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A.M. Ougouag
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Idaho National Engineering and Environmental Laboratory
Bechtel BWXT Idaho, LLC

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Idaho National Engineering and Environmental Laboratory
Idaho Falls, Idaho 83415

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1.0 Introduction

The usual strategy for solving the neutron diffusion equation in two or three dimensions by nodal methods is to reduce the multidimensional partial differential equation to a set of ordinary differential equations (ODEs) in the separate spatial coordinates. This reduction is accomplished by "transverse integration" of the equation.[1] For example, in three-dimensional Cartesian coordinates, the three-dimensional equation is first integrated over *x* and *y* to obtain an ODE in *z*, then over *x* and *z* to obtain an ODE in *y*, and finally over *y* and *z* to obtain an ODE in *x*. Then the ODEs are solved to obtain one-dimensional solutions for the neutron fluxes averaged over the other two dimensions. These solutions are found in regions ("nodes") small enough for the material properties and cross sections in them to be adequately represented by average values. Because the solution in each node is an exact analytical solution, the nodes can be much larger than the mesh elements used in finite-difference solutions. Then the solutions in the different nodes are coupled by applying interface conditions, ultimately fixing the solutions to the external boundary conditions.

However, the transverse integration procedure fails in (r,θ) or (r,θ, z) cylindrical geometry, because the transverse integration over *r* (in 2-d) or *z* and *r* (in 3-d) leads to an impasse, as shown in Section 2.0. In this report, it is shown how the impasse can be circumvented. The diffusion equation is readily integrated over *z* to obtain an equation in *r* and θ for the *z*-averaged neutron flux ${}^{z}\overline{\phi}(r,\theta)$. Then the solution for ${}^{z}\overline{\phi}(r,\theta)$ is found analytically and integrated over each remaining coordinate to obtain a solution for the neutron flux averaged over the other coordinate (and *z*). Even though the solution for ${}^{z}\overline{\phi}(r,\theta)$ has been found, it is still necessary to compute the one-dimensional solutions, because it is impractical to couple the two-dimensional solutions across node interfaces.

Thus, instead of obtaining one-dimensional differential equations and solving each of them to obtain one-dimensional solutions, a two-dimensional solution is found directly and then integrated to obtain one-dimensional solutions. The two-dimensional solution for ${}^{z}\overline{\phi}(r,\theta)$ has been found by three different methods. The first two, the method of integral transforms and the method of separation of variables, are presented formally in Sections 3.0 and 4.0, respectively. By a "formal solution," it is meant that an expression for ${}^{z}\overline{\phi}(r,\theta)$ is obtained, but the boundary conditions may not have been applied and the integration of ${}^{z}\overline{\phi}(r,\theta)$ to obtain one-dimensional solutions has not been performed. The third approach, the Green's function method, is carried out completely in Section 5.0, to the point where it is ready for coding. The implementation of the Green's function solution in a computer code is the focus of the next phase of work after that which is reported here.

In Section 6.0, the three-dimensional solution is completed by application of the usual transverse integration method to obtain a one-dimensional solution in *z* for the neutron flux averaged over *r* and θ.

The question of the equivalence of the three solutions for ${}^{z}\overline{\phi}(r,\theta)$ is addressed in Section 7.0. Section 8.0 is a summary.

2.0 The Failure of Traditional Transverse Integration

The transverse integration procedure begins with the diffusion equation (written here in multigroup form, with the energy group index omitted):

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}+\frac{\partial^2\phi}{\partial z^2}-\frac{\Sigma_R\phi}{D} = -\frac{S}{D} \qquad , \tag{1}$$

where Σ_R = removal cross section
S = volumetric source rate
D = diffusion coefficient,
and the other symbols have their usual meaning in nuclear reactor physics.

This equation is first integrated over $z_k \le z \le z_{k+1}$, the domain of z in the node, to obtain

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\, {}^z\overline{\phi}}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\, {}^z\overline{\phi}}{\partial\theta^2}-\frac{\Sigma_R\, {}^z\overline{\phi}}{D} = \frac{{}^z\overline{J}_z'}{D}-\frac{{}^z\overline{S}}{D} \qquad , \tag{2}$$

where

$${}^z\overline{\phi}(r,\theta) \equiv \frac{1}{z_{k+1}-z_k}\int_{z_k}^{z_{k+1}}\phi(r,\theta,z)dz \ ,$$

and

$${}^z\overline{S} \equiv \frac{1}{z_{k+1}-z_k}\int_{z_k}^{z_{k+1}} S\, dz$$

are the z-averaged neutron flux and neutron source, and

$${}^z\overline{J}_z' \equiv \frac{1}{z_{k+1}-z_k}\left[J_z\right]_{z_k}^{z_{k+1}}$$

is the average derivative with respect to z of the z-component of the neutron current.

Next, Eq. (2) is integrated over $r_i \le r \le r_{i+1}$, the domain of r in the node. The appropriate average over r includes the weighting factor r to account for the geometry.

$$\int_{r_i}^{r_{i+1}}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\, {}^z\overline{\phi}}{\partial r}\right)+\frac{1}{r^2}\left(\frac{\partial^2\, {}^z\overline{\phi}}{\partial\theta^2}\right)-\frac{\Sigma_R}{D}{}^z\overline{\phi}\right]r\, dr = \int_{r_i}^{r_{i+1}}\left(\frac{{}^z\overline{J}_z'}{D}-\frac{{}^z\overline{S}}{D}\right)r\, dr \quad . \tag{3}$$

The difficulty arises with the azimuthal term. This is

$$\iota_{Az}^{(1)} \equiv \int_{r_i}^{r_{i+1}} \frac{1}{r}\left(\frac{\partial^2\, {}^{z}\overline{\phi}}{\partial\theta^2}\right) dr \qquad . \tag{4}$$

The goal of the transverse integration process is to obtain an equation in the θ-dependent r- and z-averaged flux,

$$^{rz}\overline{\phi}(\theta) \equiv \frac{2}{r_{i+1}^2 - r_i^2} \int_{r_i}^{r_{i+1}} {}^{z}\overline{\phi}\, r\, dr \quad . \tag{5}$$

But the presence of $1/r$ in the integrand of Eq. (4) makes this goal unattainable. Successive integration by parts does not work, because a logarithm is obtained that prevents the eventual elimination of r-dependent factors in the integrand.

If r^2 is used as a weighting factor, the azimuthal integral becomes

$$\iota_{Az}^{(2)} = \int_{r_i}^{r_{i+1}} \frac{\partial^2\, {}^{z}\overline{\phi}}{\partial\theta^2} dr = \frac{\partial}{\partial\theta^2} \int_{r_i}^{r_{i+1}} {}^{z}\overline{\phi}\, dr \qquad , \tag{6}$$

in which the integral does not lead to the quantity $^{rz}\overline{\phi}$ defined in Eq. (5) that is being sought.

If r^3 is used as a weighting factor, Eq. (4) becomes

$$\iota_{Az}^{(3)} = \int_{r_i}^{r_{i+1}} \frac{\partial^2\, {}^{z}\overline{\phi}}{\partial\theta^2} r\, dr = \frac{\partial^2}{\partial\theta^2} \int_{r_i}^{r_{i+1}} {}^{z}\overline{\phi}\, r\, dr = \frac{\left(r_{i+1}^2 - r_i^2\right)}{2} \frac{\partial^2\, {}^{rz}\overline{\phi}}{\partial\theta^2} \qquad , \tag{7}$$

as desired, but the remaining part of the integral in Eq. (3) becomes

$$\iota_{Rad}^{(3)} = \int_{r_i}^{r_{i+1}} \left[r^2 \frac{\partial}{\partial r}\left(r \frac{\partial\, {}^{z}\overline{\phi}}{\partial r}\right) - r^3 \frac{\Sigma_R}{D}\, {}^{z}\overline{\phi} \right] dr \qquad . \tag{8}$$

The first term in Eq. (8) yields to successive integration by parts, but the second term cannot be reduced to a form involving $^{rz}\overline{\phi}(\theta)$. Nor does any other choice of weighting factor permit all of the pieces of Eq. (3) to be integrated simultaneously to usable forms.

