 - \epsilon)(q, ku - k')(k, k - 2k') + 2(q, k)k^2}{k^2 [k'^2 \bar{u} + (k - k')^2 u]} \right. \\ &\left. - \int_0^1 du \frac{2(q, k)(k', k - k')}{k'^2 [k'^2 \bar{u} + (k - k')^2 u]} \right\} = \frac{(q, k)}{4\pi} \left\{ \frac{\Gamma(\epsilon/2)}{(k^2)^{\epsilon/2}} \frac{\Gamma^2(1 - \frac{\epsilon}{2})}{\Gamma(2 - \epsilon)} \left[ \frac{11}{3} - \epsilon \frac{\pi^2}{6} \right] + \frac{1}{9} \right\} \end{aligned}$$

one reduces the r.h.s. of Eq. (82) to

$$\begin{aligned} & \frac{d}{d\eta} \langle \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle_{\text{Fig. 7}} = -\frac{\alpha_s^2 N_c}{2\pi} \mu^{2\epsilon} \int d^{2-\epsilon} z [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - \frac{1}{N_c} \text{Tr}\{U_x U_y^\dagger\}] \\ &\times \int d^{2-\epsilon} k d^{2-\epsilon} q e^{i(q, X) - i(k, Y)} \frac{(q, k)}{k^2 q^2} \left\{ \frac{\Gamma(\epsilon/2)}{(k^2)^{\epsilon/2}} \frac{\Gamma^2(1 - \frac{\epsilon}{2})}{\Gamma(2 - \epsilon)} \left[ \frac{11}{3} - \epsilon \frac{\pi^2}{6} \right] + \frac{1}{9} \right\} \end{aligned} \quad (83)$$

Next we subtract the counterterm

$$-\frac{22}{3} \frac{\alpha_s^2 N_c}{\pi \epsilon} \mu^\epsilon \int d^{2-\epsilon} z [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - \frac{1}{N_c} \text{Tr}\{U_x U_y^\dagger\}] \int d^{2-\epsilon} q d^{2-\epsilon} k e^{i(q,X)-i(k,Y)} \frac{(q,k)}{q^2 k^2} \quad (84)$$

corresponding to the poles  $1/\epsilon$  in the loop diagrams in Fig. 8 (we use the  $\overline{MS}$  scheme). We obtain

$$\begin{aligned} \frac{d}{d\eta} \langle \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle_{\text{Fig. 7}} &= -\frac{\alpha_s^2 N_c}{2\pi} \int d^{2-\epsilon} z [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - \frac{1}{N_c} \text{Tr}\{U_x U_y^\dagger\}] \\ &\times \int d^{2-\epsilon} k d^{2-\epsilon} k' d^{2-\epsilon} q e^{i(q,X)-i(k,Y)} \frac{(q,k)}{k^2 q^2} \left\{ \frac{11}{3} \ln \frac{\mu^2}{k^2} + \frac{67}{9} - \frac{\pi^2}{3} \right\} \end{aligned} \quad (85)$$

The complete set of running-coupling diagrams is presented in Fig. 8.

The contribution of diagrams in Fig. 8 VI-XII differs from Eq. (85) by the exchange  $e^{i(q,X)} \leftrightarrow e^{i(q,Y)}$  and sign. There is also a symmetric set of diagrams XII-XXIV obtained by reflection of the diagrams in Fig. 8 with respect to  $x_*$  axis. Again, the result is obtained by  $e^{-i(k,X)} \leftrightarrow e^{-i(k,Y)}$  substitution so that the contribution of these diagrams takes the form

$$\begin{aligned} \frac{d}{d\eta} \langle \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle_{\text{Fig. 8 I+...+XXIV}} &= \frac{\alpha_s^2 N_c}{2\pi} \int d^2 z [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - \frac{1}{N_c} \text{Tr}\{U_x U_y^\dagger\}] \\ &\times \int d^2 k d^2 k' d^2 q [e^{i(q,X)} - e^{i(q,Y)}] [e^{-i(k,X)} - e^{-i(k,Y)}] \frac{(q,k)}{k^2 q^2} \left\{ \frac{11}{3} \ln \frac{\mu^2}{k^2} + \frac{67}{9} - \frac{\pi^2}{3} \right\} \end{aligned} \quad (86)$$

The remaining diagrams XXV-XXVIII contribute only to the (LO)<sup>2</sup>. We have shown this for the diagram XXVII (Fig. 7g). The diagram in Fig. 8 XXV is obtained from the above equation by the replacement  $x \leftrightarrow y$ , and the diagrams in Fig. 8 XXVI and XXVIII by the replacements  $(x|\frac{p_i}{p_\perp^2} U \frac{p_i}{p_\perp^2}|y)(x|\frac{1}{p_\perp^2}|x)$  by  $(x|\frac{p_i}{p_\perp^2} U \frac{p_i}{p_\perp^2}|x)(y|\frac{1}{p_\perp^2}|y)$  and  $(y|\frac{p_i}{p_\perp^2} U \frac{p_i}{p_\perp^2}|y)(x|\frac{1}{p_\perp^2}|x)$ , respectively. Thus, the diagrams XXV-XXVII do not contribute to the NLO kernel.

There is another set of diagrams obtained by the reflection of diagrams shown in Fig. (8) with respect to the shock-wave line. It is obtained from Eq. (86) by the replacement  $q \leftrightarrow k$  in the logarithm so the final result for the sum of all ‘‘running coupling’’ diagrams of Fig. 8 type has the form

$$\begin{aligned} \frac{d}{d\eta} \text{Tr}\{U_x U_y^\dagger\}_{\text{Fig. 8 total}} &= \frac{\alpha_s^2 N_c}{2\pi} \int d^2 z [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - \frac{1}{N_c} \text{Tr}\{U_x U_y^\dagger\}] \\ &\times \int d^2 k d^2 k' d^2 q [e^{i(q,X)} - e^{i(q,Y)}] [e^{-i(k,X)} - e^{i(k,Y)}] \frac{(q,k)}{k^2 q^2} \left\{ \frac{11}{3} \ln \frac{\mu^4}{q^2 k^2} + \frac{134}{9} + \frac{2\pi^2}{3} \right\} \\ &= \frac{\alpha_s^2 N_c}{8\pi^3} \int d^2 z [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - \frac{1}{N_c} \text{Tr}\{U_x U_y^\dagger\}] \left\{ \frac{(x-y)^2}{X^2 Y^2} \left[ \frac{11}{3} \ln \frac{X^2 Y^2}{\mu^{-4}} + \frac{134}{9} - \frac{2\pi^2}{3} \right] + \frac{11}{3} \left[ \frac{1}{X^2} - \frac{1}{Y^2} \right] \ln \frac{X^2}{Y^2} \right\} \end{aligned} \quad (87)$$

## B. Diagrams for 1→2 dipoles transition

There is one more class of diagrams with one gluon-shockwave intersection shown in Fig. 9. These diagrams are UV-convergent so we do not need to change the dimension of the transverse space to  $2 - \epsilon$ . First we calculate the diagrams shown in Fig. 9a,b. (AGREES)

$$\begin{aligned} &\text{Tr}\{U_x U_y^\dagger\}_{\text{Fig. 9a+9b}} \quad (88) \\ &= 4 \text{Tr}\{t^a U_x t^b t^c t^d U_y^\dagger\} \int d^{2-\epsilon} z \int \bar{d}\alpha_1 \bar{d}\beta_1 \bar{d}\alpha_2 \bar{d}\beta_2 \int d^{2-\epsilon} k_1 d^{2-\epsilon} k_2 e^{-i(k_1,y)_\perp - i(k_2,y)_\perp} \left[ \frac{\delta^{bd} U_z^{ac}}{(\beta_1 - i\epsilon)(\beta_1 + \beta_2 - i\epsilon)} \right. \\ &+ \left. \frac{\delta^{bc} U_z^{ad}}{(\beta_2 - i\epsilon)(\beta_1 + \beta_2 - i\epsilon)} \right] \theta(\alpha_1) \int \bar{d}^2 q e^{i(q,x-z)_\perp + i(k_1,z)_\perp} \frac{(q_1, k_1)_\perp}{\alpha_1(\alpha_1 \beta_1 s - k_{1\perp}^2 + i\epsilon) q_\perp^2} \frac{e^{i(k_2,x)_\perp}}{\alpha_2(\alpha_2 \beta_2 s - k_{2\perp}^2 + i\epsilon)} \\ &= -\frac{g^4}{\pi^2} \text{Tr}\{t^a U_x t^b t^c t^d U_y^\dagger\} \int d^{2-\epsilon} z \int_0^\sigma \frac{d\alpha}{\alpha} \int_0^1 du \int d^{2-\epsilon} k d^{2-\epsilon} k' e^{-i(k,y)_\perp - i(k',y)_\perp} \\ &\quad \times \left[ \frac{\delta^{bd} U_z^{ac}}{k^2 \bar{u}(k^2 \bar{u} + k'^2 u)} + \frac{\delta^{ab} U_z^{dc}}{k'^2 u(k^2 \bar{u} + k'^2 u)} \right] \int \bar{d}^2 q e^{i(q,x-z)_\perp + i(k,z)_\perp + i(k',x)_\perp} \frac{(q,k)_\perp}{q_\perp^2} \\ &= -\frac{g^4}{\pi^2} \text{Tr}\{t^a U_x t^b t^c t^d U_y^\dagger\} \int d^2 z \int_0^\infty \frac{d\alpha}{\alpha} \int \bar{d}^2 k d^2 k' d^2 q \frac{(q,k)}{q^2} e^{i(q,X)-i(k,Y)_\perp + i(k',x-y)_\perp} \frac{\delta^{bc} U_z^{ad} - \delta^{bd} U_z^{ac}}{k^2 k'^2} \ln \frac{k^2}{k'^2} \end{aligned}$$

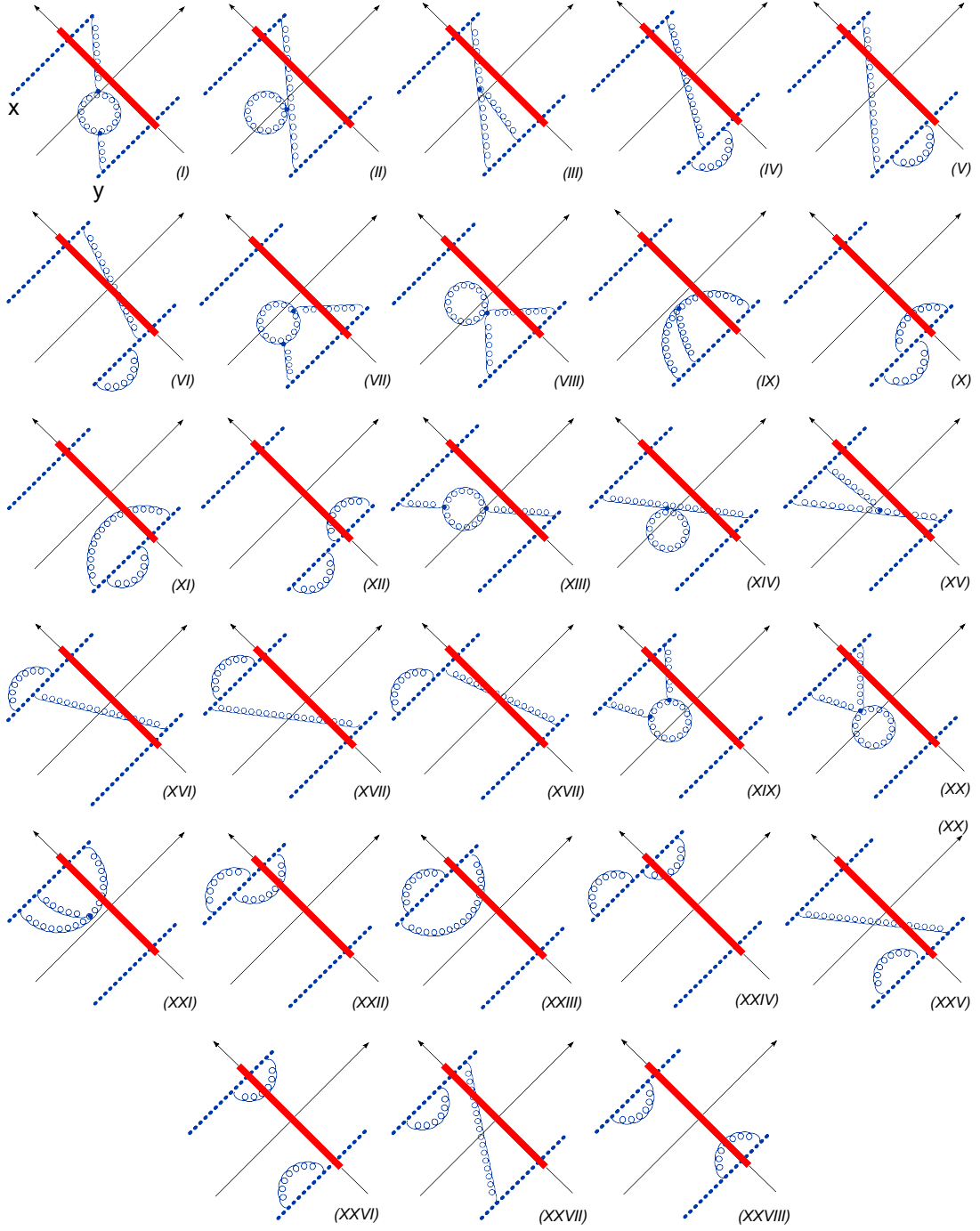


FIG. 8: The full set of “running coupling” diagrams.

and therefore (AGREES)

$$\begin{aligned}
 & \frac{d}{d\eta} \text{Tr}\{U_x U_y^\dagger\}_{\text{Fig. 9a+9b}} \\
 &= -\frac{g^4 N_c}{4\pi^2} \int d^2 z [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - \frac{1}{N_c} \text{Tr}\{U_x U_y^\dagger\}] \int \tilde{d}^2 k \tilde{d}^2 k' \tilde{d}^2 q \frac{(q, k)}{q^2 k^2 k'^2} e^{i(q, x-z) - i(k, y-z)_\perp + i(k', x-y)_\perp} \ln \frac{k^2}{k'^2}
 \end{aligned} \tag{89}$$

The contribution of diagrams shown in Fig. 9c,d is obtained from Eq. (89) by the replacement  $x \leftrightarrow y$  in the left part of the graph and the sign change so that  $e^{-ik(y-z) + i(k', x-y)} \rightarrow -e^{-ik(x-z) - i(k', x-y)}$ . The sum of the diagrams in Fig.

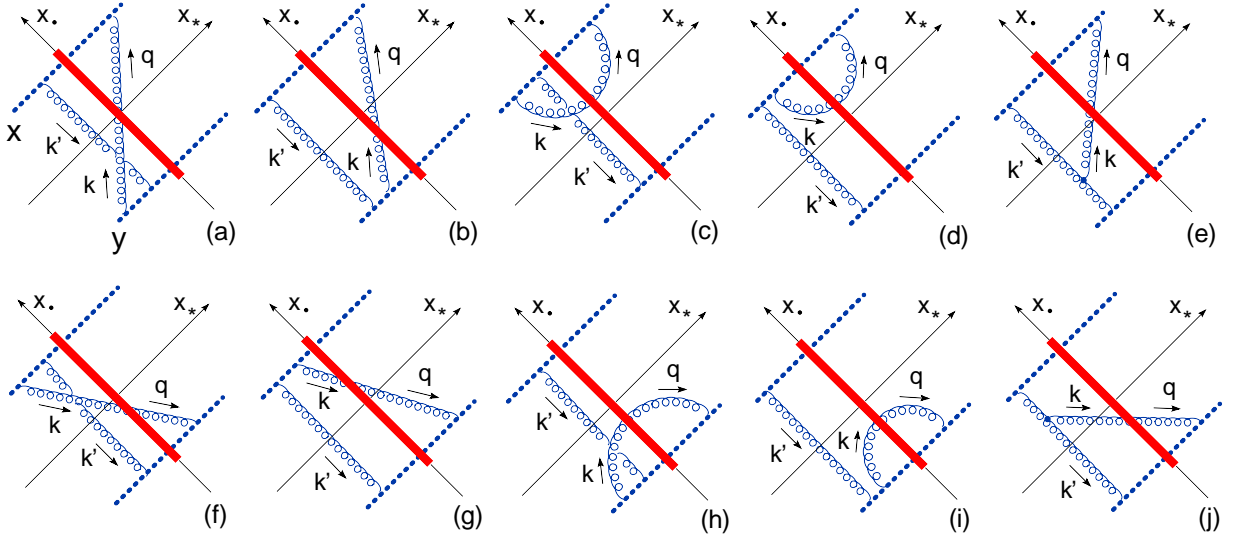


FIG. 9: 1→2 dipole transition diagrams.