3.0 Solution by the Integral Transform Method

The integral transform of a quantity Q in (r,θ) geometry is[2]

$$\widetilde{Q}_m \equiv \int_\delta Q \psi_m(r,\theta)\, r\, dr\, d\theta \qquad , \tag{9}$$

where δ is the domain of r and θ,
r is the appropriate weighting function,
and ψ_m is the m^{th} eigenfunction of the homogeneous problem related to Eq. (2).

The quantity Q may be a function, an operator, or an equation.

The eigenfunction ψ_m satisfies the homogeneous equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi_m}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \psi_m}{\partial \theta^2} + \lambda_m^2\, \psi_m = 0 \qquad , \tag{10}$$

subject to the homogeneous analogues of the boundary conditions on Eq. (2).

The solution of Eq. (2) is found by first obtaining the integral transform ${}^z\widetilde{\widetilde{\overline{\phi}}}_m$ for all m, and then by calculating the inverse transform according to the formula

$${}^z\overline{\phi} = \sum_m \frac{\psi_m(r,\theta)\; {}^z\widetilde{\widetilde{\overline{\phi}}}_m}{N_m} \qquad , \tag{11}$$

in which

$$N_m = \int_\delta [\psi_m(r,\theta)]^2\, r\, dr\, d\theta \qquad . \tag{12}$$

One begins the solution by taking the integral transform of Eq. (2):

$$\int_\delta \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\, {}^z\overline{\phi}}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2\, {}^z\overline{\phi}}{\partial \theta^2}\right) - \frac{\Sigma_R}{D}\, {}^z\overline{\phi}\right]\psi_m\, r\, dr\, d\theta = \int_\delta \left(\frac{{}^z\overline{J}'_z}{D} - \frac{{}^z\overline{S}}{D}\right)\psi_m\, r\, dr\, d\theta \qquad . \tag{13}$$

By rearranging and applying the definition of the integral transform (Eq. (9)), one obtains

$$\Sigma_R\, {}^z\widetilde{\widetilde{\overline{\phi}}}_m = D \int_\delta \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\, {}^z\overline{\phi}}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2\, {}^z\overline{\phi}}{\partial \theta^2}\right)\right]\psi_m\, r\, dr\, d\theta + {}^z\widetilde{\widetilde{\overline{S}}}_m - {}^z\widetilde{\widetilde{\overline{J}}}'_{zm} \qquad . \tag{14}$$

Eq. (10) is separable, so the eigenfunctions may be written

$$\psi_m(r,\theta) = R_m(r)T_m(\theta) \quad . \tag{15}$$

If Eq. (15) is inserted into Eq. (10), the result divided by $R_m T_m$ and multiplied by r^2, and the terms in r and θ collected, it is found that

$$\frac{r}{R_m}\frac{d}{dr}\left(r\frac{dR_m}{dr}\right) + \lambda_m^2 r^2 = -\frac{1}{T_m}\frac{d^2 T_m}{d\theta^2} \equiv \gamma_m^2 \quad , \tag{16}$$

where γ_m^2 is a constant because only then can a function only of r and a function only of θ be equal for all r and θ.

The θ-dependent side of Eq. (16) can be written

$$\frac{d^2 T_m}{d\theta^2} + \gamma_m^2 T_m = 0 \quad , \tag{17}$$

subject to the appropriate homogeneous boundary conditions (i.e., the homogeneous analogues to the actual boundary conditions on the θ-boundaries). The subscript m will be seen below to be the index for the eigenvalues of the r-equation obtained from Eq. (16). For any value of m, infinitely many values of γ_m^2 satisfy Eq. (17) with the appropriate homogeneous boundary conditions. Thus, any linear combination of the θ-eigenfunctions (indexed by n) will satisfy Eq. (17) with the homogeneous boundary conditions. Furthermore, since Eq. (17) and these boundary conditions are the same for all m, the θ-eigenfunctions are the same for all m. But there will be a different r-eigenfunction for every combination of m and n. So Eq. (15) should be rewritten as

$$\psi_m(r,\theta) = \sum_{n=0}^{\infty} c_{mn} R_{mn}(r) T_n(\theta) \quad , \tag{18}$$

where $T_n(\theta)$ is the θ-eigenfunction corresponding to the eigenvalue γ_n^2.

Now Eq. (18) is substituted into Eq. (14):

$$\Sigma_R\, {}^{z}\tilde{\tilde{\phi}}_m = D \iint_{\delta} \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\, {}^{z}\overline{\phi}}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2\, {}^{z}\overline{\phi}}{\partial\theta^2}\right)\right] \sum_{n=0}^{\infty} c_{mn} R_{mn} T_n \, r\, dr\, d\theta + {}^{z}\tilde{\tilde{S}}_m - {}^{z}\tilde{\tilde{J}}'_{zm} \quad ,$$

or

$$\Sigma_R\, {}^{z}\tilde{\tilde{\phi}}_m = \sum_{n=0}^{\infty} D \iint_{\delta} \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\, {}^{z}\overline{\phi}}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2\, {}^{z}\overline{\phi}}{\partial\theta^2}\right)\right] c_{mn} R_{mn} T_n \, r\, dr\, d\theta + {}^{z}\tilde{\tilde{S}} - {}^{z}\tilde{\tilde{J}}'_{zm} \quad . \tag{19}$$

Next, the domain of integration is written in terms of the domains in r and θ, and the integral over θ is brought inside the integral over r:

$$\Sigma_R\, {}^z\tilde{\tilde{\bar{\phi}}}_m = \sum_{n=0}^{\infty} D\, c_{mn} \int_r \left[\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\int_\theta T_n\, {}^z\bar{\phi}\, d\theta \right) + \frac{1}{r^2}\int_\theta \frac{\partial^2\, {}^z\bar{\phi}}{\partial\theta^2} T_n\, d\theta \right] R_{mn}\, r\, dr + {}^z\tilde{\tilde{\bar{S}}}_{zm} - {}^z\tilde{\tilde{\bar{J}}}'_{zm}. \quad 20)$$

In Eq. (20), the first integral over θ is simply the integral transform over θ only, which is written for economy of notation as

$$ {}^z\hat{\bar{\phi}}_n(r) \equiv \int_\theta T_n\, {}^z\bar{\phi}\, d\theta \quad . \qquad (21)$$

The second integral succumbs to integration by parts twice; then Eq. (20) is

$$\Sigma_R\, {}^z\tilde{\tilde{\bar{\phi}}}_m = \sum_{n=0}^{\infty} D\, c_{mn} \left\{ \int_r R_{mn} \frac{d}{dr}\left(r\frac{d\, {}^z\hat{\bar{\phi}}_n}{dr} \right) dr + \int_r \frac{R_{mn}}{r}\left[T_n \frac{\partial\, {}^z\bar{\phi}}{\partial\theta} \right]_\theta dr - \int_r \frac{R_{mn}}{r}\left[{}^z\bar{\phi}\frac{dT_n}{d\theta} \right]_\theta dr \right.$$

$$\left. + \iint_{r\,\theta} \frac{R_{mn}}{r}\, {}^z\bar{\phi}\, \frac{d^2 T_n}{d\theta^2}\, d\theta\, dr \right\} + {}^z\tilde{\tilde{\bar{S}}} - {}^z\tilde{\tilde{\bar{J}}}'_{zm} \quad . \qquad (22)$$

Now $d^2T_n/d\theta^2$ is substituted from Eq. (17) (with index n) in Eq. (22), with the result

$$\Sigma_R\, {}^z\tilde{\tilde{\bar{\phi}}}_m = \sum_{n=0}^{\infty} D\, c_{mn} \left\{ \int_r R_{mn} \frac{d}{dr}\left(r\frac{d\, {}^z\hat{\bar{\phi}}_n}{dr} \right) dr + \int_r \frac{R_{mn}}{r}\left[T_n \frac{\partial\, {}^z\bar{\phi}}{\partial\theta} - {}^z\bar{\phi}\frac{dT_n}{d\theta} \right]_\theta dr \right.$$

$$\left. - \gamma_n^2 \int_r \frac{R_{mn}}{r}\, {}^z\hat{\bar{\phi}}_n dr \right\} + {}^z\tilde{\tilde{\bar{S}}}_m - {}^z\tilde{\tilde{\bar{J}}}'_{zm} \quad .$$

(23)

The first integral in Eq. (23) is now integrated twice by parts, and Eq. (23) becomes

$$\Sigma_R\, {}^z\tilde{\tilde{\bar{\phi}}}_m = \sum_{n=0}^{\infty} D\, c_{mn} \left\{ \left[rR_{mn}\frac{d\, {}^z\hat{\bar{\phi}}_n}{dr} \right]_r - \left[r\, {}^z\hat{\bar{\phi}}_n \frac{dR_{mn}}{dr} \right]_r + \int_r {}^z\hat{\bar{\phi}}_n \frac{d}{dr}\left(r\frac{dR_{mn}}{dr} \right) dr \right.$$

$$\left. + \int_r \frac{R_{mn}}{r}\left[T_n \frac{\partial\, {}^z\bar{\phi}}{\partial\theta} - {}^z\bar{\phi}\frac{dT_n}{d\theta} \right]_\theta dr - \gamma_n^2 \int_r \frac{R_{mn}}{r}\, {}^z\hat{\bar{\phi}}_n dr \right\} + {}^z\tilde{\tilde{\bar{S}}}_m - {}^z\tilde{\tilde{\bar{J}}}'_{zm} \quad . \qquad (24)$$

Now the r-dependent side of Eq. (16), written as

$$\frac{d}{dr}\left(r\frac{dR_{mn}}{dr}\right)=\frac{R_{mn}}{r}\left(\gamma_n^2-\lambda_{mn}^2 r^2\right)\quad ,\, n=0,1,2,... \tag{25}$$

(because the r-eigenvalues, λ_{mn}, will be different for different values of n), is substituted into Eq. (24):

$$\Sigma_R\,{}^z\tilde{\bar{\phi}}_m=\sum_{n=0}^{\infty}D\,c_{mn}\left\{\left[rR_{mn}\frac{d\,{}^z\hat{\bar{\phi}}_n}{dr}-r\,{}^z\hat{\bar{\phi}}_n\frac{dR_{mn}}{dr}\right]_r+\int_r {}^z\hat{\bar{\phi}}_n\frac{R_{mn}}{r}\left(\gamma_n^2-\lambda_m^2 r^2\right)dr\right.$$

$$\left.+\int_r\frac{R_{mn}}{r}\left[T_n\frac{\partial\,{}^z\bar{\phi}}{\partial\theta}-{}^z\bar{\phi}\frac{dT_n}{d\theta}\right]_\theta dr-\gamma_n^2\int_r {}^z\hat{\bar{\phi}}_n\frac{R_{mn}}{r}dr\right\}+{}^z\tilde{\bar{S}}_m-{}^z\tilde{\bar{J}}'_{zm}\qquad . \tag{26}$$

After cancellation of terms, application of the definition of ${}^z\bar{\phi}_m$, use of Eqs. (18) and (21), and rearrangement, Eq. (26) becomes

$${}^z\tilde{\bar{\phi}}_m=\left(\frac{D}{\Sigma_R+D}\right)\left\langle\left(\frac{{}^z\tilde{\bar{S}}_m-{}^z\tilde{\bar{J}}'_{zm}}{D}\right)+\sum_{n=0}^{\infty}c_{mn}\left\{\left[rR_{mn}\frac{d\,{}^z\hat{\bar{\phi}}_n}{dr}-r\,{}^z\hat{\bar{\phi}}_n\frac{dR_{mn}}{dr}\right]_r\right.\right.$$

$$\left.\left.+\int_r\frac{R_{mn}}{r}\left[T_n\frac{\partial\,{}^z\bar{\phi}}{\partial\theta}-{}^z\bar{\phi}\frac{dT_n}{d\theta}\right]_\theta dr\right\}\right\rangle\qquad . \tag{27}$$

In Eq. (27), R_{mn} is the eigenfunction of Eq. (25); this equation is written in standard form as

$$r^2\frac{d^2R_{mn}}{dr^2}+r\frac{dR_{mn}}{dr}+(\lambda_{mn}^2 r^2-\gamma_n^2)R_{mn}=0\qquad , \tag{28}$$

where λ_{mn} is the eigenvalue. The eigenvalue does not appear as a coefficient in Eq. (27), because it has been absorbed into the constant of integration, c_{mn} (ψ_m must satisfy Eq. (10) for <u>any</u> linear combination of the individual solutions $R_{mn}T_n$).

Eq. (28) is Bessel's equation, and the eigenfunctions are linear combinations of the Bessel functions $J_{\gamma_n}(\lambda_{mn}r)$ and $Y_{\gamma_n}(\lambda_{mn}r)$, as determined by the appropriate homogeneous boundary conditions.

The quantities in the square brackets in Eq. (27) are evaluated at the limits of integration by applying the actual (<u>in</u>homogeneous) boundary conditions. Then the complete solution for ${}^z\overline{\phi}$ is found by applying Eq. (11) for the inverse transform:

$$
{}^z\overline{\phi}(r,\theta) = \sum_{m=0}^{\infty} \frac{\sum_{n=0}^{\infty} c_{mn} R_{mn} T_n \left(\dfrac{D}{\Sigma_R + D} \right)}{\int\limits_r \int\limits_\theta \left[\sum_{n=0}^{\infty} c_{mn} R_{mn} T_n \right]^2 r\, dr\, d\theta} \left\langle \left(\frac{{}^z\tilde{\tilde{S}}_m - {}^z\tilde{\tilde{J}}'_{zm}}{D} \right) + \sum_{n=0}^{\infty} c_{mn} \left\{ \left[r R_{mn} \frac{d\, {}^z\hat{\overline{\phi}}_n}{dr} - r\, {}^z\hat{\overline{\phi}}_n \frac{dR_{mn}}{dr} \right]_r \right. \right.
$$

$$
\left. \left. + \int\limits_r \frac{R_{mn}}{r} \left[T_n \frac{\partial\, {}^z\overline{\phi}}{\partial \theta} - {}^z\overline{\phi} \frac{dT_n}{d\theta} \right]_\theta dr \right\} \right\rangle \quad , \qquad (29)
$$

where, to save space, the functional dependence of R_{mn} on r and T_n on θ has not been written explicitly.

This formulation would be cumbersome to implement because of the double summation. Furthermore, the eigenvalues λ_{mn} are difficult to evaluate. For example, in the case where boundary conditions are specified for currents, the equivalent homogeneous boundary conditions are $dR_{mn}/dr = 0$ at $r = r_i$ and $r = r_{i+1}$. These boundary conditions lead to the following equation that must be solved for λ_{mn} :

$$
\frac{\gamma_n Y_{\gamma_n}(\lambda_{mn} r_i) - \lambda_{mn} r_i Y_{\gamma_n+1}(\lambda_{mn} r_i)}{\gamma_n J_{\gamma_n}(\lambda_{mn} r_i) - \lambda_{mn} r_i J_{\gamma_n+1}(\lambda_{mn} r_i)} = \frac{\gamma_n Y_{\gamma_n}(\lambda_{mn} r_{i+1}) - \lambda_{mn} r_{i+1} Y_{\gamma_n+1}(\lambda_{mn} r_{i+1})}{\gamma_n J_{\gamma_n}(\lambda_{mn} r_{i+1}) - \lambda_{mn} r_{i+1} J_{\gamma_n+1}(\lambda_{mn} r_{i+1})} \quad . \qquad (30)
$$

Eq. (29) is one form of the solution to the problem, but it is desirable to seek more convenient forms. Another approach is tried in the next section.