9a-d takes the form

$$\begin{aligned} \frac{d}{d\eta} \text{Tr}\{U_x U_y^\dagger\}_{\text{Fig. 9a-d}} &= -\frac{g^4 N_c}{4\pi^2} \int d^2 z [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - \frac{1}{N_c} \text{Tr}\{U_x U_y^\dagger\}] \\ &\times \int \bar{d}^2 k \bar{d}^2 k' \bar{d}^2 q \frac{(q, k)}{q^2 k^2 k'^2} e^{i(q, x-z)} \left( e^{-i(k, y-z)_\perp - i(k', x-y)_\perp} - x \leftrightarrow y \right) \ln \frac{k^2}{k'^2} \end{aligned} \quad (90)$$

Next relevant diagram is shown in Fig. 9e

$$\begin{aligned} &g^3 \int_0^\infty du A_\bullet^a(up_1 + x) \int_{-\infty}^0 dv A_\bullet^b(vp_1 + x) \int_{-\infty}^0 dt A_\bullet^c(tp_1 + y) \\ &= 2g^4 \int \bar{d} \alpha_1 \bar{d} \beta_1 \bar{d} \alpha_2 \bar{d} \beta_2 \bar{d}^2 k_1^\perp \bar{d}^2 k_2^\perp \frac{\theta(\alpha_1 + \alpha_2) f^{bcd} (k_{1\mu}^\perp + 2\beta_1 p_{2\mu}) (k_{2\nu}^\perp + 2\beta_2 p_{2\nu})}{\alpha_1 \alpha_2 (\beta_1 - i\epsilon) (\alpha_1 \beta_1 s - k_{1\perp}^2 + i\epsilon) (\beta_2 - i\epsilon) (\alpha_2 \beta_2 s - k_{2\perp}^2 + i\epsilon)} \\ &\times \int d^2 z U_z^{ad} \int \bar{d}^2 q \frac{e^{i(q-k_1, x-z)_\perp - ik_2(y-z)_\perp} \left[ q + \frac{2(k_1 + k_2, q)_\perp}{(\alpha_1 + \alpha_2)s} p_2 \right]_\lambda \Gamma^{\mu\nu\lambda}(k_1, k_2, -k_1 - k_2)}{q^2 [(\alpha_1 + \alpha_2)(\beta_1 + \beta_2)s - (k_1 + k_2)_\perp^2 + i\epsilon]} \end{aligned} \quad (91)$$

There are three regions of integration over  $\alpha$ 's:  $\alpha_1, \alpha_2 > 0$ ,  $\alpha_1 > -\alpha_2 > 0$  and  $\alpha_2 > -\alpha_1 > 0$ . Going to the variables  $\alpha = \alpha_1 + \alpha_2$ ,  $u = \alpha_2/\alpha$  in the first region,  $\alpha = \alpha_1$ ,  $u = -\alpha_2/\alpha$  in the second and  $\alpha = \alpha_2$ ,  $u = -\alpha_1/\alpha$  in the third, we obtain

$$\begin{aligned} &g^3 \int_0^\infty du A_\bullet^a(up_1 + x) \int_{-\infty}^0 dv A_\bullet^b(vp_1 + x) \int_{-\infty}^0 dt A_\bullet^c(tp_1 + y) \\ &= \frac{1}{2\pi^2} \int_0^\infty \frac{d\alpha}{\alpha} \int_0^1 du \int \bar{d}^2 k_1^\perp \bar{d}^2 k_2^\perp f^{bcd} U_z^{ad} \int \frac{\bar{d}^2 q}{q^2} e^{i(q-k_1, x-z)_\perp - i(k_2, y-z)_\perp} \\ &\times \left[ k_{1\mu}^\perp k_{2\nu}^\perp \left( q_\lambda + \frac{2(k_1 + k_2, q)_\perp}{\alpha s} p_{2\lambda} \right) \frac{\Gamma^{\mu\nu\lambda}(\alpha \bar{u} p_1 + k_1^\perp, \alpha u p_1 + k_2^\perp, -\alpha p_1 - k_1^\perp - k_2^\perp)}{\bar{u} u k_{1\perp}^2 k_{2\perp}^2 (k_1 + k_2)_\perp^2} \right. \\ &- \bar{u} k_{1\mu}^\perp \left( k_{2\nu}^\perp + 2 \frac{(k_1 + k_2)_\perp^2}{\alpha \bar{u} s} p_{2\nu} \right) \left( q_\lambda + \frac{2(k_1 + k_2, q)_\perp}{\alpha \bar{u} s} p_{2\lambda} \right) \frac{\Gamma^{\mu\nu\lambda}(\alpha p_1 + k_1^\perp, -u \alpha p_1 + k_2^\perp, -\alpha \bar{u} - k_1^\perp - k_2^\perp)}{u k_{1\perp}^2 (k_1 + k_2)_\perp^2 [u(k_1 + k_2)_\perp^2 + \bar{u} k_{2\perp}^2]} \\ &\left. - \bar{u} \left( k_{1\mu}^\perp + 2 \frac{(k_1 + k_2)_\perp^2}{\alpha \bar{u} s} p_{2\mu} \right) k_{2\nu}^\perp \left( q_\lambda + \frac{2(k_1 + k_2, q)_\perp}{\alpha \bar{u} s} p_{2\lambda} \right) \frac{\Gamma^{\mu\nu\lambda}(-u \alpha p_1 + k_1^\perp, \alpha p_1 + k_2^\perp, -\alpha \bar{u} - k_1^\perp - k_2^\perp)}{u k_{2\perp}^2 (k_1 + k_2)_\perp^2 [u(k_1 + k_2)_\perp^2 + \bar{u} k_{1\perp}^2]} \right] \end{aligned} \quad (92)$$

Using the formula

$$\left(k_1^\perp + \frac{2A}{s}p_2\right)_\mu \left(k_2^\perp + \frac{2B}{s}p_2\right)_\nu \left(q + \frac{2C}{s}p_2\right)_\lambda \Gamma^{\mu\nu\lambda}(\alpha_1 p_1 + k_{1\perp}, \alpha_2 p_1 + k_{2\perp}, -(\alpha_1 + \alpha_2)p_1 - (k_1 + k_2)_\perp) = -C(\alpha_1 - \alpha_2)(k_1, k_2)_\perp - A(\alpha_1 + 2\alpha_2)(q, k_2)_\perp + B(2\alpha_1 + \alpha_2)(q, k_1)_\perp - [(q, k_1)_\perp(k_2, k_1 + k_2)_\perp - (k_1 \leftrightarrow k_2)]$$

we get

$$\begin{aligned} \text{Tr}\{U_x U_y^\dagger\}_{\text{Fig. 9c}} &= \frac{g^4 N_c}{8\pi^2} \int \bar{d}^2 k_1 \bar{d}^2 k_2 \frac{\bar{d}^2 q}{q^2} \int d^2 z \int_0^\sigma \frac{d\alpha}{\alpha} \int_0^1 du e^{i(q-k_1, x-z) - i(k_2, y-z)} \\ &\times [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - \frac{1}{N_c} \text{Tr}\{U_x U_y^\dagger\}] \left\{ \frac{(q, k_1)(k_2, k_1 + k_2) - (q, k_2)(k_1, k_1 + k_2)}{\bar{u} u k_1^2 k_2^2 (k_1 + k_2)^2} \right. \\ &+ \frac{(1 + \bar{u})(k_1 + k_2)^2 (q, k_1) - (1 + u)(q, k_1 + k_2)(k_1, k_2) - \bar{u}[(q, k_1)(k_2, k_1 + k_2) - (q, k_2)(k_1, k_1 + k_2)]}{u k_1^2 (k_1 + k_2)^2 (k_2^2 \bar{u} + (k_1 + k_2)^2 u)} \\ &\left. + \frac{-(1 + \bar{u})(k_1 + k_2)^2 (q, k_2) + (1 + u)(q, k_1 + k_2)(k_1, k_2) - \bar{u}[(q, k_1)(k_2, k_1 + k_2) - (q, k_2)(k_1, k_1 + k_2)]}{u k_2^2 (k_1 + k_2)^2 (k_1^2 \bar{u} + (k_1 + k_2)^2 u)} \right\} \end{aligned} \quad (93)$$

Performing the integration over  $u$  (with prescription (47)) we obtain

$$\begin{aligned} &\frac{d}{d\eta} \text{Tr}\{U_x U_y^\dagger\}_{\text{Fig. 9c}} \\ &= \frac{g^4 N_c}{8\pi^2} \int d^2 z [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - \frac{1}{N_c} \text{Tr}\{U_x U_y^\dagger\}] \int \frac{\bar{d}^2 k_1 \bar{d}^2 k_2}{k_1^2 k_2^2 (k_1 + k_2)^2} \frac{\bar{d}^2 q}{q^2} e^{i(q-k_1, X) - i(k_2, Y)} U_z^{cl} \\ &\times \left\{ (k_1 + k_2)^2 \left[ (q, k_2) \ln \frac{(k_1 + k_2)^2}{k_1^2} - (q, k_1) \ln \frac{(k_1 + k_2)^2}{k_2^2} \right] - (q, k_1 + k_2)(k_1^2 + k_2^2) \ln \frac{k_1^2}{k_2^2} \right\} \\ &= \frac{g^4 N_c}{8\pi^2} \int d^2 z [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - \text{Tr}\{U_x U_y^\dagger\}] \int \bar{d}^2 k \bar{d}^2 k' \frac{\bar{d}^2 q}{q^2} e^{i(q-k', x-z) - i(k-k', y-z)} U_z^{cl} \\ &\times \left\{ \frac{(q, k-k')}{k'^2 (k-k')^2} \ln \frac{k^2}{k'^2} - \frac{(q, k')}{k'^2 (k-k')^2} \ln \frac{k^2}{(k-k')^2} + \frac{(q, k)}{k^2 k'^2} \ln \frac{(k-k')^2}{k'^2} + \frac{(q, k)}{k^2 (k-k')^2} \ln \frac{(k-k')^2}{k'^2} \right\} \end{aligned} \quad (94)$$

where we made the change of variables  $k_1 \rightarrow k'$  and  $k_2 \rightarrow k - k'$ .

The sum of the diagrams shown in Fig. 9a-e can be represented as

$$\begin{aligned} &\frac{d}{d\eta} \text{Tr}\{U_x U_y^\dagger\}_{\text{Fig. 9a-e}} = 2\alpha_s^2 N_c \int d^2 z [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - \frac{1}{N_c} \text{Tr}\{U_x U_y^\dagger\}] \int \bar{d}^2 k \bar{d}^2 k' \bar{d}^2 q \\ &\times e^{i(q, x-z)} \left( e^{-i(k, y-z)_\perp - i(k', x-y)_\perp} - x \leftrightarrow y \right) \left[ \frac{(q, k-k')}{k'^2 (k-k')^2} + \frac{(q, k)}{k^2 (k-k')^2} - \frac{(q, k)}{k^2 k'^2} \right] \ln \frac{k^2}{k'^2} \end{aligned} \quad (95)$$

Note that the expressions (90) and (94) are IR divergent as  $k' \rightarrow 0$  but their sum (95) is IR stable. Once again, the contribution of the diagrams in Fig. 9f-k are obtained by replacement  $e^{iq(x-z)} \rightarrow -e^{iq(y-z)}$  so the contribution of the diagrams of Fig. 9 a-k has the form

$$\begin{aligned} &\frac{d}{d\eta} \text{Tr}\{U_x U_y^\dagger\}_{\text{Fig. 9a-k}} = 2\alpha_s^2 N_c \int d^2 z [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - \frac{1}{N_c} \text{Tr}\{U_x U_y^\dagger\}] \int \bar{d}^2 k \bar{d}^2 k' \bar{d}^2 q \\ &\times (e^{i(q, x-z)} - e^{i(q, y-z)}) \left( e^{-i(k, y-z)_\perp - i(k', x-y)_\perp} - x \leftrightarrow y \right) \left[ \frac{(q, k-k')}{k'^2 (k-k')^2} + \frac{(q, k)}{k^2 (k-k')^2} - \frac{(q, k)}{k^2 k'^2} \right] \ln \frac{k^2}{k'^2} \end{aligned} \quad (96)$$

Performing the Fourier transformation with the help of the formula



$$\begin{aligned}
& \int \bar{d}k \bar{d}k' \frac{e^{-i(k,y)-i(k',x-y)}}{k'^2} \left( \frac{(k-k')_i}{(k-k')^2} - \frac{k_i}{k^2} + \frac{k_i k'^2}{k^2(k-k')^2} \right) \ln \frac{k^2}{k'^2} \\
&= \frac{i}{16\pi^2} \left( \frac{x_i}{x^2} - \frac{y_i}{y^2} \right) \ln \frac{(x-y)^2}{x^2} \ln \frac{(x-y)^2}{y^2} \\
&+ \frac{i}{8\pi^2} \left( \frac{(x,y)}{y^2} y_i - x_i \right) \frac{1}{i\kappa} \left\{ \int_0^1 du \left[ \frac{\ln u}{u - \frac{(x,y)-i\kappa}{x^2}} - \frac{\ln u}{u - \frac{(x,y)+i\kappa}{x^2}} \right] + \frac{1}{2} \ln \frac{x^2}{y^2} \ln \frac{(x-y,y) + i\kappa}{(x-y,y) - i\kappa} \right\} \\
&+ \frac{i}{8\pi^2} \left( \frac{(x,y)}{x^2} x_i - y_i \right) \frac{1}{i\kappa} \left\{ \int_0^1 du \left[ \frac{\ln u}{u - \frac{(x,x-y)-i\kappa}{(x-y)^2}} - \frac{\ln u}{u - \frac{(x,x-y)+i\kappa}{(x-y)^2}} \right] - \frac{1}{2} \ln \frac{(x-y)^2}{x^2} \ln \frac{(x,y) + i\kappa}{(x,y) - i\kappa} \right\}
\end{aligned} \tag{97}$$

(here  $\kappa = \sqrt{x^2 y^2 - (x,y)^2}$ ) one obtains

$$\frac{d}{d\eta} \text{Tr}\{U_x U_y^\dagger\}_{\text{Fig. 9a-k}} = -\frac{\alpha_s^2 N_c}{8\pi^3} \int d^2 z [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - \frac{1}{N_c} \text{Tr}\{U_x U_y^\dagger\}] \frac{(x-y)^2}{X^2 Y^2} \ln \frac{X^2}{(x-y)^2} \ln \frac{Y^2}{(x-y)^2} \tag{98}$$

Note that the two last terms in the r.h.s. of Eq. (97) do not contribute.

The contribution of the diagram obtained by reflection of Fig. 9 with respect to the shock wave differs from Eq. (95) by replacement  $q \leftrightarrow k$  which doubles the result (98). The final expression for the contribution of all ‘‘dipole recombination diagrams’’ of Fig. 9 type has the form

$$\frac{d}{d\eta} \text{Tr}\{U_x U_y^\dagger\}_{\text{Fig. 9 total}} = -\frac{\alpha_s^2 N_c}{4\pi^3} \int d^2 z [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - \frac{1}{N_c} \text{Tr}\{U_x U_y^\dagger\}] \frac{(x-y)^2}{X^2 Y^2} \ln \frac{X^2}{(x-y)^2} \ln \frac{Y^2}{(x-y)^2} \tag{99}$$

## V. ASSEMBLING THE NLO KERNEL

Adding the results (58), (87) and (99) one obtains the contribution of the diagrams with one and two gluon intersections with the shock wave in the form:

$$\begin{aligned}
& \frac{d}{d\eta} \text{Tr}\{U_x U_y^\dagger\}_{1\text{cut}+2\text{cuts}} = \frac{\alpha_s^2 N_c}{8\pi^3} \int d^2 z \left\{ \frac{(x-y)^2}{X^2 Y^2} \left[ \frac{11}{3} \ln(x-y)^2 \mu^2 + \frac{67}{9} - \frac{\pi^2}{3} \right] \right. \\
&+ \frac{11}{3} \left[ \frac{1}{X^2} - \frac{1}{Y^2} \right] \ln \frac{X^2}{Y^2} - 2 \frac{(x-y)^2}{X^2 Y^2} \ln \frac{X^2}{(x-y)^2} \ln \frac{Y^2}{(x-y)^2} \left. \right\} [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - \frac{1}{N_c} \text{Tr}\{U_x U_y^\dagger\}] \\
&+ \frac{\alpha_s^2}{16\pi^4} \int d^2 z d^2 z' \left[ \left( -\frac{4}{(z-z')^4} + \left\{ 2 \frac{X^2 Y'^2 + X'^2 Y^2 - 4(x-y)^2 (z-z')^2}{(z-z')^4 [X^2 Y'^2 - X'^2 Y^2]} \right. \right. \right. \\
&+ \left. \left. \left( \frac{(x-y)^4}{X^2 Y'^2 - X'^2 Y^2} \left[ \frac{1}{X^2 Y'^2} + \frac{1}{Y^2 X'^2} \right] + \frac{(x-y)^2}{(z-z')^2} \left[ \frac{1}{X^2 Y'^2} - \frac{1}{X'^2 Y^2} \right] \right) \ln \frac{X^2 Y'^2}{X'^2 Y^2} \right) \right. \\
&\quad \times [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_{z'}^\dagger\} \text{Tr}\{U_{z'} U_y^\dagger\} - \text{Tr}\{U_x U_z^\dagger U_{z'} U_y^\dagger U_z U_{z'}^\dagger\} - (z' \rightarrow z)] \\
&\left. + \left\{ -\frac{(x-y)^4}{X^2 Y'^2 X'^2 Y^2} + \frac{(x-y)^2}{(z-z')^2} \left( \frac{1}{X^2 Y'^2} + \frac{1}{Y^2 X'^2} \right) \right\} \ln \frac{X^2 Y'^2}{X'^2 Y^2} \text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_{z'}^\dagger\} \text{Tr}\{U_{z'} U_y^\dagger\} \right]
\end{aligned} \tag{100}$$

There are also diagrams without gluon-shockwave intersection like the graph shown in Fig. 10. They are proportional to the parent dipole  $\text{Tr}\{U_x U_y^\dagger\}$  and their contribution can be found from Eq. (100) using the requirement that the r.h.s. of the evolution equation must vanish at  $x = y$  (since  $U_x U_x^\dagger = 1$ ). It is easy to see that the replacement  $\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - \frac{1}{N_c} \text{Tr}\{U_x U_y^\dagger\}$  by  $\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - N_c \text{Tr}\{U_x U_y^\dagger\}$  fulfills the above requirement so one

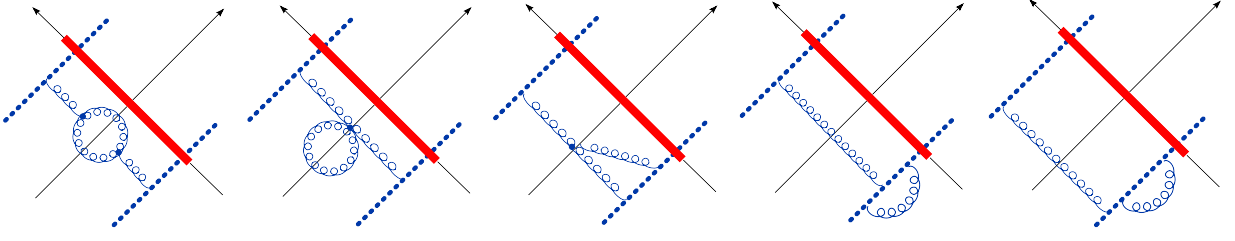


FIG. 10: Typical diagrams without the gluon-shockwave intersection.

obtains the final gluon contribution to the NLO kernel in the form

$$\begin{aligned}
\frac{d}{d\eta} \langle \text{Tr} \{ \hat{U}_x \hat{U}_y^\dagger \} \rangle &= \frac{\alpha_s^2 N_c}{8\pi^3} \int d^2 z \left\{ \frac{(x-y)^2}{X^2 Y^2} \left[ \frac{11}{3} \ln(x-y)^2 \mu^2 + \frac{67}{9} - \frac{\pi^2}{3} \right] \right. \\
&+ \frac{11}{3} \left[ \frac{1}{X^2} - \frac{1}{Y^2} \right] \ln \frac{X^2}{Y^2} - 2 \frac{(x-y)^2}{X^2 Y^2} \ln \frac{X^2}{(x-y)^2} \ln \frac{Y^2}{(x-y)^2} \left. \right\} [\text{Tr} \{ U_x U_z^\dagger \} \text{Tr} \{ U_z U_y^\dagger \} - N_c \text{Tr} \{ U_x U_y^\dagger \}] \\
&+ \frac{\alpha_s^2}{16\pi^4} \int d^2 z d^2 z' \left[ \left( -\frac{4}{(z-z')^4} + \left\{ 2 \frac{X^2 Y'^2 + X'^2 Y^2 - 4(x-y)^2 (z-z')^2}{(z-z')^4 [X^2 Y'^2 - X'^2 Y^2]} \right. \right. \right. \\
&+ \left. \left. \left( \frac{(x-y)^4}{X^2 Y'^2 - X'^2 Y^2} \left[ \frac{1}{X^2 Y'^2} + \frac{1}{Y^2 X'^2} \right] + \frac{(x-y)^2}{(z-z')^2} \left[ \frac{1}{X^2 Y'^2} - \frac{1}{X'^2 Y^2} \right] \right) \ln \frac{X^2 Y'^2}{X'^2 Y^2} \right) \right. \\
&\quad \times [\text{Tr} \{ U_x U_z^\dagger \} \text{Tr} \{ U_z U_{z'}^\dagger \} \text{Tr} \{ U_{z'} U_y^\dagger \} - \text{Tr} \{ U_x U_z^\dagger U_{z'} U_y^\dagger U_z U_{z'}^\dagger \} - (z' \rightarrow z)] \\
&\left. + \left\{ -\frac{(x-y)^4}{X^2 Y'^2 X'^2 Y^2} + \frac{(x-y)^2}{(z-z')^2} \left( \frac{1}{X^2 Y'^2} + \frac{1}{Y^2 X'^2} \right) \right\} \ln \frac{X^2 Y'^2}{X'^2 Y^2} \text{Tr} \{ U_x U_z^\dagger \} \text{Tr} \{ U_z U_{z'}^\dagger \} \text{Tr} \{ U_{z'} U_y^\dagger \} \right]
\end{aligned} \tag{101}$$

Promoting Wilson lines in the r.h.s of this equation to operators and adding the quark contribution from Ref. [9]

$$\begin{aligned}
\frac{d}{d\eta} \langle \text{Tr} \{ \hat{U}_x \hat{U}_y^\dagger \} \rangle_{\text{quark}} &= \frac{\alpha_s}{2\pi^2} \int d^2 z [\text{Tr} \{ U_x U_z^\dagger \} \text{Tr} \{ U_z U_y^\dagger \} - N_c \text{Tr} \{ U_x U_y^\dagger \}] \\
&\quad \times \left[ -\frac{\alpha_s n_f}{6\pi} \frac{(x-y)^2}{X^2 Y^2} \left( \ln(x-y)^2 \mu^2 + \frac{5}{3} \right) + \frac{\alpha_s n_f}{6\pi} \frac{X^2 - Y^2}{X^2 Y^2} \ln \frac{X^2}{Y^2} \right] \\
&+ \frac{\alpha_s^2}{\pi^4} n_f \text{Tr} \{ t^a U_x t^b U_y^\dagger \} \int d^2 z d^2 z' \text{Tr} \{ t^a U_z t^b U_{z'}^\dagger - t^a U_z t^b U_{z'}^\dagger \} \frac{1}{(z-z')^4} \\
&\quad \times \left\{ 1 - \frac{X'^2 Y^2 + Y'^2 X^2 - (x-y)^2 (z-z')^2}{2(X'^2 Y^2 - Y'^2 X^2)} \ln \frac{X'^2 Y^2}{Y'^2 X^2} \right\}
\end{aligned} \tag{102}$$

we obtain the full NLO kernel cited in Eq. (5).

## VI. COMPARISON TO NLO BFKL

### A. Linearized forward kernel

In this section we compare our kernel to the forward NLO BFKL results[12]. The linearized equation (5) has the form

$$\begin{aligned}
\frac{d}{d\eta}\mathcal{U}(x,y) &= \frac{\alpha_s N_c}{2\pi^2} \int d^2z \frac{(x-y)^2}{X^2 Y^2} \left\{ 1 + \frac{\alpha_s}{4\pi} \left[ b \ln(x-y)^2 \mu^2 - b \frac{X^2 - Y^2}{(x-y)^2} \ln \frac{X^2}{Y^2} + \left(\frac{67}{9} - \frac{\pi^2}{3}\right) N_c - \frac{10}{9} n_f \right. \right. \\
&\quad \left. \left. - 2N_c \ln \frac{X^2}{(x-y)^2} \ln \frac{Y^2}{(x-y)^2} \right] \right\} [\mathcal{U}(x,z) + \mathcal{U}(z,y) - \mathcal{U}(x,y)] \\
&+ \frac{\alpha_s^2 N_c^2}{16\pi^4} \int d^2z d^2z' \left[ -\frac{4}{(z-z')^4} + \left\{ 2 \frac{X^2 Y'^2 + X'^2 Y^2 - 4(x-y)^2 (z-z')^2}{(z-z')^4 [X^2 Y'^2 - X'^2 Y^2]} \right. \right. \\
&+ \frac{(x-y)^4}{X^2 Y'^2 - X'^2 Y^2} \left[ \frac{1}{X^2 Y'^2} + \frac{1}{Y^2 X'^2} \right] + \frac{(x-y)^2}{(z-z')^2} \left[ \frac{1}{X^2 Y'^2} - \frac{1}{X'^2 Y^2} \right] \left. \right\} \ln \frac{X^2 Y'^2}{X'^2 Y^2} \\
&\left. - \frac{n_f}{N_c^3} \left\{ \frac{4}{(z-z')^4} - 2 \frac{X'^2 Y^2 + Y'^2 X^2 - (x-y)^2 (z-z')^2}{(z-z')^4 (X'^2 Y^2 - Y'^2 X^2)} \ln \frac{X'^2 Y^2}{Y'^2 X^2} \right\} \right] \mathcal{U}(z,z') \tag{103}
\end{aligned}$$

For the case of forward scattering  $\mathcal{U}(x,y) = \mathcal{U}(x-y)$  and the linearized equation (103) reduces to

$$\begin{aligned}
\frac{d}{d\eta}\mathcal{U}(x) &= \frac{\alpha_s N_c}{2\pi^2} \int d^2z \frac{x^2}{(x-z)^2 z^2} \left\{ 1 + \frac{\alpha_s}{4\pi} \left[ b \ln x^2 \mu^2 - b \frac{(x-z)^2 - z^2}{x^2} \ln \frac{(x-z)^2}{z^2} + \left(\frac{67}{9} - \frac{\pi^2}{3}\right) N_c - \frac{10}{9} n_f \right. \right. \\
&\quad \left. \left. - 2N_c \ln \frac{(x-z)^2}{x^2} \ln \frac{z^2}{x^2} \right] \right\} [\mathcal{U}(x-z) + \mathcal{U}(z) - \mathcal{U}(x)] \\
&+ \frac{\alpha_s^2 N_c^2}{16\pi^4} \int d^2z d^2z' \left[ -\frac{4}{z^4} + \left\{ 2 \frac{(x-z-z')^2 z'^2 + (x-z')^2 (z+z')^2 - 4x^2 z^2}{z^4 [(x-z-z')^2 z'^2 - (x-z')^2 (z+z')^2]} + \frac{x^4}{(x-z-z')^2 z'^2 - (x-z')^2 (z+z')^2} \right. \right. \\
&\times \left[ \frac{1}{(x-z-z')^2 z'^2} + \frac{1}{(x-z')^2 (z+z')^2} \right] + \frac{x^2}{z^2} \left[ \frac{1}{(x-z-z')^2 z'^2} - \frac{1}{(x-z')^2 (z+z')^2} \right] \left. \right\} \ln \frac{(x-z-z')^2 z'^2}{(x-z')^2 (z+z')^2} \\
&\left. - \frac{n_f}{N_c^3} \left\{ \frac{4}{z^4} - 2 \frac{(x-z-z')^2 z'^2 + (x-z')^2 (z+z')^2 - x^2 z^2}{z^4 [(x-z-z')^2 z'^2 - (x-z')^2 (z+z')^2]} \ln \frac{(x-z-z')^2 z'^2}{(x-z')^2 (z+z')^2} \right\} \right] \mathcal{U}(z) \tag{104}
\end{aligned}$$

Using the integral  $J_{13}$  from hep-ph/9704267 [16] we get

$$\begin{aligned}
&\frac{1}{\pi} \int d^2z' \left[ \frac{x^4}{(x-z-z')^2 z'^2 - (x-z')^2 (z+z')^2} + \frac{x^2}{z^2} \right] \frac{1}{z'^2 (x-z-z')^2} \ln \frac{(x-z-z')^2 z'^2}{(x-z')^2 (z+z')^2} \\
&= \frac{2x^2}{z^2} \left\{ \frac{(x^2 - z^2)}{(x-z)^2 (x+z)^2} \left[ \ln \frac{x^2}{z^2} \ln \frac{x^2 z^2 (x-z)^4}{(x^2 + z^2)^4} + 2Li_2\left(-\frac{z^2}{x^2}\right) - 2Li_2\left(-\frac{x^2}{z^2}\right) \right] \right. \\
&\left. - \left( 1 - \frac{(x^2 - z^2)^2}{(x-z)^2 (x+z)^2} \right) \left[ \int_0^1 - \int_1^\infty \right] \frac{du}{(x-zu)^2} \ln \frac{u^2 z^2}{x^2} \right\} \tag{105}
\end{aligned}$$

$$\begin{aligned}
\frac{d}{d\eta}\mathcal{U}(x) &= \frac{\alpha_s N_c}{2\pi^2} \int d^2 z \frac{x^2}{(x-z)^2 z^2} \left\{ 1 + \frac{\alpha_s}{4\pi} \left[ b \ln x^2 \mu^2 - b \frac{(x-z)^2 - z^2}{x^2} \ln \frac{(x-z)^2}{z^2} + \left( \frac{67}{9} - \frac{\pi^2}{3} \right) N_c - \frac{10}{9} n_f \right. \right. \\
&\quad \left. \left. - 2N_c \ln \frac{(x-z)^2}{x^2} \ln \frac{z^2}{x^2} \right] \right\} [\mathcal{U}(x-z) + \mathcal{U}(z) - \mathcal{U}(x)] \\
&+ \frac{\alpha_s^2 N_c^2}{4\pi^3} \int d^2 z \frac{x^2}{z^2} \left\{ \left( 1 + \frac{n_f}{N_c^3} \right) \frac{3(x,z)^2 - 2x^2 z^2}{16x^2 z^2} \left( \frac{2}{x^2} + \frac{2}{z^2} + \frac{x^2 - z^2}{x^2 z^2} \right) \ln \frac{x^2}{z^2} \right. \\
&\quad - \left[ 3 + \left( 1 + \frac{n_f}{N_c^3} \right) \left( 1 - \frac{(x^2 + z^2)^2}{8x^2 z^2} + \frac{3x^4 + 3z^4 - 2x^2 z^2}{16x^4 z^4} (x,z)^2 \right) \right] \int_0^\infty dt \frac{1}{x^2 + t^2 z^2} \ln \frac{1+t}{|1-t|} \\
&\quad + \frac{(x^2 - z^2)}{(x-z)^2 (x+z)^2} \left[ \ln \frac{x^2}{z^2} \ln \frac{x^2 z^2 (x-z)^4}{(x^2 + z^2)^4} + 2Li_2\left(-\frac{z^2}{x^2}\right) - 2Li_2\left(-\frac{x^2}{z^2}\right) \right] \\
&\quad \left. - \left( 1 - \frac{(x^2 - z^2)^2}{(x-z)^2 (x+z)^2} \right) \left[ \int_0^1 - \int_1^\infty \right] \frac{du}{(x-zu)^2} \ln \frac{u^2 z^2}{x^2} \right\} \mathcal{U}(z) \tag{106}
\end{aligned}$$

### B. Comparison of eigenvalues

To compare the eigenvalues of the Eq. (106) with NLO BFKL we expand  $\mathcal{U}(x, 0)$  in eigenfunctions

$$\mathcal{U}(x_\perp, 0) = \sum_{n=-\infty}^{\infty} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{d\gamma}{2\pi i} e^{in\phi} (x_\perp^2 \mu^2)^\gamma \mathcal{U}(n, \gamma), \tag{107}$$

compute the evolution of  $\mathcal{U}(n, \gamma)$  from Eq. (106) and compare it to the calculation based on the NLO BFKL results from [12, 18]. (For the quark part of the NLO BK kernel the agreement with NLO BFKL was proved in Ref. [17]).

The relevant integrals have the form

$$\begin{aligned}
\frac{1}{2\pi} \int d^2 z [2(z^2/x^2)^\gamma e^{in\phi} - 1] \frac{x^2}{(x-z)^2 z^2} &= \chi(n, \gamma) \\
\frac{1}{\pi} \int d^2 z [2(z^2/x^2)^\gamma e^{in\phi} - 1] \left( \frac{1}{(x-z)^2} - \frac{1}{z^2} \right) \ln \frac{(x-z)^2}{z^2} &= \chi^2(n, \gamma) - \chi'(n, \gamma) - \frac{4\gamma\chi(\gamma)}{\gamma^2 - \frac{n^2}{4}} \\
\frac{1}{\pi} \int d^2 z (z^2/x^2)^\gamma \frac{x^2}{(x-z)^2 z^2} e^{in\phi} \ln \frac{(x-z)^2}{x^2} \ln \frac{z^2}{x^2} &= \frac{1}{2} \chi''(n, \gamma) + \chi'(n, \gamma) \chi(n, \gamma) \tag{108}
\end{aligned}$$

where  $\chi(n, \gamma) = 2\phi(1) - \psi(\gamma + \frac{n}{2}) - \psi(1 - \gamma + \frac{n}{2})$ , and

$$\begin{aligned}
\frac{1}{\pi} \int d^2 z (z^2/x^2)^{\gamma-1} e^{in\phi} \left\{ \left( 1 + \frac{n_f}{N_c^3} \right) \frac{3(x,z)^2 - 2x^2 z^2}{16x^2 z^2} \left( \frac{2}{x^2} + \frac{2}{z^2} + \frac{x^2 - z^2}{x^2 z^2} \right) \ln \frac{x^2}{z^2} \right. \\
\left. - \left[ 3 + \left( 1 + \frac{n_f}{N_c^3} \right) \left( 1 - \frac{(x^2 + z^2)^2}{8x^2 z^2} + \frac{3x^4 + 3z^4 - 2x^2 z^2}{16x^4 z^4} (x,z)^2 \right) \right] \int_0^\infty dt \frac{1}{x^2 + t^2 z^2} \ln \frac{1+t}{|1-t|} \right\} \\
= \left\{ - \left[ 3 + \left( 1 + \frac{n_f}{N_c^3} \right) \frac{2 + 3\gamma\bar{\gamma}}{(3-2\gamma)(1+2\gamma)} \right] \delta_{0n} + \left( 1 + \frac{n_f}{N_c^3} \right) \frac{\gamma\bar{\gamma}}{2(3-2\gamma)(1+2\gamma)} \delta_{2n} \right\} \frac{\pi^2 \cos \pi\gamma}{(1-2\gamma) \sin^2 \pi\gamma} \equiv F(n, \gamma) \tag{109}
\end{aligned}$$

$$\begin{aligned}
\frac{1}{2\pi} \int d^2 z (z^2/x^2)^{\gamma-1} e^{in\phi} \left\{ \frac{(x^2 - z^2)}{(x-z)^2 (x+z)^2} \left[ \ln \frac{x^2}{z^2} \ln \frac{x^2 z^2 (x-z)^4}{(x^2 + z^2)^4} + 2Li_2\left(-\frac{z^2}{x^2}\right) - 2Li_2\left(-\frac{x^2}{z^2}\right) \right] \right. \\
\left. - \left( 1 - \frac{(x^2 - z^2)^2}{(x-z)^2 (x+z)^2} \right) \left[ \int_0^1 - \int_1^\infty \right] \frac{du}{(x-zu)^2} \ln \frac{u^2 z^2}{x^2} \right\} = -\Phi(n, \gamma) - \Phi(n, 1-\gamma) \tag{110}
\end{aligned}$$

where [18]

$$\begin{aligned}
\Phi(n, \gamma) &= \int_0^1 \frac{dt}{1+t} t^{\gamma-1+\frac{n}{2}} \left\{ \frac{\pi^2}{12} - \frac{1}{2} \psi' \left( \frac{n+1}{2} \right) - Li_2(t) - Li_2(-t) \right. \\
&\quad \left. - \left( \psi(n+1) - \psi(1) + \ln(1+t) + \sum_{k=1}^{\infty} \frac{(-t)^k}{k+n} \right) \ln t - \sum_{k=1}^{\infty} \frac{t^k}{(k+n)^2} [1 - (-1)^k] \right\} \tag{111}
\end{aligned}$$

The convenient way to calculate the integrals over angle  $\phi$  is to represent  $\cos n\phi$  as  $T_n(\cos \phi)$  and use formulas for the integration of Chebyshev polynomials from Ref. [18].

Using the integrals (108) - (110) one easily obtains the evolution equation for  $\mathcal{U}(n, \gamma)$  in the form

$$\begin{aligned} \frac{d}{d\eta}\mathcal{U}(n, \gamma) &= \frac{\alpha_s N_c}{\pi} \left\{ \left[ 1 - \frac{b\alpha_s}{4\pi} \frac{d}{d\gamma} + \left( \frac{67}{9} - \frac{\pi^2}{3} \right) N_c - \frac{10}{9} \frac{n_f}{N_c^2} \right] \chi(n, \gamma) + \frac{\alpha_s b}{4\pi} \left[ \frac{1}{2} \chi^2(n, \gamma) - \frac{1}{2} \chi'(n, \gamma) - \frac{2\gamma \chi(n, \gamma)}{\gamma^2 - \frac{n^2}{4}} \right] \right. \\ &\left. + \frac{\alpha_s N_c}{4\pi} \left[ -\chi''(n, \gamma) - 2\chi(n, \gamma)\chi'(n, \gamma) + 4\zeta(3) + F(n, \gamma) - 2\Phi(n, \gamma) - 2\Phi(n, 1 - \gamma) \right] \right\} \mathcal{U}(n, \gamma) \end{aligned} \quad (112)$$

where  $\chi'(n, \gamma) \equiv \frac{d}{d\gamma}\chi(n, \gamma)$  etc.