4.0 Solution by Separation of Variables

The boundary conditions on Eq. (2) are generally inhomogeneous on all four boundaries, a circumstance that introduces complications in the standard application of the separation-of-variables technique. These complications are accommodated by the method of Grinberg.[3] Eq. (2) is cast in the form

$$\frac{1}{r(x)}\left\{\frac{\partial}{\partial x}\left[p(x)\frac{\partial u}{\partial x}\right]-q(x)u\right\}+M_y u=F(x,y) \tag{31}$$

with boundary conditions

$$\alpha_a \left.\frac{\partial u}{\partial x}\right|_{x=a}+\beta_a u\big|_{x=a}=f_a(y), \qquad \alpha_b \left.\frac{\partial u}{\partial x}\right|_{x=b}+\beta_b u\big|_{x=b}=f_b(y) \quad , \tag{32}$$

$$\gamma_c \left.\frac{\partial u}{\partial y}\right|_{y=c}+\delta_c u\big|_{y=c}=g_c(x), \qquad \text{and} \qquad \gamma_d \left.\frac{\partial u}{\partial y}\right|_{y=d}+\delta_d u\big|_{y=d}=g_d(x) \quad , \tag{33}$$

where M_y is an operator in y.

A solution of eigenfunctions is sought of form

$$u=\sum_{n=0}^{\infty} u_n(y)X_n(x) \quad , \tag{34}$$

where $X_n(x)$ are the eigenfunctions, which satisfy homogeneous boundary conditions analogous to Eqs. (32).

The inhomogeneous boundary conditions at $x = a,b$ are applied by a procedure that yields an ordinary differential equation in y,

$$M_y\bar{u}_n-\lambda_n\bar{u}_n=F_n-\frac{p(b)}{\alpha_b}X_n(b)f_b(y)+\frac{p(a)}{\alpha_a}X_n(a)f_a(y) \quad , \tag{35}$$

where

$$u_n(y)=\frac{\bar{u}_n}{\int_{x=a}^{b} rX_n^2(x)\,dx} \quad , \tag{36a}$$

$$F_n=\int_{x=a}^{b} rF(x,y)X_n(x)\,dx \quad , \tag{36b}$$

and λ_n is the eigenvalue of Eq. (31).

To find the eigenvalues and eigenfunctions for Eq. (2), examine the homogeneous equation obtained from Eq. (2):

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\, {}^{z}\overline{\phi}_h}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\, {}^{z}\overline{\phi}_h}{\partial \theta^2}-\frac{\Sigma_R}{D}\, {}^{z}\overline{\phi}_h=0 \qquad . \tag{37}$$

Eq. (37) is subject to the homogeneous boundary conditions analogous to Eqs. (32) and (33).

Next, assume that the solution of Eq. (37) is separable:

$${}^{z}\overline{\phi}_h = R(r)T(\theta) \qquad . \tag{38}$$

Substitution of Eq. (38) into Eq. (37) and collection of terms gives

$$\frac{r}{R}\left[\frac{d}{dr}\left(r\frac{dR}{dr}\right)\right]-r^2\frac{\Sigma_R}{D}=-\frac{1}{T}\frac{d^2T}{d\theta^2}\equiv\lambda^2 \qquad , \tag{39}$$

where λ^2 is constant in order for a function only of r to be always equal to a function only of θ. Then

$$\frac{d^2T}{d\theta^2}+\lambda^2T=0 \tag{40}$$

and

$$r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}-\left(r^2\frac{\Sigma_R}{D}+\lambda^2\right)R=0 \ . \tag{41}$$

Eq. (40) has solutions satisfying the appropriate homogeneous boundary conditions for infinitely many values λ_n of the factor λ^2 ; the individual values of λ_n^2 are the eigenvalues of Eq. (40). Eq. (41) is the modified, or hyperbolic, Bessel equation, which is not a proper Sturm-Liouville problem and has no eigenvalues.

Therefore, the following correspondence can be made between Eq. (31) and Eq. (2):

Eq. (31)	Eq. (2)
u	${}^{z}\overline{\phi}$
x	θ
y	r
$r(x)$	1
$p(x)$	1
$q(x)$	0
$F(x,y)$	$r^2\left({}^{z}\overline{J}'_z - {}^{z}\overline{S}\right)/D$
M_y	$r\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)-\frac{\Sigma_R}{D}r^2$

Eq. (34) corresponds to

$$ {}^{z}\overline{\phi}(r,\theta)=\sum_{n=0}^{\infty}R_n(r)T_n(\theta) \qquad , \tag{42}$$

where $T_n(\theta)$ is the eigenfunction of Eq. (40) corresponding to the eigenvalue λ_n^2 (instead of λ_n as in Eq. (35)).

Then Eq. (35) corresponds to the equation

$$ r\frac{d}{dr}\left(r\frac{d\overline{R}_n}{dr}\right)-\frac{\Sigma_R}{D}r^2\overline{R}_n-\lambda_n^2\overline{R}_n=S_n(r)-\frac{1}{\alpha_{j+1}}T_n(\theta_{j+1})f_{j+1}(r)+\frac{1}{\alpha_j}T_n(\theta_j)f_j(r)\equiv\xi_n(r), \tag{43}$$

where $S_n(r)$ is the n^{th} expansion coefficient of the source-and-transverse-leakage term $r^2\left({}^{z}\overline{J}'_z-{}^{z}\overline{S}\right)/D$ expanded in the eigenfunctions T_n , where $\theta_j\leq\theta\leq\theta_{j+1}$ is the domain of θ in the node, and where $f_{j+1}(r)$ and $f_j(r)$ are the boundary functions on $\theta=\theta_{j+1}$ and $\theta=\theta_j$, respectively.

Eq. (43) is solved by first finding the solution of the related homogeneous equation, next finding a particular solution to the inhomogeneous equation, and then applying the boundary conditions on the r-boundaries.

The homogeneous equation related to Eq. (43) is

$$r^2 \frac{d^2 \overline{R}_n}{dr^2} + r\frac{d\overline{R}_n}{dr} - \left(r^2 \frac{\Sigma_R}{D} + \lambda_n^2 \right) \overline{R}_n = 0 \qquad , \tag{44}$$

and the solutions are the modified Bessel functions I_{λ_n} and K_{λ_n}.

From these solutions, the method of variation of parameters is used to construct a particular solution. The complete solution is

$$\overline{R}_n(r) = c_{1n} I_{\lambda_n}(kr) + c_{2n} K_{\lambda_n}(kr) + I_{\lambda_n}(kr) \int_{r'=r_o}^{r} k \frac{\xi_n(r')}{r'} K_{\lambda_n}(kr') dr' - K_{\lambda_n}(kr) \int_{r'=r_o}^{r} k \frac{\xi_n(r')}{r'} I_{\lambda_n}(kr') dr' \tag{45}$$

where r_o is any point in the interval $r_i \le r \le r_{i+1}$, and where

$$k \equiv \sqrt{\Sigma_R / D} \,. \tag{46}$$

It remains to compute the integration constants c_{1n} and c_{2n} . First, the solutions $\overline{R}_n(r)$ are normalized according to Eq. (36a):

$$R_n(r) = \frac{\overline{R}_n(r)}{\int_{\theta=\theta_j}^{\theta_{j+1}} T_n^2(\theta) d\theta} \qquad . \tag{47}$$

Then the boundary conditions corresponding to Eqs. (33) are expanded in series of eigenfunctions. In terms of r and θ, these boundary conditions are

$$\gamma_i \left. \frac{\partial \overline{\phi}^z}{\partial r} \right|_{r=r_i} + \delta_i \overline{\phi}^z \Big|_{r=r_i} = g_i(\theta) \tag{48a}$$

and

$$\gamma_{i+1} \left. \frac{\partial \overline{\phi}^z}{\partial r} \right|_{r=r_{i+1}} + \delta_{i+1} \overline{\phi}^z \Big|_{r=r_i+1} = g_{i+1}(\theta) \qquad . \tag{48b}$$

The expansion may be written

$$g_i(\theta) = \sum_{n=0}^{\infty} g_{in} T_n(\theta) \quad \text{and} \quad g_{i+1}(\theta) = \sum_{n=0}^{\infty} g_{(i+1)n} T_n(\theta) \tag{49a}$$

where

$$g_{in} = \int_{\theta=\theta_j}^{\theta_{j+1}} g_i(\theta) T_n(\theta) d\theta \qquad \text{and} \qquad g_{(i+1)n} = \int_{\theta=\theta_j}^{\theta_{j+1}} g_{(i+1)}(\theta) T_n(\theta) d\theta \quad . \tag{49b}$$