Next we calculate the same thing using NLO BFKL results [12, 18]. The impact factor  $\Phi_A(q)$  for the color dipole  $\mathcal{U}(x, y)$  is proportional to  $\alpha_s(q)(e^{iqx} - e^{iqy})(e^{-iqx} - e^{-iqy})$  so one obtains the cross section of the scattering of color dipole in the form

$$\langle \mathcal{U}(x, 0) \rangle = \frac{1}{4\pi^2} \int \frac{d^2 q}{q^2} \frac{d^2 q'}{q'^2} \alpha_s(q) (e^{iqx} - 1)(e^{-iqx} - 1) \Phi_B(q') \int_{a-i\infty}^{a+i\infty} \frac{d\omega}{2\pi i} \left( \frac{s}{qq'} \right)^\omega G_\omega(q, q') \quad (113)$$

where  $G_\omega(q, q')$  is the partial wave of the forward reggeized gluon scattering amplitude satisfying the equation

$$\omega G_\omega(q, q') = \delta^{(2)}(q - q') + \int d^2 p K(q, p) G_\omega(p, q') \quad (114)$$

and  $\Phi_B(q')$  is the target impact factor. The kernel  $K(q, p)$  is symmetric with respect to  $q \leftrightarrow p$  and the eigenvalues are

$$\begin{aligned} \int d^2 p \left( \frac{p^2}{q^2} \right)^{\gamma-1} e^{in\phi} K(q, p) &= \frac{\alpha_s(q)}{\pi} N_c \left[ \chi(n, \gamma) + \frac{\alpha_s N_c}{4\pi} \delta(n, \gamma) \right], \\ \delta(n, \gamma) &= -\frac{b}{2} [\chi'(n, \gamma) + \chi^2(n, \gamma)] + \left( \frac{67}{9} - \frac{\pi^2}{3} - \frac{10}{9} \frac{n_f}{N_c^2} \right) \chi(n, \gamma) + 6\zeta(3) \\ &\quad - \chi''(n, \gamma) + F(n, \gamma) - 2\Phi(n, \gamma) - 2\Phi(n, 1 - \gamma) \end{aligned} \quad (115)$$

The corresponding expression for  $\mathcal{U}(n, \gamma)$  takes the form

$$\langle \mathcal{U}(n, \gamma) \rangle = -\frac{1}{2\pi^2} \cos \frac{\pi n}{2} \frac{\Gamma(-\gamma + \frac{n}{2})}{\Gamma(1 + \gamma + \frac{n}{2})} \int \frac{d^2 q}{q^2} \frac{d^2 q'}{q'^2} e^{-in\theta} \alpha_s(q) \left( \frac{q^2}{4\mu^2} \right)^\gamma \Phi_B(q') \int_{a-i\infty}^{a+i\infty} \frac{d\omega}{2\pi i} \left( \frac{s}{qq'} \right)^\omega G_\omega(q, q') \quad (116)$$

where  $\theta$  is the angle between  $\vec{q}$  and  $x$  axis. Using Eq. (114) we obtain

$$\begin{aligned} s \frac{d}{ds} \langle \mathcal{U}(n, \gamma) \rangle &= -\frac{1}{2\pi^2} \cos \frac{\pi n}{2} \frac{\Gamma(-\gamma + \frac{n}{2})}{\Gamma(1 + \gamma + \frac{n}{2})} \int \frac{d^2 q}{q^2} \frac{d^2 q'}{q'^2} e^{-in\theta} \alpha_s(q) \left( \frac{q^2}{4\mu^2} \right)^\gamma \Phi_B(q') \int_{a-i\infty}^{a+i\infty} \frac{d\omega}{2\pi i} \left( \frac{s}{qq'} \right)^\omega \int d^2 p K(q, p) G_\omega(p, q') \end{aligned} \quad (117)$$

The integration over  $q$  can be performed using

$$\int d^2 q \alpha_s(q) \left( \frac{q^2}{p^2} \right)^{\gamma-1} e^{in\phi} K(q, p) = \frac{\alpha_s^2(p)}{\pi} N_c \left[ \chi(n, \gamma) - \frac{b\alpha_s}{4\pi} \chi'(n, \gamma) + \frac{\alpha_s N_c}{4\pi} \delta(n, \gamma) \right] \quad (118)$$

(recall that  $K(q, p) = K(p, q)$  and  $\alpha_s(p) = \alpha_s - \frac{b\alpha_s^2}{4\pi} \ln \frac{p^2}{\mu^2}$  with our accuracy). The result is

$$\begin{aligned} s \frac{d}{ds} \langle \mathcal{U}(n, \gamma) \rangle &= -\frac{\alpha_s}{2\pi^2} \cos \frac{\pi n}{2} \frac{\Gamma(-\gamma + \frac{n}{2})}{\Gamma(1 + \gamma + \frac{n}{2})} \int \frac{d^2 p}{p^2} \frac{d^2 q'}{q'^2} e^{-in\varphi} \left( \frac{p^2}{4\mu^2} \right)^\gamma \Phi_B(q') \\ &\times \int_{a-i\infty}^{a+i\infty} \frac{d\omega}{2\pi i} \left( \frac{s}{pq'} \right)^\omega G_\omega(p, q') \frac{\alpha_s(p)}{\pi} N_c \left[ \chi(n, \gamma - \frac{\omega}{2}) - \frac{b\alpha_s}{4\pi} \chi'(n, \gamma - \frac{\omega}{2}) + \frac{\alpha_s N_c}{4\pi} \delta(n, \gamma - \frac{\omega}{2}) \right] \end{aligned} \quad (119)$$

where the angle  $\varphi$  corresponds to  $\vec{p}$ . Since  $\omega \sim \alpha_s$  we can neglect terms  $\sim \omega$  in the argument of  $\delta$  and expand  $\chi(n, \gamma - \frac{\omega}{2}) \simeq \chi(n, \gamma) - \frac{\omega}{2} \chi'(n, \gamma)$ . Using again Eq. (114) in the leading order we can replace extra  $\omega$  by  $\frac{\alpha_s}{\pi} N_c \chi(n, \gamma)$  and obtain

$$s \frac{d}{ds} \langle \mathcal{U}(n, \gamma) \rangle = - \frac{\alpha_s}{2\pi^2} \cos \frac{\pi n}{2} \frac{\Gamma(-\gamma + \frac{n}{2})}{\Gamma(1 + \gamma + \frac{n}{2})} \int \frac{d^2 p}{p^2} \frac{d^2 q'}{q'^2} e^{-in\varphi} \left( \frac{p^2}{4\mu^2} \right)^\gamma \\ \times \Phi_B(q') \int_{a-i\infty}^{a+i\infty} \frac{d\omega}{2\pi i} \left( \frac{s}{pq'} \right)^\omega G_\omega(p, q') \frac{\alpha_s^2(p)}{\pi} N_c \left[ \chi(n, \gamma) - \frac{b\alpha_s}{4\pi} \chi'(n, \gamma) + \frac{\alpha_s N_c}{4\pi} [\delta(n, \gamma) - 2\chi(n, \gamma)\chi'(n, \gamma)] \right] \quad (120)$$

Finally, expanding  $\alpha_s^2(p) \simeq \alpha_s(p) \left( \alpha_s - \frac{b\alpha_s^2}{4\pi} \ln \frac{p^2}{\mu^2} \right) \alpha_s(p)$  we obtain

$$s \frac{d}{ds} \langle \mathcal{U}(n, \gamma) \rangle = - \frac{\alpha_s N_c}{2\pi^3} \cos \frac{\pi n}{2} \frac{\Gamma(-\gamma + \frac{n}{2})}{\Gamma(1 + \gamma + \frac{n}{2})} \left\{ \chi(n, \gamma) \left( 1 - \frac{b\alpha_s}{4\pi} \frac{d}{d\gamma} \right) - \frac{b\alpha_s}{4\pi} \chi'(n, \gamma) \right. \\ \left. + \frac{\alpha_s N_c}{4\pi} [\delta(n, \gamma) - 2\chi(n, \gamma)\chi'(n, \gamma)] \right\} \int \frac{d^2 p}{p^2} \frac{d^2 q'}{q'^2} e^{-in\varphi} \alpha_s(p) \left( \frac{p^2}{4\mu^2} \right)^\gamma \Phi_B(q') \int_{a-i\infty}^{a+i\infty} \frac{d\omega}{2\pi i} \left( \frac{s}{pq'} \right)^\omega G_\omega(p, q') \quad (121)$$

which can be rewritten as an evolution equation

$$s \frac{d}{ds} \langle \mathcal{U}(n, \gamma) \rangle = \frac{\alpha_s N_c}{\pi} \left\{ \left( 1 + \frac{b\alpha_s}{4\pi} \left[ \chi(n, \gamma) - \frac{2\gamma}{\gamma^2 - \frac{n^2}{4}} - \frac{d}{d\gamma} \right] \right) \chi(n, \gamma) + \frac{\alpha_s N_c}{4\pi} [\delta(n, \gamma) - 2\chi(n, \gamma)\chi'(n, \gamma)] \right\} \langle \mathcal{U}(n, \gamma) \rangle \\ = \frac{\alpha_s N_c}{\pi} \left\{ \left[ 1 - \frac{b\alpha_s}{4\pi} \frac{d}{d\gamma} + \left( \frac{67}{9} - \frac{\pi^2}{3} \right) N_c - \frac{10 n_f}{9 N_c^2} \right] \chi(n, \gamma) + \frac{\alpha_s b}{4\pi} \left[ \frac{1}{2} \chi^2(n, \gamma) - \frac{1}{2} \chi'(n, \gamma) - \frac{2\gamma}{\gamma^2 - \frac{n^2}{4}} \chi(n, \gamma) \right] \right. \\ \left. + \frac{\alpha_s N_c}{4\pi} \left[ -\chi''(n, \gamma) - 2\chi(n, \gamma)\chi'(n, \gamma) + 6\zeta(3) + F(n, \gamma) - 2\Phi(n, \gamma) - 2\Phi(n, 1 - \gamma) \right] \right\} \langle \mathcal{U}(n, \gamma) \rangle \quad (122)$$

This eigenvalue coincides with Eq. (112) up to the extra term  $2\zeta(3)$ . It would correspond to the additional contribution to the r.h.s. of eq. (5) in the form of  $\frac{\alpha_s^2 N_c^2}{4\pi^2} \zeta(3) \text{Tr} U_x U_y^\dagger$  which contradicts the requirement  $\frac{d}{d\eta} U_x U_y^\dagger = 0$  at  $x = y$ . A possible reason for the disagreement is the connection between the matrix element of the color dipole with a rigid cutoff  $\alpha < \sigma$  and the cutoff by energy  $s$  in Eq. (113). It is worth noting that the coefficient  $6\zeta(3)$  in Eq. (122) agrees with the  $j \rightarrow 1$  asymptotics of the three-loop anomalous dimensions of leading-twist gluon operators [19].

It should be emphasized that the coincidence of terms with the nontrivial  $\gamma$  dependence proves that there is no additional  $O(\alpha_s)$  correction to the vertex of the gluon - shock wave interaction coming from the small loop inside the shock wave, see Fig. 11 (In other words, all the effects coming from the small loop in the shock wave are absorbed in the renormalization of coupling constant in the definition of the U operator (6)). In the case of quark loop, we proved that by the comparison of our results for  $\text{Tr}\{U_x U_y^\dagger\}$  in the shock-wave background with explicit light-cone calculation of the behavior of  $\text{Tr}\{U_x U_y^\dagger\}$  as  $x \rightarrow y$  [9]. For the gluon loop, we can use the NLO BFKL results as an independent calculation. Let us repeat the arguments of Ref. [9] for this case. The characteristic transverse scale inside the shock wave is small (see the discussion in Ref. [9]) and therefore the contribution of the diagram in Fig. 11 reduces to the contribution of some operator *local* in the transverse space. This would bring the additional terms with the nontrivial  $z$  dependence to the kernel which translates into the nontrivial additional  $\gamma$ -dependent term in the eigenvalues. Such terms do not exist and therefore the gluon interaction with the shock wave does not get an extra  $O(\alpha_s)$  correction.

## VII. ARGUMENT OF THE COUPLING CONSTANT IN THE BK EQUATION

In this section we briefly summarize the results of the renormalon-based analysis of the argument of the coupling constant carried in Refs. [9, 10]

To get an argument of coupling constant we can trace the quark part of the  $\beta$ -function (proportional to  $n_f$ ). In the leading log approximation  $\alpha_s \ln \frac{p^2}{\mu^2} \sim 1$ ,  $\alpha_s \ll 1$  the quark part of the  $\beta$ -function comes from the bubble chain of quark loops in the shock-wave background. We can either have no intersection of quark loop with the shock wave (see Fig. 12a) or we may have one of the loops in the shock-wave background (see Fig. 12b).

The sum of these diagrams yields

$$\frac{d}{d\eta} \text{Tr}\{U_x U_y^\dagger\} = 2\alpha_s \text{Tr}\{t^a U_x t^b U_y^\dagger\} \int \tilde{d}^2 p \tilde{d}^2 l [e^{i(p,x)_\perp} - e^{i(p,y)_\perp}] [e^{-i(p-l,x)_\perp} - e^{-i(p-l,y)_\perp}] \\ \times \frac{1}{p^2 (1 + \frac{\alpha_s}{6\pi} \ln \frac{\mu^2}{p^2})} \left( 1 - \frac{\alpha_s n_f}{6\pi} \ln \frac{l^2}{\mu^2} \right) \partial_\perp^2 U^{ab}(l) \frac{1}{(p-l)^2 (1 + \frac{\alpha_s}{6\pi} \ln \frac{\mu^2}{(p-l)^2})} \quad (123)$$

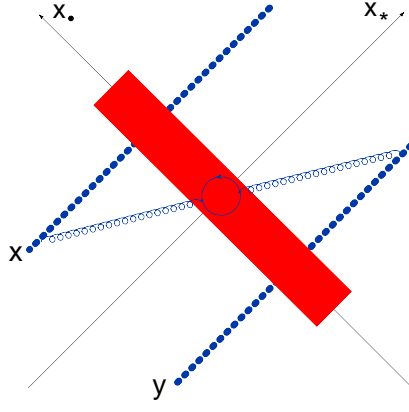


FIG. 11: Gluon loop inside the shock wave .

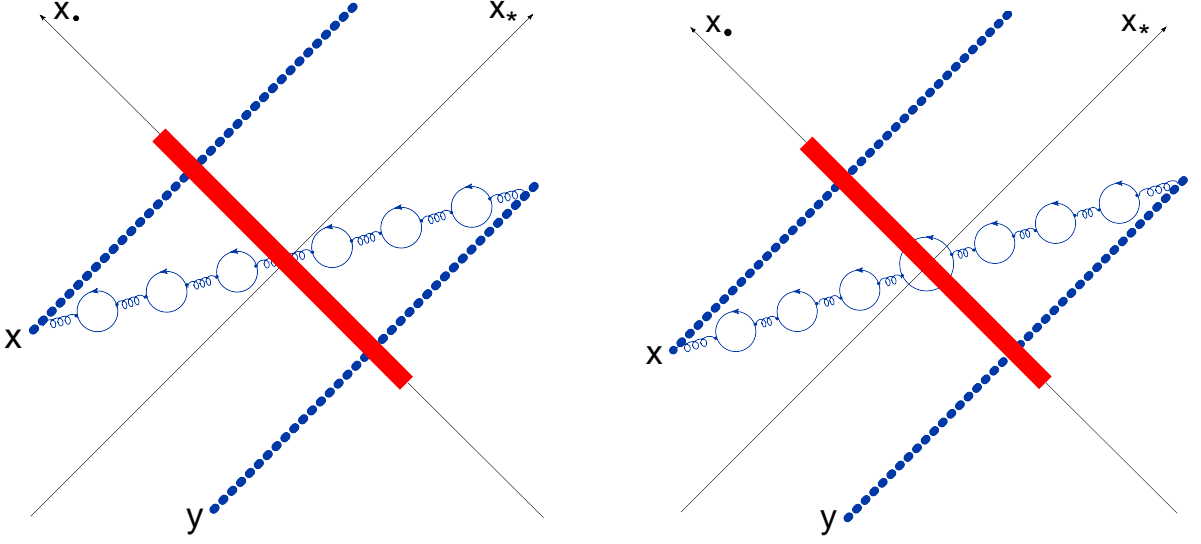


FIG. 12: Renormalon bubble chain of quark loops.

where we have left only the  $\beta$ -function part of the quark loop. Replacing the quark part of the  $\beta$ -function  $-\frac{\alpha_s}{6\pi}n_f \ln \frac{p^2}{\mu^2}$  by the total contribution  $\frac{\alpha_s}{4\pi}b \ln \frac{p^2}{\mu^2}$  we get

$$\begin{aligned} \frac{d}{d\eta} \text{Tr}\{U_x U_y^\dagger\} &= 2\text{Tr}\{t^a U_x t^b U_y^\dagger\} \\ &\times \int \tilde{d}^2 p \tilde{d}^2 q [e^{i(p,x)_\perp} - e^{i(p,y)_\perp}][e^{-i(p-l,x)_\perp} - e^{-i(p-l,y)_\perp}] \frac{\alpha_s(p^2)}{p^2} \alpha_s^{-1}(l^2) \partial_\perp^2 U^{ab}(q) \frac{\alpha_s((p-l)^2)}{(p-l)^2} \end{aligned} \quad (124)$$

In principle, one should also include the ‘‘renormalon dressing’’ of the double-log and conformal terms in Eq. (5). We think, however, that they form a separate contribution which has nothing to do with the argument of the BK equation.

To go to the coordinate space, we expand the coupling constants in Eq. (124) in powers of  $\alpha_s = \alpha_s(\mu^2)$ , i.e. return back to Eq. (123) with  $\frac{\alpha_s}{6\pi}n_f \rightarrow -b\frac{\alpha_s}{4\pi}$ . Unfortunately, the Fourier transformation to the coordinate space can be performed explicitly only for a couple of first terms of the expansion  $\alpha_s(p^2) \simeq \alpha_s - \frac{b\alpha_s}{4\pi} \ln p^2/\mu^2 + (\frac{b\alpha_s}{4\pi} \ln p^2/\mu^2)^2$ . In the first order we get the running-coupling part of the NLO BK equation (5)

$$\begin{aligned} \frac{d}{d\eta} \text{Tr}\{U_x U_y^\dagger\} & \\ &= \frac{\alpha_s}{2\pi^2} \int d^2 z [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - N_c \text{Tr}\{U_x U_y^\dagger\}] \left[ \frac{(x-y)^2}{X^2 Y^2} \left(1 + b\frac{\alpha_s}{4\pi} \ln(x-y)^2 \mu^2\right) - b\frac{\alpha_s}{4\pi} \frac{X^2 - Y^2}{X^2 Y^2} \ln \frac{X^2}{Y^2} \right] \end{aligned} \quad (125)$$

The result of the Fourier transformation up to the second order has the form[9, 10]

$$\begin{aligned} \frac{d}{d\eta} \text{Tr}\{U_x U_y^\dagger\} &= \frac{\alpha_s}{2\pi^2} \int d^2 z [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - N_c \text{Tr}\{U_x U_y^\dagger\}] \left\{ \frac{(x-y)^2}{X^2 Y^2} \left[ 1 + \frac{b\alpha_s}{4\pi} \left( \ln(x-y)^2 \mu^2 + \frac{5}{3} \right) \right. \right. \\ &+ \left. \left. \left( \frac{b\alpha_s}{4\pi} \right)^2 \ln^2(x-y)^2 \mu^2 \right] + \frac{b\alpha_s}{4\pi} \frac{1}{X^2} \ln \frac{X^2}{Y^2} \left[ 1 + \frac{b\alpha_s}{4\pi} \ln(x-y)^2 \mu^2 + \frac{b\alpha_s}{4\pi} \ln X^2 \mu^2 \right] \right. \\ &\left. - \frac{b\alpha_s}{4\pi} \frac{1}{Y^2} \ln \frac{X^2}{Y^2} \left[ 1 + \frac{b\alpha_s}{4\pi} \ln(x-y)^2 \mu^2 + \frac{b\alpha_s}{4\pi} \ln Y^2 \mu^2 \right] \right\} + \dots \end{aligned} \quad (126)$$

We extrapolate the  $\ln + \ln^2$  terms in the above equation as follows:

$$\frac{d}{d\eta} \text{Tr}\{U_x U_y^\dagger\} = \frac{\alpha_s ((x-y)^2)}{2\pi^2} \int d^2 z [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - N_c \text{Tr}\{U_x U_y^\dagger\}] \quad (127)$$

$$\times \left[ \frac{(x-y)^2}{X^2 Y^2} + \frac{1}{X^2} \left( \frac{\alpha_s(X^2)}{\alpha_s(Y^2)} - 1 \right) + \frac{1}{Y^2} \left( \frac{\alpha_s(Y^2)}{\alpha_s(X^2)} - 1 \right) \right] + \dots \quad (128)$$

where dots stand for the remaining conformal terms and  $\ln^2$  term.

When the sizes of the dipoles are very different the kernel of the above equation reduces to

$$\begin{aligned} \frac{\alpha_s ((x-y)^2)}{2\pi^2} \frac{(x-y)^2}{X^2 Y^2} &|x-y| \ll |x-z|, |y-z| \\ \frac{\alpha_s(X^2)}{2\pi^2 X^2} &|x-z| \ll |x-y|, |y-z| \\ \frac{\alpha_s(Y^2)}{2\pi^2 Y^2} &|y-z| \ll |x-y|, |x-z| \end{aligned} \quad (129)$$

In the earlier paper[9] the Eq. (127) was interpreted as an indication that the argument of the coupling constant is the size of the parent dipole  $x-y$ . We are grateful to G. Salam for pointing out that the proper interpretation is the size of the smallest dipole as follows from Eq. (129).

It is instructive to compare our result to the paper [10] where the NLO BK equation is rewritten in terms of three effective coupling constants. The authors of Ref. [10] extrapolate Eq. (126) in a different way

$$\begin{aligned} \frac{d}{d\eta} \text{Tr}\{U_x U_y^\dagger\} &= \frac{1}{2\pi^2} \int d^2 z [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - N_c \text{Tr}\{U_x U_y^\dagger\}] \\ &\times \left[ \frac{1}{X^2} \alpha_s(X^2 e^{5/3}) + \frac{1}{Y^2} \alpha_s(Y^2 e^{5/3}) - \frac{2(x-z, y-z)}{X^2 Y^2} \frac{\alpha_s(X^2 e^{5/3}) \alpha_s(Y^2 e^{5/3})}{\alpha_s(R^2)} \right] \\ &+ \frac{\alpha_s^2}{\pi^4} n_f \text{Tr}\{t^a U_x t^b U_y^\dagger\} \int \frac{d^2 z d^2 z'}{(z-z')^4} \text{Tr}\{t^a U_z t^b U_z^\dagger, -t^a U_z t^b U_z^\dagger\} \left\{ 1 - \frac{X'^2 Y^2 + Y'^2 X^2 - (x-y)^2 (z-z')^2}{2(X'^2 Y^2 - Y'^2 X^2)} \ln \frac{X'^2 Y^2}{Y'^2 X^2} \right\} \end{aligned} \quad (130)$$

where  $R^2$  is some scale interpolating between  $X^2$  and  $Y^2$  (the explicit form can be found in Ref. [10]). Theoretically, until the Fourier transformations in all orders in  $\ln p^2/\mu^2$  are performed, both of these interpretations are models of the high-order behavior of running coupling constant. The convenience of these models can be checked by the numerical estimates of the size of the neglected term(s) in comparison to terms taken into account by the model, see the discussion in Refs. [22]

## VIII. CONCLUSIONS AND OUTLOOK

We have calculated the NLO kernel for the evolution of the color dipole. It consists of three parts: the running-coupling part proportional to  $\beta$ -function (see diagrams shown in Fig. 8), the conformal part describing  $1 \rightarrow 3$  dipoles transition (diagrams in Fig. 6) and the non-conformal term coming from the diagrams in Fig. (9). The result agrees with the forward NLO BFKL kernel [12] up to a term proportional  $\alpha_s^2 \zeta(3)$  times the original dipole. We think that the difference could be due to different definitions of the cutoff in the longitudinal momenta (see the discussion in previous Section). It would be instructive to get the  $j \rightarrow 1$  asymptotics of the anomalous dimensions of gluon operators directly from Eq. (5), without a Fourier transformation of our result to the momentum space and comparing to NLO BFKL as it is done in Sect. VI. The study is in progress.

There is a recent paper [20] where the dipole form of the non-forward NLO BFKL kernel is calculated using the non-forward NLO BFKL kernel[21]. The kernel obtained in [20] is different from our result (and not conformally invariant). We think that at least part of the difference is coming from the fact that the evolution kernel (5) should



be compared to the non-symmetric “evolution” NLO BFKL kernel  $K^{\text{evol}}(q, p)$  rather than to the symmetric kernel  $K(q, p)$  defined by Eq. (113). The kernel  $K^{\text{evol}}$  corresponds to the Green function  $\tilde{G}_\omega$  defined by Eq. (113) with different lower cutoff for the longitudinal integration

$$\langle \mathcal{U}(x, 0) \rangle = \frac{1}{4\pi^2} \int \frac{d^2 q}{q^2} \frac{d^2 q'}{q'^2} \alpha_s(q) (e^{iqx} - 1) (e^{-iqx} - 1) \Phi_B(q') \int_{a-i\infty}^{a+i\infty} \frac{d\omega}{2\pi i} \left(\frac{s}{q'^2}\right)^\omega \tilde{G}_\omega(q, q') \quad (131)$$

The  $\tilde{G}_\omega(q, q')$  satisfies the equation (114) with the kernel  $K^{\text{evol}}$

$$\omega \tilde{G}_\omega(q, q') = \delta^{(2)}(q - q') + \int d^2 p K^{\text{evol}}(q, p) \tilde{G}_\omega(p, q') \quad (132)$$

and the relation between  $K^{\text{evol}}(q, p)$  and  $K(q, p)$  has the form (cf. Ref. [12])

$$K^{\text{evol}}(q, p) = K(q, p) - \frac{1}{2} \int d^2 q' K(q, q') \ln \frac{q^2}{q'^2} K(q', p) \quad (133)$$

It is easy to see that the structure (131) repeats itself after differentiation with respect to  $s$  so it can be rewritten as an evolution equation for  $\mathcal{U}(x)$  (whereas the derivative of the original formula (113) does not have the structure of the evolution equation due to an extra  $\frac{1}{|q|^\omega}$ ). In terms of eigenvalues, the modified kernel (133) lead to the shifts of the type  $\chi(n, \gamma) \rightarrow \chi(n, \gamma - \frac{\omega}{2})$  which we saw in Sect. VIB.

It should be emphasized that the conformally invariant NLO kernel describes the evolution of the light-like Wilson lines with the “rigid” cutoff in the longitudinal momenta (17). On the contrary, for dipoles with the non-light-like slope the sum of the diagrams in Fig. 6 is not conformally invariant (see Appendix). The reason is that a general Wilson line is a non-local operator which is not conformally invariant to begin with - for example, the non-light-like Wilson line turns into a circle under the inversion  $x^\mu \rightarrow x^\mu/x^2$ . With the light-like Wilson lines, the situation is different. Formally, a Wilson line

$$[\infty p_1 + x_\perp, -\infty p_1 + x_\perp] = \text{Pexp } ig \int_{-\infty}^{\infty} dx^+ A_+(x^+, x_\perp) \quad (134)$$

is invariant under the inversion  $x^\mu \rightarrow x^\mu/x^2$  (with respect to the point with zero (-) component). Indeed,  $(x^+, x_\perp)^2 = -x_\perp^2$  so after the inversion  $x_\perp \rightarrow x_\perp/x_\perp^2$  and  $x_+ \rightarrow x^+/x_\perp^2$  and therefore

$$[\infty p_1 + x_\perp, -\infty p_1 + x_\perp] \rightarrow \text{Pexp } ig \int_{-\infty}^{\infty} d\frac{x^+}{x_\perp^2} A_+\left(\frac{x^+}{x_\perp^2}, x_\perp\right) = [\infty p_1 + x_\perp, -\infty p_1 + x_\perp] \quad (135)$$

Thus, it is not surprising that the bulk of our NLO kernel for the light-like dipoles is conformally invariant in the transverse space. The part proportional to the  $\beta$ -function is not conformally invariant and should not be, but there is another term  $\sim \ln \frac{(x-y)^2}{(x-z)^2} \ln \frac{(x-y)^2}{(y-z)^2}$  which is not invariant. The reason for that is probably the cutoff  $|\alpha| < \sigma$  which can be expressed as a cutoff in longitudinal coordinate  $x^+$ , and therefore under the inversion  $x^+ \rightarrow x^+/x_\perp^2$  the cutoff can pick up some logs of transverse separations. It is worth noting that conformal and non-conformal terms come from graphs with different topology: the conformal terms come from 1→3 dipoles diagrams in Fig. (6) which describe the dipole creation while the non-conformal double-log term comes from the 1→2 dipole transitions (see Fig. 9) which can be regarded as a combination of dipole creation and dipole recombination. It is possible that in the effective action language, symmetric with respect to the projectile and the target [23], the evolution kernel is conformally invariant. We hope to study this problem in a separate publication.

### Acknowledgments

The authors are indebted to Yu.V. Kovchegov, L.N. Lipatov, G. Salam and H. Weigert for valuable discussions. I.B. would like to thank E. Iancu and other members of theory group at CEA Saclay for for valuable discussions and kind hospitality. This work was supported by contract DE-AC05-06OR23177 under which the Jefferson Science Associates, LLC operate the Thomas Jefferson National Accelerator Facility.

### IX. APPENDIX A: UV PART OF THE ONE-TO-THREE DIPOLES KERNEL

As we mentioned above, it is convenient to separate the UV-divergent and UV-finite parts of the Eq. (43) by writing down  $U_z^{mm'}U_{z'}^{nn'} = (U_z^{mm'}U_{z'}^{mm'} - U_z^{mm'}U_z^{mm'}) + U_z^{mm'}U_{z'}^{nn'}$ . The contribution of the first part leads to Eq. (58) while the second UV-divergent term have the same color structure as the leading-order BK equation. After replacing  $U_z^{mm'}U_{z'}^{nn'}$  by  $U_z^{mm'}U_z^{nn'}$ , integrating over  $u$  with the prescription (47) and changing variables to  $k_2 = q_2 = k'$ ,  $p = q_1 + q_2$ ,  $l = q_1 - k_1$  (so that  $q_1 = p - k'$ ,  $k_1 = p - l - k'$  and  $k_1 + k_2 = p - l$ ) the Eq. (43) turns into

$$\begin{aligned}
\text{Tr}\{U_x U_y^\dagger\}_{2\text{cut } z' \rightarrow z} &= \frac{g^4}{8\pi^2} \int_0^\sigma \frac{d\alpha}{\alpha} \int d^2z \left( \frac{N_c}{2} \text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - \frac{1}{2} \text{Tr}\{U_x U_y^\dagger\} \right) \\
&\times \left[ \int \bar{d}^{2-\varepsilon} p \bar{d}^{2-\varepsilon} F_1(p, l) + \int \bar{d}^2 p \bar{d}^2 l F_2(p, l) \right] (e^{i(p, X)} - e^{i(p, Y)})(e^{-i(p-l, X)} - e^{-i(p-l, Y)}) \\
&+ (e^{-i(p-l, X)} - e^{-i(p-l, Y)})(e^{i(p-k', X)+i(k', Y)} - e^{i(p-k', Y)+i(k', X)}) \\
&\quad \times \left. \frac{(k', p-k')(p-k')^2 - 2(p-k', p-l-k')(k', p-l-k')}{(p-l)^2(p-k')^2 k'^2 (p-l-k')^2} \ln \frac{(p-l-k')^2}{k'^2} \right] \\
&+ \int \bar{d}^2 p \bar{d}^2 l \bar{d}^2 k' \left\{ (e^{i(p, X)} - e^{i(p, Y)})(e^{-i(p-l-k', X)-i(k', Y)} - e^{-i(p-l-k', Y)-i(k', X)}) \right. \\
&\quad \times \left. \frac{(k', p-l-k')(p-k')^2 - 2(p-k', p-l-k')(k', p-k')}{p^2(p-k')^2 k'^2 (p-l-k')^2} \ln \frac{(p-k')^2}{k'^2} \right\} \tag{136}
\end{aligned}$$

where

$$\begin{aligned}
F_1(p, l) &= \int \bar{d}^{2-\varepsilon} k' \left( \frac{1-\varepsilon}{p^2(p-l)^2} \left\{ -2 - \frac{(p-k')^2 + (p-k'-l)^2}{(p-k')^2 - (p-k'-l)^2} \ln \frac{(p-k')^2}{(p-k'-l)^2} + \frac{k'^2 + (p-k')^2}{(p-k')^2 - k'^2} \ln \frac{(p-k')^2}{k'^2} \right. \right. \\
&+ \left. \frac{(p-k'-l)^2 + k'^2}{(p-k'-l)^2 - k'^2} \ln \frac{(p-k'-l)^2}{k'^2} \right\} + \frac{2(p, p-l)}{p^2(p-l)^2} \left\{ \left( \frac{(p-k', p-k'-l)}{(p-k')^2(p-k'-l)^2} - \frac{1}{k'^2} \right) \ln \frac{(p-k')^2(p-k'-l)^2}{k'^4} \right. \\
&- \left. \left( \frac{(p-k', p-k'-l)}{(p-k')^2} + \frac{(p-k', p-k'-l)}{(p-k'-l)^2} + 2 \right) \frac{\ln(p-k')^2/(p-k'-l)^2}{(p-k')^2 - (p-k'-l)^2} \right\} + \frac{2}{p^2} \frac{(p, p-l-k')}{(p-l-k')^2 k'^2} \ln \frac{p^2}{k'^2} \\
&+ \left. \frac{2}{(p-l)^2} \frac{(p-l, p-k')}{(p-k')^2 k'^2} \ln \frac{(p-l)^2}{k'^2} \right) \tag{137}
\end{aligned}$$

and

$$\begin{aligned}
F_2(p, l) = & \mu^{2\epsilon} \int \bar{d}^2 k' \left( 2l_i \left[ \left( -\frac{p_i}{p^2} + \frac{(p-k')_i}{(p-k')^2} \right) \frac{\ln(p-k')^2/(p-l-k')^2}{k'^2[(p-k')^2 - (p-k'-l)^2]} \right. \right. \\
& + \left. \left. \left( \frac{(p-l)_i}{(p-l)^2} - \frac{(p-l-k')_i}{(p-k'-l)^2} \right) \frac{1/k'^2}{(p-k')^2 - (p-k'-l)^2} \ln \frac{(p-k')^2}{(p-l-k')^2} \right] \right. \\
& + \frac{2}{p^2} \left[ \frac{2(l, p-k'-l)(p, k')/k'^2}{(p-k'-l)^2[(p-k')^2 - (p-k'-l)^2]} \ln \frac{(p-k')^2}{(p-k'-l)^2} - \frac{(p-k', p-l-k')(p, k')}{k'^2(p-k')^2(p-k'-l)^2} \ln \frac{(p-k')^2}{k'^2} \right. \\
& + \frac{(l, k')/k'^2}{(p-k')^2 - (p-k'-l)^2} \ln \frac{(p-k')^2}{(p-l-k')^2} - \frac{(p-k'-l, k')}{2k'^2(p-l-k')^2} \ln \frac{(p-k')^2}{k'^2} + \frac{(p-l, k')/k'^2}{(p-l-k')^2 - k'^2} \ln \frac{(p-k'-l)^2}{k'^2} \\
& + \left. \frac{2}{(p-k')^2 - (p-k'-l)^2} \ln \frac{(p-k')^2}{(p-l-k')^2} - \frac{2 \ln(p-l-k')^2/k'^2}{(p-l-k')^2 - k'^2} + \frac{(p, p-l-k')}{(p-l-k')^2 k'^2} \ln \frac{(p-k')^2}{p^2} \right] \\
& + \frac{2}{(p-l)^2} \left[ \frac{-2(l, p-k')(p-l, k')/k'^2}{(p-k')^2[(p-k')^2 - (p-k'-l)^2]} \ln \frac{(p-k')^2}{(p-k'-l)^2} - \frac{(p-k', p-l-k')(p-l, k')}{k'^2(p-k')^2(p-k'-l)^2} \ln \frac{(p-k'-l)^2}{k'^2} \right. \\
& - \frac{(l, k')/k'^2}{(p-k')^2 - (p-k'-l)^2} \ln \frac{(p-k')^2}{(p-l-k')^2} + \frac{(p, k')/k'^2}{(p-k')^2 - k'^2} \ln \frac{(p-k')^2}{k'^2} - \frac{(p-k', k')}{2k'^2(p-k')^2} \ln \frac{(p-l-k')^2}{k'^2} \\
& \left. \left. + \frac{2}{(p-k')^2 - (p-k'-l)^2} \ln \frac{(p-k')^2}{(p-l-k')^2} - \frac{2 \ln(p-k')^2/k'^2}{(p-k')^2 - k'^2} + \frac{(p-l, p-k')}{(p-k')^2 k'^2} \ln \frac{(p-l-k')^2}{(p-l)^2} \right] \right) \quad (138)
\end{aligned}$$

We need to perform the integration over  $k'$ . Let us start with the UV-divergent term  $\sim F_1$ . Using the integrals

$$\begin{aligned}
4\pi \int \bar{d}^d k' \frac{(l-k')_i}{k'^2(l-k')^2} \ln \frac{p^2}{k'^2} &= l_i \frac{\Gamma(\frac{d}{2})\Gamma(\frac{d}{2}-1)}{\Gamma(d-1)} \frac{\Gamma(2-\frac{d}{2})}{l^{4-d}} \left[ \ln \frac{p^2}{l^2} + \frac{2}{d-2} + \psi(2-\frac{d}{2}) + \psi(d-1) - \psi(\frac{d}{2}) - \psi(1) \right] \\
4\pi \int \bar{d}^d k' \frac{(k', k'-l)}{k'^2(l-k')^2} \ln \frac{(p-k')^2}{k'^2} &= \frac{1}{2} \ln \frac{p^2}{l^2} \ln \frac{(p-l)^2}{l^2} \\
4\pi \int \bar{d}^d k' \frac{(p-k', p-k'-l)}{(p-k')^2(p-k'-l)^2} \ln \frac{(p-k')^2(p-k'-l)^2}{k'^4} &= -\ln \frac{p^2}{l^2} \ln \frac{(p-l)^2}{l^2} \\
4\pi \int \bar{d}^d k' \frac{(p, p-l-k')}{(p-l-k')^2 k'^2} \ln \frac{p^2}{k'^2} \\
&= (p, p-l) \frac{\Gamma(\frac{d}{2})\Gamma(\frac{d}{2}-1)}{\Gamma(d-1)} \frac{\Gamma(2-\frac{d}{2})}{|p-l|^{4-d}} \left[ \ln \frac{p^2}{(p-l)^2} + \frac{2}{d-2} + \psi(2-\frac{d}{2}) + \psi(d-1) - \psi(\frac{d}{2}) - \psi(1) \right] \quad (140)
\end{aligned}$$

one obtains

$$\begin{aligned}
F_1(p, l) = & \frac{1}{4\pi} \left\{ \frac{2(p, p-l)}{p^2(p-l)^2} \frac{\Gamma^2(1-\frac{\epsilon}{2})}{\Gamma(2-\epsilon)} \Gamma(\epsilon/2) \left( -4 + \frac{1-\frac{\epsilon}{2}}{3-\epsilon} \right) + \frac{2(p, p-l)}{p^2(p-l)^2} \left( \frac{11}{3} \ln \frac{l^2}{\mu^2} \right. \right. \\
& \left. \left. - \ln \frac{p^2}{l^2} \ln \frac{(p-l)^2}{l^2} - \ln^2 \frac{(p-l)^2}{p^2} + \frac{\pi^2}{3} \right) - \frac{\ln p^2/l^2}{3(p-l)^2} - \frac{\ln(p-l)^2/l^2}{3p^2} + O(\epsilon) \right\} \quad (141)
\end{aligned}$$

Let us at first consider the UV-divergent contribution

$$\begin{aligned}
\frac{d}{d\sigma} \text{Tr}\{U_x U_y^\dagger\}_{\text{UV}} &= \frac{\alpha_s^2}{\pi} \mu^{2\epsilon} \int d^{2-\epsilon} z \left( \frac{N_c}{2} \text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - \frac{1}{2} \text{Tr}\{U_x U_y^\dagger\} \right) \int \bar{d}^{2-\epsilon} p \bar{d}^{2-\epsilon} l \\
&\times (e^{i(p, X)} - e^{i(p, Y)}) (e^{-i(p-l, X)} - e^{-i(p-l, Y)}) \frac{(p, p-l)}{p^2(p-l)^2} \left[ \frac{\Gamma^2(1-\frac{\epsilon}{2})}{\Gamma(2-\epsilon)} \Gamma(\epsilon/2) \left( -4 + \frac{1-\frac{\epsilon}{2}}{3-\epsilon} \right) + \frac{11}{3} \ln \frac{l^2}{\mu^2} + O(\epsilon) \right] \quad (142)
\end{aligned}$$