In the case of interest here, the boundary conditions on the node interfaces are expressed in terms of currents, so that $\delta_i = \delta_{i+1} = 0$. Also, in this case, the eigenvalues are

$$\lambda_n = n\pi / (\theta_{j+1} - \theta_j) \tag{50a}$$

and the eigenfunctions are

$$T_n(\theta) = \cos\left[\frac{n\pi(\theta - \theta_j)}{\theta_{j+1} - \theta_j}\right] \quad . \tag{50b}$$

The solution expressed in Eq. (42), with $R_n(r)$ supplied by Eqs. (45) and (47), is applied to Eqs. (48), with the boundary conditions $g_i(\theta)$ and $g_{i+1}(\theta)$ expanded according to Eqs. (49), with $\delta_i = \delta_{i+1} = 0$. The resulting equation comprises a series of eigenfunctions on each side. Because the eigenfunctions are linearly independent, the series must be equal term by term. The equation for the n$^{\text{th}}$ term is

$$c_{1n}\left[\frac{\lambda_n}{r_i} I_{\lambda_n}(kr_i) + k\, I_{\lambda_n+1}(kr_i)\right] + c_{2n}\left[\frac{\lambda_n}{r_i} K_{\lambda_n}(kr_i) - k\, K_{\lambda_n+1}(kr_i)\right]$$

$$= -\left[\frac{\lambda_n}{r_i} I_{\lambda_n}(kr_i) + k\, I_{\lambda_n+1}(kr_i)\right] \int_{r'=r_o}^{r_i} \frac{k\xi_n(r')}{r'} K_{\lambda_n}(kr') dr' - I_{\lambda_n}(kr_i)\left[\frac{k\xi_n(r_i)}{r_i} K_{\lambda_n}(kr_i)\right]$$

$$+ \left[\frac{\lambda_n}{r_i} K_{\lambda_n}(kr_i) - k\, K_{\lambda_n+1}(kr_i)\right] \int_{r'=r_o}^{r_i} \frac{k\xi_n(r')}{r'} I_{\lambda_n}(kr') dr' + K_{\lambda_n}(kr_i)\left[\frac{k\xi_n(r_i)}{r_i} I_{\lambda_n}(kr_i)\right]$$

$$+ \left[\int_{\theta=\theta_j}^{\theta_{j+1}} T_n^2(\theta)\, d\theta\right] g_{in} / \gamma_i \quad . \tag{51}$$

and a similar equation evaluated at $r = r_{i+1}$. (In the integral from r_o to r_i, $r_i < r_o$, so the limits of integration may be reversed and the sign of the integral changed accordingly.)

Eq. (51) and its companion form a pair of equations that may be solved for c_{1n} and c_{2n}. Finding these constants completes the solution of Eq. (3) by separation of variables. This solution is easier to implement than the one obtained by integral transforms, because the doubled summation is eliminated. But a third approach, described in the next section, seems even more convenient.

5.0 Solution by the Green's Function Method

The Green's function solution is easier to implement than the solutions presented above. Therefore, only the Green's function solution is integrated to produce one-dimensional solutions. The two-dimensional solution is obtained in Section 5.1. The integration of this solution over θ to produce the one-dimensional solution in *r* is demonstrated in Section 5.2. The integration over *r* is demonstrated in Section 5.3; this leads to the one-dimensional solution in θ.

5.1 The Green's Function Solution for ${}^{z}\overline{\phi}(r,\theta)$

It is convenient to subsume the axial-current term in Eq. (2) into the source term:

$$\frac{{}^{z}\overline{J}'_{z}}{D} - \frac{{}^{z}\overline{S}}{D} \equiv S(r,\theta) \quad . \tag{52}$$

Then Eq. (2) becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\, {}^{z}\overline{\phi}}{\partial r} \right) + \frac{1}{r^2}\frac{\partial^2\, {}^{z}\overline{\phi}}{\partial \theta^2} - \frac{\Sigma_R\, {}^{z}\overline{\phi}}{D} = S(r,\theta) \quad , \tag{53}$$

subject to the boundary conditions

$$-D\frac{\partial\, {}^{z}\overline{\phi}}{\partial r}\bigg|_{r=r_i} = {}^{z}\overline{J}_r(r_i,\theta) \tag{54a}$$

$$-D\frac{\partial\, {}^{z}\overline{\phi}}{\partial r}\bigg|_{r=r_{i+1}} = {}^{z}\overline{J}_r(r_{i+1},\theta) \tag{54b}$$

$$-\frac{D}{r}\frac{\partial\, {}^{z}\overline{\phi}}{\partial \theta}\bigg|_{\theta=\theta_j} = {}^{z}\overline{J}_\theta(r,\theta_j) \tag{54c}$$

$$-\frac{D}{r}\frac{\partial\, {}^{z}\overline{\phi}}{\partial \theta}\bigg|_{\theta=\theta_{j+1}} = {}^{z}\overline{J}_\theta(r,\theta_{j+1}) \ , \tag{54d}$$

in which the boundary currents are averaged over *z* in analogy with the definitions for ${}^{z}\overline{\phi}$ and ${}^{z}\overline{S}$ following Eq. (2). These are true averaged currents, unlike ${}^{z}\overline{J}'_{z}$.

The solution of Eq. (53) is to be generated from the Green's function, $G(r,\theta;\, r_o,\theta_o)$, which satisfies the equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial G}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 G}{\partial\theta^2}-k^2G=\frac{\delta(r-r_o)}{r}\delta(\theta-\theta_o) \tag{55}$$

(recalling Eq. (46)), subject to the homogeneous analogues of Eqs. (54).

The solution for G is sought as a series of the eigenfunctions of the auxiliary problem

$$\frac{d^2\Theta}{d\theta^2}+\lambda^2\Theta=0, \tag{56}$$

subject to the homogeneous boundary conditions

$$\left.\frac{d\Theta}{d\theta}\right|_{\theta=\theta_j}=0 \quad \text{and} \quad \left.\frac{d\Theta}{d\theta}\right|_{\theta=\theta_{j+1}}=0 \quad . \tag{57}$$

The eigenvalues and eigenfunctions are the same as those found in the previous solution:

$$\lambda_n=\frac{n\pi}{\theta_{j+1}-\theta_j}\equiv\frac{n\pi}{2\alpha_o} \tag{58a}$$

and

$$\Theta_o(\tilde{\theta})=\frac{1}{\sqrt{2\alpha_o}} \quad , \tag{58b}$$

$$\Theta_n(\tilde{\theta})=\frac{1}{\sqrt{\alpha_o}}\cos\left[\frac{n\pi}{2\alpha_o}(\tilde{\theta}+\alpha_o)\right] \quad , \quad n=1,2,\ldots \quad , \tag{58c}$$

where $\tilde{\theta}\equiv\theta-\theta_j-\alpha_o$ and $2\alpha_o\equiv\theta_{j+1}-\theta_j$. The factor $1/\sqrt{\alpha_o}$ in Eq. (58c) is imposed to make the functions $\Theta_n(\tilde{\theta})$ orthonormal.