To this contribution we should add the counterterm corresponding to quark and gluon loops lying inside the shock wave. The rigorous calculation of the counterterm was performed in Ref. [9] and the result is

$$\begin{aligned} \frac{d}{d\sigma} \text{Tr}\{U_x U_y^\dagger\}_{\text{CT}} &= -b \frac{\alpha_s^2}{\pi} \frac{2}{\varepsilon} \int d^{2-\varepsilon} z_\perp \left( \frac{N_c}{2} \text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - \frac{1}{2} \text{Tr}\{U_x U_y^\dagger\} \right) \\ &\times \int \tilde{d}^d p \tilde{d}^d l (e^{i(p,X)} - e^{i(p,Y)}) (e^{-i(p-l,X)} - e^{-i(p-l,Y)}) \frac{(p,p-l)}{p^2(p-l)^2} \end{aligned} \quad (143)$$

where we need the gluon part of  $b$  ( $= \frac{11}{3} N_c$ ). After subtraction of the counterterm (143) the UV-divergent contribution (142) reduces to

$$\begin{aligned} \frac{d}{d\sigma} \text{Tr}\{U_x U_y^\dagger\}_{\text{UV-CT}} &= \frac{\alpha_s^2}{\pi} \int d^2 z \left( \frac{N_c}{2} \text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - \frac{1}{2} \text{Tr}\{U_x U_y^\dagger\} \right) \\ &\times \int \tilde{d}^2 p \tilde{d}^2 l (e^{i(p,X)} - e^{i(p,Y)}) (e^{-i(p-l,X)} - e^{-i(p-l,Y)}) \frac{(p,p-l)}{p^2(p-l)^2} \left[ \frac{11}{3} \ln \frac{l^2}{\mu^2} - \frac{67}{9} \right] \end{aligned} \quad (144)$$

so one obtains the regularized  $F_1$  in the form

$$F_1^{\text{reg}}(p, l) = \frac{1}{4\pi} \left[ \frac{2(p,p-l)}{p^2(p-l)^2} \left( \frac{11}{3} \ln \frac{l^2}{\mu^2} - \frac{67}{9} - \ln \frac{p^2}{l^2} \ln \frac{(p-l)^2}{l^2} - \ln^2 \frac{(p-l)^2}{p^2} + \frac{\pi^2}{3} \right) - \frac{\ln p^2/l^2}{3(p-l)^2} - \frac{\ln(p-l)^2/l^2}{3p^2} \right] \quad (145)$$

It is convenient to calculate first the Fourier transform with  $e^{i(p,X)-i(p-l,Y)}$ . Using the integrals

$$\begin{aligned} \int \tilde{d}^2 p \tilde{d}^2 l e^{i(p,\Delta)+i(l,Y)} \frac{(p,p-l)}{p^2(p-l)^2} \ln \frac{l^2}{\mu^2} &= -\frac{1}{4\pi^2} \frac{(X,Y)}{X^2 Y^2} \ln \frac{X^2 Y^2}{\Delta^2} \mu^2 \\ \int \tilde{d}^2 p \tilde{d}^2 l e^{i(p,\Delta)+i(l,Y)} \frac{(p,p-l)}{p^2(p-l)^2} \ln \frac{p^2}{l^2} \ln \frac{(p-l)^2}{l^2} &= \frac{1}{4\pi^2} \frac{(X,Y)}{X^2 Y^2} \ln \frac{X^2}{\Delta^2} \ln \frac{Y^2}{\Delta^2} \\ \int \tilde{d}^2 p \tilde{d}^2 l e^{i(p,\Delta)+i(l,Y)} \frac{(p,p-l)}{p^2(p-l)^2} \ln^2 \frac{(p-l)^2}{p^2} &= \frac{1}{4\pi^2} \frac{(X,Y)}{X^2 Y^2} \ln^2 \frac{X^2}{Y^2} \end{aligned} \quad (146)$$

we get

$$\begin{aligned} \int \tilde{d}^2 p \tilde{d}^2 l e^{i(p,\Delta)+i(l,Y)} F_1^{\text{reg}}(p, l) &= -\frac{11}{24\pi^3} \frac{(X,Y)}{X^2 Y^2} \left( \ln \frac{X^2 Y^2}{\Delta^2} \mu^2 + \frac{67}{33} \right) \\ &- \frac{1}{16\pi^3} \frac{(X,Y)}{X^2 Y^2} \left[ \ln^2 \frac{X^2}{\Delta^2} + \ln^2 \frac{Y^2}{\Delta^2} + \ln^2 \frac{X^2}{Y^2} - \frac{2\pi^2}{3} \right] - \frac{1}{48\pi^3} \left[ \frac{1}{X^2} \ln \frac{Y^2}{\Delta^2} + \frac{1}{Y^2} \ln \frac{X^2}{\Delta^2} \right] \end{aligned} \quad (147)$$

Hereafter we use the notation  $\Delta \equiv X - Y = x - y$ .

Next we calculate the  $F_2$  contribution. We need the following Fourier integrals:

$$\begin{aligned} \int \tilde{d}^2 p \tilde{d}^2 l e^{i(p,\Delta)+i(l,Y)} \int \tilde{d}^2 k' 2l_i \left[ \left( -\frac{p_i}{p^2} + \frac{(p-k')_i}{(p-k')^2} \right) \frac{\ln(p-k')^2/(p-l-k')^2}{k'^2[(p-k')^2 - (p-k'-l)^2]} \right. \\ \left. + \left( \frac{(p-l)_i}{(p-l)^2} - \frac{(p-l-k')_i}{(p-l-k')^2} \right) \frac{1/k'^2}{(p-k')^2 - (p-k'-l)^2} \ln \frac{(p-k')^2}{(p-l-k')^2} \right] &= \frac{(X,Y)}{16\pi^3 X^2 Y^2} \left[ \ln^2 \frac{X^2}{Y^2} + \frac{2\pi^2}{3} \right] \end{aligned} \quad (148)$$

$$\begin{aligned} \int \tilde{d}^2 p \tilde{d}^2 l e^{i(p,\Delta)+i(l,Y)} \left[ \frac{4}{p^2} \int \tilde{d} k' \frac{(l, p-l-k')(p, k')}{k'^2(p-k'-l)^2((p-k')^2 - (p-k'-l)^2)} \ln \frac{(p-k')^2}{(p-k'-l)^2} \right. \\ \left. - \frac{4}{(p-l)^2} \int \tilde{d} k' \frac{(l, p-k')(p-l, k')}{k'^2(p-k')^2((p-k')^2 - (p-k'-l)^2)} \ln \frac{(p-k')^2}{(p-k'-l)^2} \right] \\ = \frac{1}{16\pi^3} \left\{ -\frac{\pi^2}{3} \frac{(X+Y)^2}{X^2 Y^2} + \frac{2}{Y^2} \int_0^1 du \frac{\ln u}{u - \frac{X^2}{X^2 - Y^2}} + \frac{2}{X^2} \int_0^1 du \frac{\ln u}{u + \frac{Y^2}{X^2 - Y^2}} \right. \\ \left. + \frac{i\kappa}{X^2 Y^2} \left( 2 \int_0^1 du \left[ \frac{\ln u}{u - \frac{(X,\Delta) - i\kappa}{\Delta^2}} + \frac{\ln u}{u + \frac{(Y,\Delta) + i\kappa}{\Delta^2}} - c.c. \right] + \ln \frac{X^2 Y^2}{\Delta^4} \ln \frac{(X,Y) + i\kappa}{(X,Y) - i\kappa} \right) \right\} \end{aligned} \quad (149)$$

where  $\kappa = \sqrt{X^2 Y^2 - (X, Y)^2}$ , and

$$\int \bar{d}^2 p \bar{d}^2 l e^{i(p, \Delta) + i(l, Y)} \int \bar{d}^2 k' \left[ -\frac{2(p-k', p-l-k')(p, k')}{p^2 k'^2 (p-k')^2 (p-k'-l)^2} \ln \frac{(p-k')^2}{k'^2} - \frac{2(p-k', p-l-k')(p-l, k')}{(p-l)^2 k'^2 (p-k')^2 (p-k'-l)^2} \ln \frac{(p-k'-l)^2}{k'^2} \right] \quad (150)$$

$$\begin{aligned} &= \frac{1}{32\pi^3 X^2 Y^2} \left[ X^2 \ln^2 \frac{X^2}{\Delta^2} + Y^2 \ln^2 \frac{Y^2}{\Delta^2} + 2(X, Y) \ln \frac{X^2}{\Delta^2} \ln \frac{Y^2}{\Delta^2} \right] \\ &\quad + \frac{i\kappa}{16\pi^3 X^2 Y^2} \int_0^1 du \left[ \frac{\ln u}{u - \frac{(\Delta, X) + i\kappa}{\Delta^2}} + \frac{\ln u}{u + \frac{(\Delta, Y) - i\kappa}{\Delta^2}} - c.c. - \frac{1}{2} \ln \frac{X^2 Y^2}{\Delta^4} \ln \frac{(X, Y) + i\kappa}{(X, Y) - i\kappa} \right] \\ &\int \bar{d}^2 p \bar{d}^2 l e^{i(p, \Delta) + i(l, Y)} \int \bar{d}^2 k' \left( \frac{2(l, k') p^{-2}}{k'^2 [(p-k')^2 - (p-k'-l)^2]} \ln \frac{(p-k')^2}{(p-l-k')^2} + \frac{2(p-l, k') p^{-2}}{k'^2 [(p-k'-l)^2 - k'^2]} \right. \\ &\times \ln \frac{(p-l-k')^2}{k'^2} - \frac{2(l, k')(p-l)^{-2}}{k'^2 [(p-k')^2 - (p-k'-l)^2]} \ln \frac{(p-k')^2}{(p-l-k')^2} + \frac{2(p, k')(p-l)^{-2}}{k'^2 [(p-k')^2 - k'^2]} \ln \frac{(p-k')^2}{k'^2} \left. \right) \\ &= -\frac{1}{16\pi^3} \left( \frac{1}{Y^2} \ln \frac{X^2}{\Delta^2} - \frac{1}{X^2} \ln \frac{Y^2}{\Delta^2} \right) \ln \frac{X^2}{Y^2} + \frac{1}{8\pi^3} \left[ \frac{\pi^2}{6X^2} + \frac{\pi^2}{6Y^2} - \frac{1}{X^2} \int_0^1 \frac{du \ln u}{u + \frac{Y^2}{X^2 - Y^2}} - \frac{1}{Y^2} \int_0^1 \frac{du \ln u}{u - \frac{X^2}{X^2 - Y^2}} \right] \quad (151) \end{aligned}$$

$$\begin{aligned} &\int \bar{d}^2 p \bar{d}^2 l e^{i(p, \delta) + i(l, Y)} \int \bar{d}^2 k' \left[ -\frac{(p-k'-l, k')}{p^2 k'^2 (p-l-k')^2} \ln \frac{(p-k')^2}{k'^2} - \frac{(p-k', k')}{(p-l)^2 k'^2 (p-k')^2} \ln \frac{(p-l-k')^2}{k'^2} \right] \\ &= \frac{1}{32\pi^3} \ln \frac{X^2}{Y^2} \left[ \frac{1}{Y^2} \ln \frac{X^2}{\Delta^2} - \frac{1}{X^2} \ln \frac{Y^2}{\Delta^2} \right] \quad (152) \end{aligned}$$

$$\begin{aligned} &\int \bar{d}^2 p \bar{d}^2 l e^{i(p, \Delta) + i(l, Y)} \int \bar{d}^2 k' \left[ \frac{4/p^2}{(p-k')^2 - (p-k'-l)^2} \ln \frac{(p-k')^2}{(p-l-k')^2} - \frac{4 \ln(p-l-k')^2/k'^2}{p^2 [(p-l-k')^2 - k'^2]} \right] \quad (153) \\ &+ \frac{4}{(p-l)^2 [(p-k')^2 - (p-k'-l)^2]} \ln \frac{(p-k')^2}{(p-l-k')^2} - \frac{4 \ln(p-k')^2/k'^2}{(p-l)^2 [(p-k')^2 - k'^2]} = \frac{1}{4\pi^3 Y^2} \ln \frac{X^2}{\Delta^2} + \frac{1}{4\pi^3 X^2} \ln \frac{Y^2}{\Delta^2} \end{aligned}$$

$$\begin{aligned} &\int \bar{d}^2 p \bar{d}^2 l e^{i(p, \Delta) + i(l, Y)} \int \bar{d}^2 k' \left[ \frac{2(p, p-l-k')}{p^2 (p-l-k')^2 k'^2} \ln \frac{(p-k')^2}{p^2} + \frac{2(p-l, p-k')}{(p-k')^2 k'^2 (p-l)^2} \ln \frac{(p-l-k')^2}{(p-l)^2} \right] \quad (154) \\ &= \frac{i\kappa}{16\pi^3 X^2 Y^2} \left\{ 2 \int_0^1 du \left[ \frac{\ln u}{u - \frac{(X, \Delta) + i\kappa}{\Delta^2}} + \frac{\ln u}{u + \frac{(Y, \Delta) - i\kappa}{\Delta^2}} - c.c. \right] - \ln \frac{X^2 Y^2}{\Delta^4} \ln \frac{(X, Y) + i\kappa}{(X, Y) - i\kappa} \right\} + \frac{(X, Y)}{16\pi^3 X^2 Y^2} \ln^2 \frac{X^2}{Y^2} \end{aligned}$$

Adding the integrals (148) - (152) we obtain

$$\begin{aligned} &\int \bar{d}^2 p \bar{d}^2 l e^{i(p, \Delta) + i(l, Y)} F_2(p, l) \quad (155) \\ &= \frac{1}{4\pi^3} \left[ \frac{1}{X^2} \ln \frac{Y^2}{\Delta^2} + \frac{1}{Y^2} \ln \frac{X^2}{\Delta^2} \right] + \frac{(X+Y)^2}{32\pi^3 X^2 Y^2} \ln \frac{X^2}{\Delta^2} \ln \frac{Y^2}{\Delta^2} + \frac{(X, Y)}{8\pi^3 X^2 Y^2} \ln^2 \frac{X^2}{Y^2} \\ &\quad + \frac{i\kappa}{16\pi^3 X^2 Y^2} \left\{ \int_0^1 du \left[ \frac{\ln u}{u - \frac{(\Delta, X) + i\kappa}{\Delta^2}} + \frac{\ln u}{u + \frac{(\Delta, Y) - i\kappa}{\Delta^2}} - c.c. \right] - \frac{1}{2} \ln \frac{X^2 Y^2}{\Delta^4} \ln \frac{(X, Y) + i\kappa}{(X, Y) - i\kappa} \right\} \end{aligned}$$

and therefore

$$\begin{aligned} &\int \bar{d}^2 p \bar{d}^2 l e^{i(p, \Delta) + i(l, Y)} [F_1^{\text{reg}}(p, l) + F_2(p, l)] \\ &= -\frac{1}{8\pi^3} \frac{(X, Y)}{X^2 Y^2} \left[ \frac{11}{3} \ln \frac{X^2 Y^2}{\Delta^2} \mu^2 + \frac{67}{9} - \frac{\pi^2}{3} \right] + \frac{1}{32\pi^3} \frac{\Delta^2}{X^2 Y^2} \ln \frac{X^2}{\Delta^2} \ln \frac{Y^2}{\Delta^2} + \frac{11}{48\pi^3} \left[ \frac{1}{X^2} \ln \frac{Y^2}{\Delta^2} + \frac{1}{Y^2} \ln \frac{X^2}{\Delta^2} \right] \\ &\quad + \frac{i\kappa}{16\pi^3 X^2 Y^2} \left\{ \int_0^1 du \left[ \frac{\ln u}{u - \frac{(\Delta, X) + i\kappa}{\Delta^2}} + \frac{\ln u}{u + \frac{(\Delta, Y) - i\kappa}{\Delta^2}} - c.c. \right] - \frac{1}{2} \ln \frac{X^2 Y^2}{\Delta^4} \ln \frac{(X, Y) + i\kappa}{(X, Y) - i\kappa} \right\} \quad (156) \end{aligned}$$

Note that the r.h.s. of this equation is finite as  $X \rightarrow Y$  (taken separately, the contributions of  $F_1$  and  $F_2$  are singular in this limit):

$$\int \tilde{d}^2 p \tilde{d}^2 l e^{i(l,X)} [F_1^{\text{reg}}(p, l) + F_2(p, l)] = -\frac{1}{8\pi^3} \frac{1}{X^2} \left[ \frac{11}{3} \ln X^2 \mu^2 + \frac{67}{9} - \frac{\pi^2}{3} \right] \quad (157)$$

Using Eqs. (156) and (157) we obtain

$$\begin{aligned} & \int \tilde{d}^2 p \tilde{d}^2 l e^{i(p,\Delta)+i(l,Y)} [F_1^{\text{reg}}(p, l) + F_2(p, l)] (e^{i(p,X)} - e^{i(p,Y)})(e^{-i(p-l,X)} - e^{-i(p-l,Y)}) \\ &= -\frac{1}{8\pi^3} \frac{\Delta^2}{X^2 Y^2} \left[ \frac{11}{3} \ln \frac{X^2 Y^2}{\Delta^2} \mu^2 + \frac{67}{9} - \frac{\pi^2}{3} \right] - \frac{1}{16\pi^3} \frac{\Delta^2}{X^2 Y^2} \ln \frac{X^2}{\Delta^2} \ln \frac{Y^2}{\Delta^2} \\ & - \frac{i\kappa}{8\pi^3 X^2 Y^2} \left\{ \int_0^1 du \left[ \frac{\ln u}{u - \frac{(\Delta, X) + i\kappa}{\Delta^2}} + \frac{\ln u}{u + \frac{(\Delta, Y) - i\kappa}{\Delta^2}} - c.c. \right] - \frac{1}{2} \ln \frac{X^2 Y^2}{\Delta^4} \ln \frac{(X, Y) + i\kappa}{(X, Y) - i\kappa} \right\} \end{aligned} \quad (158)$$

Now we turn our attention to the two last terms in Eq. (136). Using Fourier transformation

$$\begin{aligned} & \int \tilde{d}^2 k_1 \tilde{d}^2 k_2 e^{-i(k_1, x_1) - i(k_2, x_2)} \frac{k_{2i}}{(k_1 + k_2)^2 k_1^2 k_2^2} \ln \frac{k_1^2}{k_2^2} \\ &= \frac{i}{8\pi^2} \left( x_{1i} - \frac{(x_1, x_{12})}{x_{12}^2} x_{12i} \right) \frac{1}{i\kappa_{12}} \left\{ \int_0^1 du \left[ \frac{\ln u}{u - \frac{(x_1, x_{12}) - i\kappa}{x_1^2}} - c.c. \right] \right. \\ & \quad \left. + \frac{1}{2} \ln \frac{x_1^2}{x_{12}^2} \ln \frac{(x_2, x_{12}) + i\kappa_{12}}{(x_2, x_{12}) - i\kappa_{12}} \right\} + \frac{ix_{12i}}{16\pi^2 x_{12}^2} \ln \frac{x_1^2}{x_{12}^2} \ln \frac{x_1^2}{x_2^2} \end{aligned} \quad (159)$$