Thus, solutions are being sought of form

$$G(r,\theta;r_o,\theta_o)=\sum_{n=0}^{\infty}\psi_n(r;r_o,\tilde{\theta}_o)\Theta_n(\tilde{\theta}) \quad . \tag{59}$$

Eq. (59) is inserted into Eq. (55), the result is multiplied by $\Theta_m(\tilde{\theta})$ and integrated over $\tilde{\theta}$ from $-\alpha_o$ to $+\alpha_o$, and the orthonormality of the eigenfunctions is invoked. The eventual result is an ordinary differential equation for $\psi_m(r;r_o,\tilde{\theta}_o)$:

$$\frac{1}{r}\frac{d}{dr}\left\{r\frac{d}{dr}\left[\frac{\psi_m(r;r_o,\widetilde{\theta}_o)}{\Theta_m(\widetilde{\theta}_o)}\right]\right\}-\left(\frac{m^2\pi^2}{4\alpha_o^2}\right)\frac{1}{r^2}\left[\frac{\psi_m(r;r_o,\widetilde{\theta}_o)}{\Theta_m(\widetilde{\theta}_o)}\right]-k^2\left[\frac{\psi_m(r;r_o,\widetilde{\theta}_o)}{\Theta_m(\widetilde{\theta}_o)}\right]=\frac{\delta(r-r_o)}{r} \quad . \tag{60}$$

The quantity $\frac{\psi_m(r;r_o,\widetilde{\theta}_o)}{\Theta_m(\widetilde{\theta}_o)}$ is recognized as a Green's function in r, in which $\widetilde{\theta}_o$ appears only as a parameter. Let this quantity be denoted by

$$g_m(r;r_o,\widetilde{\theta}_o)=\frac{\psi_m(r;r_o,\widetilde{\theta}_o)}{\Theta_m(\widetilde{\theta}_o)} \quad , \tag{61}$$

and rewrite Eq. (60) as

$$r^2\frac{d^2 g_m}{dr^2}+r\frac{dg_m}{dr}-\left(k^2r^2+\frac{m^2\pi^2}{2\alpha_o}\right)g_m=0, \quad r\neq r_o \quad , \tag{62}$$

subject to the boundary conditions

$$\left.\frac{dg_m}{dr}\right|_{r=r_i}=0 \qquad \text{and} \qquad \left.\frac{dg_m}{dr}\right|_{r=r_{+1i}}=0 \quad , \tag{63}$$

and the jump condition

$$\left.\frac{dg_m}{dr}\right|_{r=r_o^+}-\left.\frac{dg_m}{dr}\right|_{r=r_o^-}=\frac{1}{r_o} \quad . \tag{64}$$

Eq. (62) is, once again, the modified, or hyperbolic, Bessel equation, and the solutions are $I_{\lambda_m}(kr)$ and $K_{\lambda_m}(kr)$. Because of the jump condition, the solution takes slightly different forms in the regions $r < r_0$ and $r > r_0$. After much algebra, the solution satisfying the boundary conditions and the jump condition is found to be

$$g_\nu^I(r;r_o,\widetilde{\theta}_o)=\frac{\left[I_\nu(kr)K'_\nu(kr_i)-I'_\nu(kr_i)K_\nu(kr)\right]\left[I_\nu(kr_o)K'_\nu(kr_{i+1})-I'_\nu(kr_{i+1})K_\nu(kr_o)\right]}{K'_\nu(kr_i)I'_\nu(kr_{i+1})-K'_\nu(kr_{i+1})I'_\nu(kr_i)}, \tag{65a}$$

which is valid in the domain $r_i \leq r \leq r_0$,
and

$$g_\nu^{II}(r;r_o,\widetilde{\theta}_o)=\frac{\left[I_\nu(kr)K'_\nu(kr_{i+1})-I'_\nu(kr_{i+1})K_\nu(kr)\right]\left[I_\nu(kr_o)K'_\nu(kr_i)-I'_\nu(kr_i)K_\nu(kr_o)\right]}{K'_\nu(kr_i)I'_\nu(kr_{i+1})-K'_\nu(kr_{i+1})I'_\nu(kr_i)}, \tag{65b}$$

which is valid in the domain $r_o \le r \le r_{i+1}$,
where $\nu = m\ell/2$, in which ℓ is the number of (equal) angular zones into which the reactor is nodalized.

In Eqs. (65), $\tilde{\theta}_o$ does not actually appear. Thus, $g_\nu(r;r_o,\tilde{\theta}_o)$ may be acknowledged as a function of r and r_o only, as $g_\nu(r;r_o)$. Then Eq. (59) may be rewritten as

$$G(r,\tilde{\theta};r_o\tilde{\theta}_o) = \sum_{m=0}^{\infty} g_{m\ell/2}(r;r_o)\Theta_m(\tilde{\theta})\Theta_m(\tilde{\theta}_o) \ . \tag{66}$$

Now Eq. (53) is multiplied by G, Eq. (55) is multiplied by ${}^z\bar{\phi}$, the resulting equations are subtracted, and the result of this subtraction is integrated over the domain of r and $\tilde{\theta}$. It is observed that Eq. (66) is symmetric in r and r_o and in $\tilde{\theta}$ and $\tilde{\theta}_o$. This symmetry property is applied, and use is made of Green's identity,

$$\int_\delta \left(f\nabla^2 g - g\nabla^2 f\right) dA = \oint_\Gamma \left(f\vec{\nabla}g - g\vec{\nabla}f\right)\bullet \hat{n}\, d\ell \ , \tag{67}$$

where dA = area element of the domain δ on a plane surface,
Γ = curve enclosing δ,
$d\ell$ = arclength element of Γ,
$\hat{n}$ = unit outer normal vector on Γ,
and f and g are functions defined in δ.

These steps, together with the use of the homogeneous boundary conditions on G and the inhomogeneous boundary conditions on ${}^z\bar{\phi}$ (Eqs. (54)), yield the final result

$$\begin{aligned} {}^z\bar{\phi}(r,\theta) = & \int_{r_o=r_i}^{r_{i+1}} \int_{\theta_o=\theta_j}^{\theta_{j+1}} G(r,\theta;r_o\theta_o)S(r_o,\theta_o)r_o dr_o d\theta_o + \int_{r_o=r_i}^{r_{i+1}} \frac{G(r,\theta;r_o,\theta_{j+1})}{D}\, {}^z\bar{J}_\theta(r_o,\theta_{j+1})\, dr_o \\ & + \int_{\theta_o=\theta_j}^{\theta_{j+1}} \frac{G(r,\theta;r_{i+1},\theta_o)}{D}\, {}^z\bar{J}_r(r_{i+1},\theta_o) r_{i+1} d\theta_o - \int_{r_o=r_i}^{r_{i+1}} \frac{G(r,\theta;r_o,\theta_j)}{D}\, {}^z\bar{J}_\theta(r_o,\theta_j)\, dr_o \\ & - \int_{\theta_o=\theta_j}^{\theta_{j+1}} \frac{G(r,\theta;r_i,\theta_o)}{D}\, {}^z\bar{J}_r(r_i,\theta_o)\, r_i\, d\theta_o \qquad , \end{aligned}$$

(68)

where the angular variable in Eq. (66) has reverted to θ (and θ_o) instead of $\tilde{\theta}$ (and $\tilde{\theta}_o$).

5.2 Derivation of One-Dimensional Solutions in r and θ

5.2.1 Integration over θ

The average of ${}^{z}\overline{\phi}$ over θ is

$$ {}^{\theta z}\overline{\phi}(r) = \frac{1}{\theta_{j+1}-\theta_j}\int_{\theta=\theta_j}^{\theta_{j+1}} {}^{z}\overline{\phi}(r,\theta)\,d\theta \qquad . \tag{69} $$

Then, from Eq. (68),

$$ \left(\theta_{j+1}-\theta_j\right){}^{\theta z}\overline{\phi} = \int_{r_o=r_i}^{r_{i+1}}\int_{\theta_o=\theta_j}^{\theta_{j+1}} S(r_o,\theta_o)\left[\int_{\theta=\theta_j}^{\theta_{j+1}} G(r,\theta;r_o,\theta_o)\,d\theta\right] r_o\,dr_o\,d\theta_o $$

$$ + \int_{r_o=r_i}^{r_{i+1}}\left[{}^{z}\overline{J}_\theta(r_o,\theta_{j+1})\int_{\theta=\theta_j}^{\theta_{j+1}}\frac{G(r,\theta;r_o,\theta_{j+1})}{D}\,d\theta\right]dr_o + \int_{\theta_o=\theta_j}^{\theta_{j+1}}\left[{}^{z}\overline{J}_r(r_{i+1},\theta_o)r_{i+1}\int_{\theta=\theta_j}^{\theta_{j+1}}\frac{G(r,\theta;r_{i+1},\theta_o)}{D}\,d\theta\right]d\theta_o $$

$$ - \int_{r_o=r_i}^{r_{i+1}}\left[{}^{z}\overline{J}_\theta(r_o,\theta_j)\int_{\theta=\theta_j}^{\theta_{j+1}}\frac{G(r,\theta;r_o,\theta_j)}{D}\,d\theta\right]dr_o - \int_{\theta_o=\theta_j}^{\theta_{j+1}}\left[{}^{z}\overline{J}_r(r_i,\theta_o)r_i\int_{\theta=\theta_j}^{\theta_{j+1}}\frac{G(r,\theta;r_i,\theta_o)}{D}\,d\theta\right]d\theta_o $$

.

(70)

When Eq. (66) is inserted for G into Eq. (70) and Eqs. (58) are used for the eigenfunctions, it is found that all the eigenfunctions except the fundamental one ($m = 0$) integrate to zero, and Eq. (70) becomes

$$ D\left(\theta_{j+1}-\theta_j\right){}^{\theta z}\overline{\phi}(r) = \int_{r_o=r_i}^{r}\int_{\theta_o=\theta_j}^{\theta_{j+1}} Dr_o S(r_o,\theta_o)g_o^{II}(r;r_o)\,dr_o\,d\theta_o + \int_{r_o=r}^{r_{i+1}}\int_{\theta_o=\theta_j}^{\theta_{j+1}} Dr_o S(r_o,\theta_o)g_o^{I}(r;r_o)\,dr_o\,d\theta_o $$

$$ + \int_{r_o=r_i}^{r} g_o^{II}(r;r_o)\,{}^{z}\overline{J}_\theta(r_o,\theta_{j+1})\,dr_o + \int_{r_o=r}^{r_{i+1}} g_o^{I}(r;r_o)\,{}^{z}\overline{J}_\theta(r_o,\theta_{j+1})\,dr_o - \int_{r_o=r_i}^{r} g_o^{II}(r;r_o)\,{}^{z}\overline{J}_\theta(r_o,\theta_j)\,dr_o $$

$$ - \int_{r_o=r}^{r_{i+1}} g_o^{I}(r;r_o)\,{}^{z}\overline{J}_\theta(r_o,\theta_j)\,dr_o + r_{i+1}g_o(r;r_{i+1})\int_{\theta_o=\theta_j}^{\theta_{j+1}}{}^{z}\overline{J}_r(r_{i+1},\theta_o)\,d\theta_o - r_i g_o(r;r_i)\int_{\theta_o=\theta_j}^{\theta_{j+1}}{}^{z}\overline{J}_r(r_i,\theta_o)\,d\theta_o \tag{71} $$

where $g_o^{I}(r;r_o)$ is the version of Eq. (65a) that applies when $m = 0$,
and $g_o^{II}(r;r_o)$ is the version of Eq. (65b) that applies when $m = 0$,

and whichever of Eqs. (65a) or (65b) is appropriate is used when the superscript is unspecified.

To proceed further, one must know specific information about the boundary currents and the source term $S(r,\theta)$ (which, it will be recalled, contains the average axial current term ${}^{z}\bar{J}_z'(r,\theta)$). On the boundaries at r_i and r_{i+1}, assume a uniform r-component of the current:

$$\frac{\partial\, {}^{z}\bar{J}_r}{\partial\theta} = 0 \qquad \text{on } r = r_i\,,\ r = r_{i+1}\,. \tag{72}$$

Then
$$\int_{\theta_o=\theta_j}^{\theta_{j+1}} {}^{z}\bar{J}_r(r_i,\theta_o)\,d\theta_o = \left(\theta_{j+1}-\theta_j\right)\,{}^{z}\bar{J}_r(r_i) \tag{73a}$$

and
$$\int_{\theta_o=\theta_j}^{\theta_{j+1}} {}^{z}\bar{J}_r(r_{i+1},\theta_o)\,d\theta_o = \left(\theta_{j+1}-\theta_j\right)\,{}^{z}\bar{J}_r(r_{i+1}) \qquad . \tag{73b}$$

Similarly, assume a uniform θ-component of the current on the boundaries $\theta = \theta_j$ and $\theta = \theta_{j+1}$. Then

$${}^{z}\bar{J}_\theta(r_o,\theta_j) = {}^{z}\bar{J}_\theta(\theta_j) \tag{74a}$$

and

$${}^{z}\bar{J}_\theta(r_o,\theta_{j+1}) = {}^{z}\bar{J}_\theta(\theta_{j+1}) \qquad , \tag{74b}$$

so that

$$\int_{r_o} g_o^{I,II}(r;r_o)\ {}^{z}\bar{J}_\theta(r_o,\theta_{j,j+1})\,dr_o = {}^{z}\bar{J}_\theta(\theta_{j,j+1})\int_{r_o} g_o^{I,II}(r;r_o)\,dr_o \tag{75}$$

over any of the integration intervals in r_o indicated in Eq. (71). In Eq. (75), the symbols I,II and $\theta_{j,j+1}$ indicate alternatives.

The source integrals in Eq. (71) are addressed by expanding $S(r_o,\theta_o)$ in a series of Legendre polynomials and retaining only the first three terms. The standard Legendre polynomials $P_n(x)$ are defined on the interval $-1 \le x \le 1$, which is not appropriate for the present problem. Here the shifted Legendre polynomials, $P_n^*(x)$, are appropriate, where[4]

$$P_o^*(x) = 1 \qquad , \tag{76a}$$

$$P_1^*(x) = 2x - 1 \qquad , \tag{76b}$$

and

$$P_2^*(x) = \frac{3}{2}(2x-1)^2 - \frac{1}{2} \qquad , \tag{76c}$$

which are defined on the interval $0 \le x \le 1$. The variable x is related to r_o by

$$x \equiv \frac{r_o - r_i}{r_{i+1} - r_i} \qquad . \tag{76d}$$

Then the source term is expanded first in a series of eigenfunctions,

$$S(r,\theta) = \sum_{m=0}^{\infty} f_m(r_o)\Theta_m(\theta_o) \tag{77a}$$

and next in a series of Legendre polynomials, truncated at the third term:

$$f_m(r_o) = c_{om}P_o^*\left[x(r_o)\right] + c_{1m}P_1^*\left[x(r_o)\right] + c_{2m}P_2^*\left[x(r_o)\right] \ , \tag{77b}$$

where x is defined in Eq. (76d).

When the assumptions on the boundary currents and the source term, along with the explicit form for g_o from Eq. (65), are inserted into Eq. (71), and the result is evaluated at $r = r_i$ and the integrals are carried out as far as practical, the following expression is derived:

$$\begin{aligned}(\theta_{j+1} - \theta_j)r_{i+1}g_o(r_i;r_{i+1})\ {}^z\bar{J}_r(r_{i+1}) - (\theta_{j+1} - \theta_j)r_i g_o(r_i;r_i)\ {}^z\bar{J}_r(r_i) \\ = D(\theta_{j+1} - \theta_j)\ {}^{\theta z}\bar{\phi}(r_i) - A_1 - B_1\ {}^z\bar{J}_\theta(\theta_{j+1}) + B_1\ {}^z\bar{J}_\theta(\theta_j) \qquad ,\end{aligned} \tag{78}$$

where

$$A_1 = \frac{D\sqrt{\theta_{j+1}-\theta_j}\left[I_o(kr_i)K'_o(kr_i)-I'_o(kr_i)K_o(kr_i)\right]}{\left[I'_o(kr_{i+1})K'_o(kr_i)-I'_o(kr_i)K'_o(kr_{i+1})\right]}\left\{c_{oo}\left[K'_o(kr_{i+1})\int_{r_o=r_i}^{r_{i+1}} r_o I_o(kr_o)dr_o\right.\right.$$

$$\left. - I'_o(kr_{i+1})\int_{r_o=r_i}^{r_{i+1}} r_o K_o(kr_o)\,dr_o\right] + c_{1o}\left[K'_o(kr_{i+1})\int_{r_o=r_i}^{r_{i+1}} r_o^2 I_o(kr_o)\,dr_o\right.$$