(where  $x_{12} \equiv x_1 - x_2$  and  $\kappa_{12} \equiv \sqrt{x_1^2 x_2^2 - (x_1, x_2)^2}$ ) one easily obtains

$$\begin{aligned} & \int \tilde{d}^2 p \tilde{d}^2 l \tilde{d}^2 k' (e^{-i(p-l,X)} - e^{-i(p-l,Y)})(e^{i(p-k',X)+i(k',Y)} - e^{i(p-k',Y)+i(k',X)}) \frac{(k', p - k')}{(p-l)^2 k'^2 (p-l-k')^2} \ln \frac{(p-l-k')^2}{k'^2} \\ &= \frac{i\kappa}{16\pi^3 X^2 Y^2} \left[ \int_0^1 du \left( \frac{\ln u}{u - \frac{(X,Y) - i\kappa}{X^2}} + \frac{\ln u}{u - \frac{(X,Y) - i\kappa}{Y^2}} - c.c. \right) + \frac{1}{2} \ln \frac{X^2}{Y^2} \ln \frac{[(\Delta, X) + i\kappa][(\Delta, Y) + i\kappa]}{[(\Delta, X) - i\kappa][(\Delta, Y) - i\kappa]} \right] \\ &= \frac{i\kappa}{16\pi^3 X^2 Y^2} \left[ -\int_0^1 du \left[ \frac{\ln u}{u - \frac{(\Delta, X) + i\kappa}{\Delta^2}} + \frac{\ln u}{u + \frac{(\Delta, Y) - i\kappa}{\Delta^2}} - c.c. \right] - \frac{1}{2} \ln \frac{X^2 Y^2}{\Delta^4} \ln \frac{(X, Y) + i\kappa}{(X, Y) - i\kappa} \right] - \frac{(X, Y)}{32\pi^3 X^2 Y^2} \ln^2 \frac{X^2}{Y^2} \end{aligned} \quad (160)$$

Similarly,

$$\begin{aligned} & \int \tilde{d}^2 k_1 \tilde{d}^2 k_2 e^{-i(k_1, x_1) - i(k_2, x_2)} \frac{(k_1, k_2) k_{1i}}{(k_1 + k_2)^2 k_1^2 k_2^2} \ln \frac{k_1^2}{k_2^2} \\ &= \frac{i}{16\pi^2} \left( x_{1i} - \frac{(x_1, x_{12})}{x_{12}^2} x_{12i} \right) \frac{1}{i\kappa} \left[ \int_0^1 du \left[ \frac{\ln u}{u - \frac{(x_1, x_{12}) - i\kappa}{x_1^2}} - c.c. \right] + \frac{1}{2} \ln \frac{x_1^2}{x_{12}^2} \ln \frac{(x_2, x_{12}) + i\kappa}{(x_2, x_{12}) - i\kappa} \right] \\ & - \frac{i}{16\pi^2} \left( x_{2i} - \frac{(x_2, x_{12})}{x_{12}^2} x_{12i} \right) \frac{1}{i\kappa} \left[ \int_0^1 du \left[ \frac{\ln u}{u + \frac{(x_2, x_{12}) - i\kappa}{x_2^2}} - c.c. \right] + \frac{1}{2} \ln \frac{x_2^2}{x_{12}^2} \ln \frac{(x_1, x_{12}) + i\kappa}{(x_1, x_{12}) - i\kappa} \right] + \frac{ix_{12i}}{32\pi^2 x_{12}^2} \ln \frac{x_1^2 x_2^2}{x_{12}^4} \ln \frac{x_1^2}{x_2^2} \\ & + \frac{i}{16\pi^2} \left( x_{12i} - \frac{(x_1, x_{12})}{x_1^2} x_{1i} \right) \frac{1}{i\kappa} \left[ \int_0^1 du \left[ \frac{\ln u}{u - \frac{(x_1, x_{12}) - i\kappa}{x_{12}^2}} - c.c. \right] - \frac{1}{2} \ln \frac{x_1^2}{x_{12}^2} \ln \frac{(x_1, x_2) + i\kappa}{(x_1, x_2) - i\kappa} \right] - \frac{ix_{1i}}{32\pi^2 x_1^2} \ln \frac{x_1^2}{x_2^2} \ln \frac{x_2^2}{x_{12}^2} \end{aligned} \quad (161)$$

and therefore

$$\begin{aligned}
& \int \tilde{d}^2 p \tilde{d}^2 l \tilde{d}^2 k' (e^{-i(p-l,X)} - e^{-i(p-l,Y)}) (e^{i(p-k',X)+i(k',Y)} - e^{i(p-k',Y)+i(k',X)}) \\
& \quad \times \frac{2(p-k', p-l-k')(k', p-l-k')}{(p-l)^2 (p-k')^2 k'^2 (p-l-k')^2} \ln \frac{(p-l-k')^2}{k'^2} \\
& = \frac{i\kappa Y^{-2}}{16\pi^3 X^2} \left[ \int_0^1 du \left( -2 \frac{\ln u}{u - \frac{(\Delta, X) + i\kappa}{\Delta^2}} - 2 \frac{\ln u}{u + \frac{(\Delta, Y) - i\kappa}{\Delta^2}} - c.c. \right) + \ln \frac{X^2 Y^2}{\Delta^4} \ln \frac{(X, Y) + i\kappa}{(X, Y) - i\kappa} \right] \\
& \quad - \frac{(X, Y)}{32\pi^3 X^2 Y^2} \ln \frac{X^2 \Delta^2}{Y^4} \ln \frac{X^2}{\Delta^2} - \frac{(X, Y)}{32\pi^3 X^2 Y^2} \ln \frac{Y^2 \Delta^2}{X^4} \ln \frac{Y^2}{\Delta^2} - \frac{1}{32\pi^3} \left( \frac{1}{X^2} + \frac{1}{Y^2} \right) \ln \frac{X^2}{Y^2} \ln \frac{Y^2}{\Delta^2} \quad (162)
\end{aligned}$$

Adding the equations (160) and (162) we obtain

$$\begin{aligned}
& \int \tilde{d}^2 p \tilde{d}^2 l \tilde{d}^2 k' (e^{-i(p-l,X)} - e^{-i(p-l,Y)}) (e^{i(p-k',X)+i(k',Y)} - e^{i(p-k',Y)+i(k',X)}) \quad (163) \\
& \quad \times \frac{(k', p-k')(p-k')^2 - 2(p-k', p-l-k')(k', p-l-k')}{(p-l)^2 (p-k')^2 k'^2 (p-l-k')^2} \ln \frac{(p-l-k')^2}{k'^2} \\
& = \frac{i\kappa}{16\pi^3 X^2 Y^2} \left[ \int_0^1 du \left( \frac{\ln u}{u - \frac{(\Delta, X) + i\kappa}{\Delta^2}} + \frac{\ln u}{u + \frac{(\Delta, Y) - i\kappa}{\Delta^2}} - c.c. \right) - \frac{1}{2} \ln \frac{X^2 Y^2}{\Delta^4} \ln \frac{(X, Y) + i\kappa}{(X, Y) - i\kappa} \right] + \frac{\Delta^2}{32\pi^3 X^2 Y^2} \ln \frac{X^2}{\Delta^2} \ln \frac{Y^2}{\Delta^2}
\end{aligned}$$

It is easy to see that the contribution of the last term in Eq. (136) is equal to (163) so we get

$$\begin{aligned}
\langle \text{Tr} \{ \hat{U}_x \hat{U}_y^\dagger \} \rangle_{\text{Fig. 6 } z \rightarrow z'} & = \frac{g^4}{8\pi^2} \int_0^\sigma \frac{d\alpha}{\alpha} \int d^2 z \left[ \frac{N_c}{2} \text{Tr} \{ U_x U_z^\dagger \} \text{Tr} \{ U_z U_y^\dagger \} - \frac{1}{2} \text{Tr} \{ U_x U_y^\dagger \} \right] \\
& \times \left[ \int \tilde{d}^2 p \tilde{d}^2 l [F_1^{\text{reg}}(p, l) + F_2(p, l)] (e^{i(p,X)} - e^{i(p,Y)}) (e^{-i(p-l,X)} - e^{-i(p-l,Y)}) \right. \\
& \left. + 2 \int \tilde{d}^2 p \tilde{d}^2 l \tilde{d}^2 k' (e^{-i(p-l,X)} - e^{-i(p-l,Y)}) (e^{i(p-k',X)+i(k',Y)} - e^{i(p-k',Y)+i(k',X)}) \right. \\
& \quad \left. \times \frac{(k', p-k')(p-k')^2 - 2(p-k', p-l-k')(k', p-l-k')}{(p-l)^2 (p-k')^2 k'^2 (p-l-k')^2} \ln \frac{(p-l-k')^2}{k'^2} \right] \\
& = -\frac{\alpha_s^2 N_c}{8\pi^3} \int d^2 z \left[ \text{Tr} \{ U_x U_z^\dagger \} \text{Tr} \{ U_z U_y^\dagger \} - \frac{1}{N_c} \text{Tr} \{ U_x U_y^\dagger \} \right] \frac{\Delta^2}{X^2 Y^2} \left[ \frac{11}{3} \ln \frac{X^2 Y^2}{\Delta^2} \mu^2 + \frac{67}{9} - \frac{\pi^2}{3} \right] \quad (164)
\end{aligned}$$

Note that the dilogarithms and products of logarithms have canceled. The simplicity of the final result indicates that there should be a less tedious derivation but we were not able to find it.

## X. APPENDIX B: CUTOFF DEPENDENCE OF THE NLO KERNEL.

We will repeat the procedure from Sect. (III C), this time using the cutoff by the slope.

$$\langle K_{\text{NLO}} \text{Tr} \{ \hat{U}_x \hat{U}_y^\dagger \} \rangle_{\text{shockwave}} = \frac{\partial}{\partial \eta} \langle \text{Tr} \{ \hat{U}_x \hat{U}_y^\dagger \} \rangle_{\text{shockwave}} - \langle K_{\text{LO}} \text{Tr} \{ \hat{U}_x \hat{U}_y^\dagger \} \rangle_{\text{shockwave}} \quad (165)$$

Instead of Eq. (18) we get

$$\begin{aligned}
& g^4 \int_0^\infty du \int_{-\infty}^0 dv \langle \hat{A}_\bullet^a(un + x_\perp) \hat{A}_\bullet^b(vn + y_\perp) \rangle \quad (166) \\
& = \frac{1}{2} g^2 \frac{s^2}{4} f^{anl} f^{bn'l'} \int \tilde{d}\alpha \tilde{d}\alpha_1 \tilde{d}\beta \tilde{d}\beta' \tilde{d}\beta_1 \tilde{d}\beta_1' \tilde{d}\beta_2 \tilde{d}\beta_2' \int d^2 z d^2 z' \int \tilde{d}^2 q_1 \tilde{d}^2 q_2 \tilde{d}^2 k_1 \tilde{d}^2 k_2 e^{i(q_1+q_2, x)_\perp - i(k_1+k_2, y)_\perp} \\
& \quad \frac{4\alpha_1(\alpha - \alpha_1) U_z^{nm'} U_z^{l'l'} e^{-i(q_1-k_1, z)_\perp - i(q_2-k_2, z')_\perp} d_{\bullet\lambda}(\alpha p_1 + \beta p_2 + q_{1\perp} + k_{1\perp}) d_{\lambda'}(\alpha p_1 + \beta' p_2 + q_{2\perp} + k_{2\perp})}{(\beta - \beta_1 - \beta_2 + i\epsilon)(\beta' - \beta_1' - \beta_2' + i\epsilon)(\beta + \xi\alpha - i\epsilon)(\beta' + \xi\alpha' - i\epsilon)[\alpha\beta s - (q_1 + q_2)_\perp^2 + i\epsilon][\alpha\beta' s - (k_1 + k_2)_\perp^2 + i\epsilon]} \\
& \quad \frac{d_{\mu\xi}(\alpha_1 p_1 + \beta_1 p_2 + q_{1\perp}) d_{\nu'}^{\xi}(\alpha_1 p_1 + \beta_1' p_2 + k_{1\perp}) d_{\nu\eta}((\alpha - \alpha_1)p_1 + \beta_2 p_2 + q_{2\perp}) d_{\nu'}^{\eta}((\alpha - \alpha_1)p_1 + \beta_2' p_2 + k_{2\perp})}{\alpha_1 \beta_1 s - q_{1\perp}^2 + i\epsilon} \frac{d_{\nu'}^{\xi}(\alpha_1 p_1 + \beta_1' p_2 + k_{1\perp})}{\alpha_1 \beta_1' s - k_{1\perp}^2 + i\epsilon} \frac{d_{\nu\eta}((\alpha - \alpha_1)p_1 + \beta_2 p_2 + q_{2\perp})}{(\alpha - \alpha_1)\beta_2 s - q_{2\perp}^2 + i\epsilon} \frac{d_{\nu'}^{\eta}((\alpha - \alpha_1)p_1 + \beta_2' p_2 + k_{2\perp})}{(\alpha - \alpha_1)\beta_2' s - k_{2\perp}^2 + i\epsilon} \\
& \Gamma^{\mu\nu\lambda}(\alpha p_1 + q_{1\perp}, (\alpha - \alpha_1)p_1 + q_{2\perp}, -\alpha p_1 - q_{1\perp} - q_{2\perp}) \Gamma^{\mu'\nu'\lambda'}(\alpha p_1 + k_{1\perp}, (\alpha - \alpha_1)p_1 + k_{2\perp}, -\alpha p_1 - k_{1\perp} - k_{2\perp})
\end{aligned}$$

where  $\xi = e^{-2m}$ . In this formula  $\frac{1}{\beta + \xi\alpha - i\epsilon}$  comes from the integration over  $u$  parameter in the l.h.s. and  $\frac{1}{\beta' + \xi\alpha' - i\epsilon}$  from the integration over  $v$  parameter.

Taking residues at  $\beta = -\xi\alpha$  and  $\beta' = -\xi\alpha'$  and  $\beta_2 = -\beta_1$ ,  $\beta'_2 = -\beta'_1$  we obtain

$$\begin{aligned}
& \int_0^\infty du \int_{-\infty}^0 dv \langle \hat{A}_\bullet^a(un + x_\perp) \hat{A}_\bullet^b(vn + y_\perp) \rangle \\
&= \frac{1}{2} g^2 \frac{s^2}{4} f^{anl} f^{bn'l'} \int \dot{d}\alpha \dot{d}\alpha_1 \dot{d}\beta_1 \dot{d}\beta_1' \int d^2z d^2z' \int \dot{d}^2q_1 \dot{d}^2q_2 \dot{d}^2k_1 \dot{d}^2k_2 e^{i(q_1+q_2, x)_\perp - i(k_1+k_2, y)_\perp} \\
& 4 \frac{\alpha_1(\alpha - \alpha_1)}{\alpha^2} U_z^{nn'} U_{z'}^{ll'} e^{-i(q_1-k_1)z - i(q_2-k_2)z'} \frac{(q_{1\perp} + q_{2\perp})_\lambda}{(q_1 + q_2)_\perp^2 + \xi\alpha^2} \frac{(k_{1\perp} + k_{2\perp})_{\lambda'}}{(k_1 + k_2)_\perp^2 + \xi\alpha'^2} \\
& \frac{d_\mu^\xi(\alpha_1 p_1 + q_{1\perp})}{\alpha_1 \beta_1 s - q_{1\perp}^2 + i\epsilon} \frac{d_{\xi\mu'}(\alpha_1 p_1 + k_{1\perp})}{\alpha_1 \beta_1' s - k_{1\perp}^2 + i\epsilon} \frac{d_\eta^\eta((\alpha - \alpha_1)p_1 + q_{2\perp})}{-(\alpha - \alpha_1)\beta_1 s - q_{2\perp}^2 + i\epsilon} \frac{d_{\eta\nu'}((\alpha - \alpha_1)p_1 + k_{2\perp})}{-(\alpha - \alpha_1)\beta_1' s - k_{2\perp}^2 + i\epsilon} \\
& \Gamma^{\mu\nu\lambda}(\alpha_1 p_1 + q_{1\perp}, (\alpha - \alpha_1)p_1 + q_{2\perp}, -\alpha p_1 - q_{1\perp} - q_{2\perp}) \Gamma^{\mu'\nu'\lambda'}(\alpha_1 p_1 + k_{1\perp}, (\alpha - \alpha_1)p_1 + k_{2\perp}, -\alpha p_1 - k_{1\perp} - k_{2\perp})
\end{aligned} \tag{167}$$

which leads to (cf. Eq. (23))

$$\begin{aligned}
& \frac{d}{d\eta} \langle \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle_{\text{Fig.3a}} = \frac{g^4}{4\pi^2} \text{Tr}\{t^a U_x t^b U_y^\dagger\} f^{anl} f^{bn'l'} \int d^2z d^2z' U_z^{nn'} U_{z'}^{ll'} \\
& \times \xi \frac{d}{d\xi} \int_0^\infty \frac{d\alpha}{\alpha} \int_0^1 du \bar{u} u \int \dot{d}^2q_1 \dot{d}^2q_2 \dot{d}^2k_1 \dot{d}^2k_2 \frac{e^{i(q_1, X)_\perp + i(q_2, X')_\perp - i(k_1, Y)_\perp - i(k_2, Y')_\perp}}{[(q_1 + q_2)^2 + \xi\alpha^2][(k_1 + k_2)^2 + \xi\alpha'^2](q_1^2 \bar{u} + q_2^2 u)(k_1^2 \bar{u} + k_2^2 u)} \\
& \times \left[ (q_1^2 - q_2^2) \delta_{ij} - \frac{2}{u} q_{1i}(q_1 + q_2)_j + \frac{2}{\bar{u}} (q_1 + q_2)_i q_{2j} \right] \left[ (k_1^2 - k_2^2) \delta_{ij} - \frac{2}{u} k_{1i}(k_1 + k_2)_j + \frac{2}{\bar{u}} (k_1 + k_2)_i k_{2j} \right] \\
& = \frac{g^4}{4\pi^2} \text{Tr}\{t^a U_x t^b U_y^\dagger\} f^{anl} f^{bn'l'} \int d^2z d^2z' U_z^{nn'} U_{z'}^{ll'} \\
& \times \xi \frac{d}{d\xi} \int_0^\infty d\alpha \int_0^\alpha d\alpha' \int \dot{d}^2q_1 \dot{d}^2q_2 \dot{d}^2k_1 \dot{d}^2k_2 \frac{(\alpha - \alpha')\alpha' e^{i(q_1, x-z)_\perp + i(q_2, x-z')_\perp - i(k_1, y-z)_\perp - i(k_2, y-z')_\perp}}{[(q_1 + q_2)^2 + \xi\alpha^2][(k_1 + k_2)^2 + \xi\alpha'^2](q_1^2(\alpha - \alpha') + q_2^2\alpha')(k_1^2(\alpha - \alpha') + k_2^2\alpha')} \\
& \times \left[ \frac{\delta_{ij}}{\alpha} (q_1^2 - q_2^2) - \frac{2}{\alpha'} q_{1i}(q_1 + q_2)_j + \frac{2}{\alpha - \alpha'} (q_1 + q_2)_i q_{2j} \right] \left[ \frac{\delta_{ij}}{\alpha'} (k_1^2 - k_2^2) - \frac{2}{\alpha'} k_{1i}(k_1 + k_2)_j + \frac{2}{\alpha - \alpha'} (k_1 + k_2)_i k_{2j} \right]
\end{aligned} \tag{168}$$

(recall that  $\frac{d}{d\eta} = -2\xi \frac{d}{d\xi}$ ). The contribution which is sensitive to the subtraction of (LO)<sup>2</sup> is

$$\begin{aligned}
& \frac{d}{d\eta} \langle \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle_{\text{Fig.3a}} = \frac{g^4}{\pi^2} \text{Tr}\{t^a U_x t^b U_y^\dagger\} f^{anl} f^{bn'l'} \int d^2z d^2z' U_z^{nn'} U_{z'}^{ll'} \\
& \times \xi \frac{d}{d\xi} \int_0^\infty d\alpha \int_0^\alpha d\alpha' \int \dot{d}^2q_1 \dot{d}^2q_2 \dot{d}^2k_1 \dot{d}^2k_2 \frac{e^{i(q_1, X)_\perp + i(q_2, X')_\perp - i(k_1, Y)_\perp - i(k_2, Y')_\perp}}{[(q_1 + q_2)^2 + \xi\alpha^2][(k_1 + k_2)^2 + \xi\alpha'^2](q_1^2(\alpha - \alpha') + q_2^2\alpha')(k_1^2(\alpha - \alpha') + k_2^2\alpha')} \\
& \times \left[ \frac{\alpha}{\alpha'} (q_1, k_1) + \frac{\alpha}{\alpha - \alpha'} (q_2, k_2) \right] (q_1 + q_2, k_1 + k_2)
\end{aligned} \tag{169}$$

The “+”-prescription (47) leads to the subtraction

$$\begin{aligned}
& \frac{d}{d\eta} \langle \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle_{\text{Fig.3a}} \\
&= \xi \frac{d}{d\xi} \frac{g^4}{\pi^2} \text{Tr}\{t^a U_x t^b U_y^\dagger\} f^{anl} f^{bn'l'} \int d^2z d^2z' U_z^{nn'} U_{z'}^{ll'} \int_0^\infty \frac{d\alpha}{\alpha} \int \dot{d}^2q_1 \dot{d}^2q_2 \dot{d}^2k_1 \dot{d}^2k_2 (q_1 + q_2, k_1 + k_2) \\
& \times \frac{e^{i(q_1, X)_\perp + i(q_2, X')_\perp - i(k_1, Y)_\perp - i(k_2, Y')_\perp}}{[(q_1 + q_2)^2 + \xi\alpha^2][(k_1 + k_2)^2 + \xi\alpha'^2]} \left\{ \int_0^\alpha \frac{d\alpha'}{\alpha'} \left[ \frac{\alpha^2(q_1, k_1)}{[q_1^2(\alpha - \alpha') + q_2^2\alpha'] [k_1^2(\alpha - \alpha') + k_2^2\alpha']} - \frac{(q_1, k_1)}{q_1^2 k_1^2} \right] \right. \\
& \quad \left. + \int_0^\alpha \frac{d\alpha'}{\alpha - \alpha'} \left[ \frac{\alpha^2(q_2, k_2)}{[q_1^2(\alpha - \alpha') + q_2^2\alpha'] [k_1^2(\alpha - \alpha') + k_2^2\alpha']} - \frac{(q_2, k_2)}{q_2^2 k_2^2} \right] \right\}
\end{aligned} \tag{170}$$

The details of the upper cutoff in  $\alpha$  do not matter since they correspond to changes in the impact factor which do



not affect the evolution. For example,

$$\begin{aligned}
& -2\xi \frac{d}{d\xi} \int_0^\infty \frac{d\alpha}{\alpha} \frac{1}{[(q_1+q_2)^2 + \xi\alpha^2][(k_1+k_2)^2 + \xi\alpha^2]} \int_0^\alpha \frac{d\alpha'}{\alpha'} \left[ \frac{\alpha^2(q_1, k_1)}{(q_1^2(\alpha-\alpha') + q_2^2\alpha')[k_1^2(\alpha-\alpha') + k_2^2\alpha']} - \frac{(q_1, k_1)}{q_1^2 k_1^2} \right] \\
&= \frac{1}{(q_1+q_2)^2(k_1+k_2)^2} \int_0^1 \frac{du}{u} \left[ \frac{(q_1, k_1)}{(q_1^2\bar{u} + q_2^2u)[k_1^2\bar{u} + k_2^2u]} - \frac{(q_1, k_1)}{q_1^2 k_1^2} \right] \\
&= \frac{d}{d\sigma} \int_0^\sigma \frac{d\alpha}{\alpha} \frac{1}{(q_1+q_2)^2(k_1+k_2)^2} \int_0^\alpha \frac{d\alpha'}{\alpha'} \left[ \frac{\alpha^2(q_1, k_1)}{(q_1^2(\alpha-\alpha') + q_2^2\alpha')[k_1^2(\alpha-\alpha') + k_2^2\alpha']} - \frac{(q_1, k_1)}{q_1^2 k_1^2} \right] \tag{171}
\end{aligned}$$

where the last line is exactly our ‘‘rigid cutoff’’ with ‘‘+’’ subtraction (47).

On the contrary, the details of the upper cutoff in  $\alpha'$  are essential for the evolution equation (165). The contribution to  $\langle K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle$  corresponding to the ‘‘slope’’ cutoff (6) has the form

$$\begin{aligned}
& \langle K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle_{\text{Fig.3a}}^{\text{slope}} \\
&= -\frac{g^4}{2\pi^2} \text{Tr}\{t^a U_x t^b U_y^\dagger\} f^{anl} f^{bn'l'} \int d^2 z d^2 z' U_z^{nn'} U_{z'}^{ll'} \int \bar{d}^2 q_1 \bar{d}^2 q_2 \bar{d}^2 k_1 \bar{d}^2 k_2 \frac{(q_1+q_2, k_1+k_2)}{(q_1+q_2)^2(k_1+k_2)^2} \\
&\times e^{i(q_1, X)_\perp + i(q_2, X')_\perp - i(k_1, Y)_\perp - i(k_2, Y')_\perp} \int_0^\infty \frac{d\alpha'}{\alpha'} \left[ \frac{(q_1, k_1)}{(q_1^2 + \xi\alpha'^2)(k_1^2 + \xi\alpha'^2)} + \frac{(q_2, k_2)}{(q_2^2 + \xi\alpha'^2)(k_2^2 + \xi\alpha'^2)} \right]
\end{aligned}$$

and therefore the difference between the subtractions in ‘‘rigid cutoff’’ (170) and ‘‘slope cutoff’’ (171) prescriptions can be written as

$$\begin{aligned}
& \langle K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle_{\text{Fig.3a}}^{\text{rigid}} - \langle K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle_{\text{Fig.3a}}^{\text{slope}} \\
&= \frac{g^4}{2\pi^2} \text{Tr}\{t^a U_x t^b U_y^\dagger\} f^{anl} f^{bn'l'} \int d^2 z d^2 z' U_z^{nn'} U_{z'}^{ll'} \int \bar{d}^2 q_1 \bar{d}^2 q_2 \bar{d}^2 k_1 \bar{d}^2 k_2 e^{i(q_1, X)_\perp + i(q_2, X')_\perp - i(k_1, Y)_\perp - i(k_2, Y')_\perp} \\
&\times \left\{ \frac{(q_1+q_2, k_1+k_2)}{(q_1+q_2)^2(k_1+k_2)^2} \int_0^\infty \frac{d\alpha'}{\alpha'} \left[ \frac{(q_1, k_1)}{(q_1^2 + \xi\alpha'^2)(k_1^2 + \xi\alpha'^2)} + \frac{(q_2, k_2)}{(q_2^2 + \xi\alpha'^2)(k_2^2 + \xi\alpha'^2)} \right] \right. \\
&+ 2\xi \frac{d}{d\xi} \int_0^\infty \frac{d\alpha}{\alpha} \frac{(q_1+q_2, k_1+k_2)}{[(q_1+q_2)^2 + \xi\alpha^2][(k_1+k_2)^2 + \xi\alpha^2]} \left[ \frac{(q_1, k_1)}{q_1^2 k_1^2} + \frac{(q_2, k_2)}{q_2^2 k_2^2} \right] \int_0^\alpha \frac{d\alpha'}{\alpha'} \left. \right\} \\
&= -\frac{g^4}{2\pi^2} \text{Tr}\{t^a U_x t^b U_y^\dagger\} f^{anl} f^{bn'l'} \int d^2 z d^2 z' U_z^{nn'} U_{z'}^{ll'} \int \bar{d}^2 q_1 \bar{d}^2 q_2 \bar{d}^2 k_1 \bar{d}^2 k_2 e^{i(q_1, X)_\perp + i(q_2, X')_\perp - i(k_1, Y)_\perp - i(k_2, Y')_\perp} \\
&\times \int_0^\infty \frac{d\alpha'}{\alpha'} \left\{ \frac{(q_1+q_2, k_1+k_2)}{(q_1+q_2)^2(k_1+k_2)^2} \left[ \frac{(q_1, k_1)}{(q_1^2 + \xi\alpha'^2)(k_1^2 + \xi\alpha'^2)} + \frac{(q_2, k_2)}{(q_2^2 + \xi\alpha'^2)(k_2^2 + \xi\alpha'^2)} \right] \right. \\
&\left. - \frac{(q_1+q_2, k_1+k_2)}{[(q_1+q_2)^2 + \xi\alpha'^2][(k_1+k_2)^2 + \xi\alpha'^2]} \left[ \frac{(q_1, k_1)}{q_1^2 k_1^2} + \frac{(q_2, k_2)}{q_2^2 k_2^2} \right] \right\}
\end{aligned}$$

It is instructive to rewrite this result in Schwinger’s notations

$$\begin{aligned}
& \langle K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle_{\text{Fig.3a}}^{\text{rigid}} - \langle K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle_{\text{Fig.3a}}^{\text{slope}} \\
&= \frac{g^4}{2\pi^2} \text{Tr}\{t^a U_x t^b U_y^\dagger\} f^{anl} f^{bn'l'} \int d^2 z \int_0^\infty \frac{d\alpha'}{\alpha'} \left[ (x | \frac{p_i}{p^2 + \xi\alpha'^2} | z) U_z^{nn'} (z | \frac{p_i}{p^2 + \xi\alpha'^2} | y) (z | \frac{p_j}{p^2} U^{ll'} \frac{p_j}{p^2} | y) \right. \\
&\quad \left. - (x | \frac{p_i}{p^2} | z) U_z^{nn'} (z | \frac{p_i}{p^2} | y) (z | \frac{p_j}{p^2 + \xi\alpha'^2} U^{ll'} \frac{p_j}{p^2 + \xi\alpha'^2} | y) \right]
\end{aligned}$$

We see now that the difference between the two regularizations of the longitudinal divergence is given by the difference of  $(\text{LO})^2$  contributions with cutoffs in  $\alpha$  determined by the momenta on the first and on the second step of  $(\text{LO})^2$  evolution.

It is easy to see that for the sum of all diagrams this yields (see eq. (55))

$$\begin{aligned}
& \langle K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\}^{\text{rigid}} - K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\}^{\text{slope}} \rangle \\
&= \alpha_s^2 \int d^2 z d^2 z' [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_{z'}^\dagger\} \text{Tr}\{U_{z'} U_y^\dagger\} - \text{Tr}\{U_x U_z^\dagger U_{z'} U_y^\dagger U_z U_{z'}^\dagger\} + (z \leftrightarrow z')] \int_0^\infty \frac{dt}{t} \\
&\times \left\{ \left[ \left( x \left| \frac{p_i}{p^2+t} \right| z \right) - \left( y \left| \frac{p_i}{p^2+t} \right| z \right) \right]^2 \left[ \left( z \left| \frac{p_i}{p^2} \right| z' \right) - \left( y \left| \frac{p_i}{p^2} \right| z' \right) \right]^2 - \left[ \left( x \left| \frac{p_i}{p^2} \right| z \right) - \left( y \left| \frac{p_i}{p^2} \right| z \right) \right]^2 \left[ \left( z \left| \frac{p_i}{p^2+t} \right| z' \right) - \left( y \left| \frac{p_i}{p^2+t} \right| z' \right) \right]^2 \right. \\
&+ \left. \left[ \left( x \left| \frac{p_i}{p^2+t} \right| z' \right) - \left( y \left| \frac{p_i}{p^2+t} \right| z' \right) \right]^2 \left[ \left( z' \left| \frac{p_i}{p^2} \right| z \right) - \left( x \left| \frac{p_i}{p^2} \right| z \right) \right]^2 - \left[ \left( x \left| \frac{p_i}{p^2} \right| z' \right) - \left( y \left| \frac{p_i}{p^2} \right| z' \right) \right]^2 \left[ \left( z' \left| \frac{p_i}{p^2+t} \right| z \right) - \left( x \left| \frac{p_i}{p^2+t} \right| z \right) \right]^2 \right\}
\end{aligned}$$

Using the integral

$$\begin{aligned}
& \int_0^\infty \frac{dt}{t^{1-\epsilon}} \left( x \left| \frac{p_i}{p^2+t} \right| z \right) \left( y \left| \frac{p_i}{p^2+t} \right| z \right) = -\frac{(X, Y)}{4\pi^2} \int_0^1 du \frac{\Gamma(\epsilon) \Gamma(2-\epsilon) (4\bar{u}u)^{-\epsilon}}{(X^2 \bar{u} + Y^2 u)^{2-\epsilon}} \\
&= \frac{(X, Y)}{4\pi^2 X^2 Y^2} \left( -\frac{1}{\epsilon} - \ln 4 + \frac{X^2 \ln Y^2 - Y^2 \ln X^2}{X^2 - Y^2} - \ln X^2 Y^2 \right) + O(\epsilon)
\end{aligned}$$

$$\begin{aligned}
& \langle K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\}^{\text{rigid}} - K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\}^{\text{slope}} \rangle \tag{172} \\
&= -\frac{\alpha_s^2}{16\pi^4} \int d^2 z d^2 z' \frac{1}{(z-z')^2 X^2 Y^2} [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_{z'}^\dagger\} \text{Tr}\{U_{z'} U_y^\dagger\} - \text{Tr}\{U_x U_z^\dagger U_{z'} U_y^\dagger U_z U_{z'}^\dagger\} + z \leftrightarrow z'] \\
&\times \left( \frac{Y^2}{Y'^2} \left\{ (X, Y) \left[ \frac{X^2 + Y^2}{X^2 - Y^2} \ln \frac{X^2}{Y^2} + 2 \right] + (\Delta, Y) \ln X^2 - (\Delta, X) \ln Y^2 \right\} \right. \\
&\left. - \frac{\Delta^2}{Y'^2} \left\{ (z-z', Y') \left[ \frac{(z-z')^2 + Y'^2}{(z-z')^2 - Y'^2} \ln \frac{(z-z')^2}{Y'^2} + 2 \right] - (Y, Y') \ln (z-z')^2 + (Y, z-z') \ln Y'^2 \right\} + x \leftrightarrow y \right)
\end{aligned}$$

The NLO kernel for the evolution of color dipoles with respect to the slope is the sum of Eq. (5) and the correction (172). Note that the correction term (172) is not conformally invariant (cf. Ref. [24]). This is hardly surprising since the non-light-like Wilson line turns into a circle under the inversion  $x_\mu \rightarrow x_\mu/x^2$ .

## References

- 
- [1] I. Balitsky, “*High-Energy QCD and Wilson Lines*”, In \*Shifman, M. (ed.): At the frontier of particle physics, vol. 2\*, p. 1237-1342 (World Scientific, Singapore, 2001) [hep-ph/0101042]
  - [2] A.H. Mueller, *Nucl. Phys.* **B415**, 373 (1994); A.H. Mueller and Bimal Patel, *Nucl. Phys.* **B425**, 471 (1994).
  - [3] N.N. Nikolaev and B.G. Zakharov, *Phys. Lett.* **B 332**, 184 (1994); *Z. Phys.* **C64**, 631 (1994); N.N. Nikolaev B.G. Zakharov, and V.R. Zoller, *JETP Letters* **59**, 6 (1994).
  - [4] I. Balitsky, *Nucl. Phys.* **B463**, 99 (1996); “*Operator expansion for diffractive high-energy scattering*”, [hep-ph/9706411];
  - [5] Yu.V. Kovchegov, *Phys. Rev.* **D60**, 034008 (1999); *Phys. Rev.* **D61**, 074018 (2000).
  - [6] V.S. Fadin, E.A. Kuraev, and L.N. Lipatov, *Phys. Lett.* **B 60**, 50 (1975); I. Balitsky and L.N. Lipatov, *Sov. Journ. Nucl. Phys.* **28**, 822 (1978).
  - [7] L.V. Gribov, E.M. Levin, and M.G. Ryskin, *Phys. Rept.* **100**, 1 (1983), A.H. Mueller and J.W. Qiu, *Nucl. Phys.* **B268**, 427 (1986); A.H. Mueller, *Nucl. Phys.* **B335**, 115 (1990).
  - [8] E. Iancu and R. Venugopalan, In \*Hwa, R.C. (ed.) et al.: Quark gluon plasma\* 249-3363, [e-Print: hep-ph/0303204]; H. Weigert, *Prog. Part. Nucl. Phys.* **55**, 461 (2005); J. Jalilian-Marian and Yu.V. Kovchegov, *Prog. Part. Nucl. Phys.* **56**, 104 (2006).
  - [9] I. Balitsky, *Phys. Rev.* **D75**, 014001 (2007).
  - [10] Yu. V. Kovchegov and H. Weigert, *Nucl. Phys.* **A784**, 188 (2007),
  - [11] M. Beneke, *Phys. Rept.* **317**, 1 (1999); M. Beneke and V.M. Braun, “*Renormalons and power corrections.*”, In \*Shifman, M. (ed.): At the frontier of particle physics, vol. 3\*, p. 1719-1773 (World Scientific, Singapore, 2001) [hep-ph/0010208]
  - [12] V.S. Fadin and L.N. Lipatov, *Phys. Lett.* **B429**, 127 (1998); G. Camici and M. Ciafaloni, *Phys. Lett.* **B430**, 349 (1998).
  - [13] I. Balitsky, *Phys. Rev.* **D60**, 014020 (1999).
  - [14] I. Balitsky and A.V. Belitsky, *Nucl. Phys.* **B629**, 290 (2002).

- [15] I. Balitsky, *Phys.Lett.***B518**, 235(2001).
- [16] V.S. Fadin , M.I. Kotsky, and L.N. Lipatov “*Gluon pair production in the quasimulti - Regge kinematics*”, [hep-ph/9704267].
- [17] Yu.V. Kovchegov and H. Weigert *Nucl.Phys.***A789**, 260(2007).
- [18] A.V. Kotikov and L.N. Lipatov, *Nucl. Phys.* **B582**, 19 (2000).
- [19] A. Vogt, S. Moch, and J.A.M. Vermaseren, *Nucl.Phys.***B691**, 129 (2004)
- [20] V. S. Fadin, R. Fiore, A.V. Grabovsky, and A. Papa, *Nucl.Phys.***B784**, 49(2007).
- [21] V.S. Fadin and R. Fiore, *Phys. Rev.* **D72**, 014018 (2005).
- [22] E. Gardi, J. Kuokkanen, K. Rummukainen, and H. Weigert, *Nucl.Phys.***A784**, 282(2007); J.L. Albacete and Yu.V. Kovchegov, *Phys.Rev.***D75**,125021 (2007).
- [23] A. Kovner and M. Lublinsky, *Phys.Rev.***D71**, 085004(2005); *Phys.Rev.Lett.***94**, 181603(2005); *JHEP*, 0503:001(2005); Y. Hatta, E. Iancu, L. McLerran, A. Stasto and D.N. Triantafyllopoulos, *Nucl.Phys.***A764**, 423 (2006); I. Balitsky, *Phys. Rev.* **D72**, 074027 (2005); A.H. Mueller, A.I. Shoshi, and S.M.H. Wong, *Nucl.Phys.***B715**, 440(2005).
- [24] A. Babansky and I. Balitsky, *Phys. Rev.* **D67**, 054026 (2003).