$$\left. - I'_o(kr_{i+1})\int_{r_o=r_i}^{r_{i+1}} r_o^2 K_o(kr_o)\,dr_o\right] + c_{2o}\left[K'_o(kr_{i+1})\int_{r_o=r_i}^{r_{i+1}} r_o^3 I_o(kr_o)\,dr_o\right.$$

$$\left.\left. - I'_o(kr_{i+1})\int_{r_o=r_i}^{r_{i+1}} r_o^3 K_o(kr_o)\,dr_o\right]\right\} \qquad , \tag{79}$$

and

$$B_1 = \frac{\left[I_o(kr_i)K'_o(kr_i)-I'_o(kr_i)K_o(kr_i)\right]}{\left[I'_o(kr_{i+1})K'_o(kr_i)-I'_o(kr_i)K'_o(kr_{i+1})\right]}\left\langle\frac{\pi K'_o(kr_{i+1})}{2k}\left\{r_{i+1}\left[I_o(kr_{i+1})L_{-1}(kr_{i+1})\right.\right.\right.$$

$$\left.\left.\left. - I_1(kr_{i+1})L_o(kr_{i+1})\right] - r_i\left[I_o(kr_i)L_{-1}(kr_i) - I_1(kr_i)L_o(kr_i)\right]\right\} - I'_o(kr_{i+1})\int_{r_o=r_i}^{r_{i+1}} K_o(kr_o)dr_o\right\rangle, \tag{80}$$

in which $L_\mu(x)$ is the modified Struve function,

$$L_\mu(x) = \frac{x^\mu}{\Gamma(½)\Gamma(\mu+½)2^{\mu-1}}\int_{z=0}^{\pi/2}\sinh\left[x\cos(z)\right]\sin^{2\mu}(z)dz \qquad . \tag{81}$$

The integrals of modified Bessel functions that remain in Eq. (79) will be evaluated in the numerical solution by Gaussian quadrature.

Eq. (78) is written for each node in the reactor. A system of equations is developed that can be written in matrix form as

$$\underline{\underline{C}}\,\underline{J}_r = \underline{R}\,. \tag{82}$$

In Eq. (82), the radial interface currents are regarded as the unknowns, $\underline{\underline{C}}$ is a coefficient matrix that can be evaluated from the geometry and Eq. (65), and $\underline{R}$ is regarded as a known matrix function. Actually, $\underline{R}$ depends on the unknowns ${}^{\theta z}\bar{\phi}$, ${}^{z}\bar{J}_\theta(\theta_{j+1})$, and

${}^{z}\bar{J}_{\theta}(\theta_j)$. However, in an iterative solution scheme, these quantities can be evaluated from the solution of the previous iteration and used to find new values of $\underline{J}_r$.

Eq. (78) has two unknown ${}^{z}\bar{J}_r$ values, and as Eq. (78) is written for more and more nodes, each node introduces one additional unknown. The system is closed by application of the external boundary condition, which in effect introduces another equation without another unknown quantity. The closed system of equations is solved by standard matrix inversion techniques.

5.2.2 Integration over *r*

The integration over θ was simplified because all the terms in the series of eigenfunctions integrated to zero except the fundamental term, for which $m = 0$. No such simplification happens in the integration over *r*, and the infinite series remains. Otherwise, the procedure for the integration over *r* is the same as the procedure displayed in the previous subsection. The analogue of Eq. (71) is

$$\frac{\left(r_{i+1}^2 - r_i^2\right)}{2} D\, {}^{rz}\bar{\phi}(\theta) = \int_{r_o=r_i}^{r_{i+1}} \int_{\theta_o=\theta_j}^{\theta_{j+1}} D\left[S(r_o,\theta_o) \sum_{m=0}^{\infty} \Theta_m(\theta)\Theta_m(\theta_o) \int_{r=r_i}^{r_{i+1}} r g_{m\ell/2}(r;r_o)\, dr \right] r_o\, dr_o\, d\theta_o$$

$$+ \int_{r_o=r_i}^{r_{i+1}} \left[{}^{z}\bar{J}_{\theta}(r_o,\theta_{j+1}) \sum_{m=0}^{\infty} \Theta_m(\theta)\Theta_m(\theta_{j+1}) \int_{r=r_i}^{r_{i+1}} r g_{m\ell/2}(r;r_o) dr \right] dr_o$$

$$+ \int_{\theta_o=\theta_j}^{\theta_{j+1}} \left[{}^{z}\bar{J}_r(r_{i+1},\theta_o) r_{i+1} \sum_{m=0}^{\infty} \Theta_m(\theta)\Theta_m(\theta_o) \int_{r=r_i}^{r_{i+1}} r g_{m\ell/2}(r;r_{i+1}) dr \right] d\theta_o$$

$$- \int_{r_o=r_i}^{r_{i+1}} \left[{}^{z}\bar{J}_{\theta}(r_o,\theta_j) \sum_{m=0}^{\infty} \Theta_m(\theta)\Theta_m(\theta_j) \int_{r=r_i}^{r_{i+1}} r g_{m\ell/2}(r;r_o) dr \right] dr_o$$

$$- \int_{\theta_o=\theta_j}^{\theta_{j+1}} \left[{}^{z}\bar{J}_r(r_i,\theta_o) r_i \sum_{m=0}^{\infty} \Theta_m(\theta)\Theta_m(\theta_o) \int_{r=r_i}^{r_{i+1}} r g_{m\ell/2}(r;r_i) dr \right] d\theta_o \quad . \tag{83}$$

Again, the assumption of "flat" boundary currents is made, and the source term is expanded in accordance with Eqs. (77). The result is evaluated at $\theta = \theta_j$ and the integrals are carried out as far as practical. The eventual result is

$$-\left[\sum_{m=0}^{\infty}\Theta_m^2(\theta_j)A_{2\nu}\right]{}^z\bar{J}_\theta(\theta_j)+\left[\sum_{m=0}^{\infty}\Theta_m(\theta_j)\Theta_m(\theta_{j+1})A_{2\nu}\right]{}^z\bar{J}_\theta(\theta_{j+1})$$

$$=\frac{\left(r_{i+1}^2-r_i^2\right)}{2}D\,{}^{rz}\bar{\phi}(\theta_j)-\frac{\Theta_o(\theta_j)B_2D}{D_1}+\frac{\Theta_1(\theta_j)C_2D}{D_2}$$

$$-\sqrt{\theta_{j+1}-\theta_j}\;\Theta_o(\theta_j)\left[r_{i+1}\,{}^z\bar{J}_r(r_{i+1})E_2+r_i\,{}^z\bar{J}_r(r_i)F_2\right] \quad , \tag{84}$$

where

$$A_{2\nu}=\frac{1}{K_o'(kr_i)I_o'(kr_{i+1})-K_o'(kr_{i+1})I_o'(kr_i)}\int_{r_o=r_i}^{r_{i+1}}\left\{\frac{K_o'(kr_i)}{k}\left[K_o'(kr_{i+1})\,r_oI_o(kr_o)I_1(kr_o)\right.\right.$$

$$\left.-I_o'(kr_{i+1})\,r_oK_o(kr_o)I_1(kr_o)-r_iK_o'(kr_{i+1})I_1(kr_i)I_o(kr_o)+r_iI_o'(kr_{i+1})I_1(kr_i)K_o(kr_o)\right]$$

$$+\frac{I_o'(kr_i)}{k}\left[K_o'(kr_{i+1})\,r_oI_o(kr_o)K_1(kr_o)-I_o'(kr_{i+1})\,r_oK_o(kr_o)K_1(kr_o)-r_iK_o'(kr_{i+1})K_1(kr_i)I_o(kr_o)\right.$$

$$\left.+r_iI_o'(kr_{i+1})K_1(kr_i)K_o(kr_o)\right]+\frac{K_o'(kr_{i+1})}{k}\left[r_{i+1}K_o'(kr_i)I_1(kr_{i+1})I_o(kr_o)-r_{i+1}I_o'(kr_i)I_1(kr_{i+1})K_o(kr_o)\right.$$

$$\left.-K_o'(kr_i)\,r_oI_o(kr_o)I_1(kr_o)+I_o'(kr_i)\,r_oK_o(kr_o)I_1(kr_o)\right]+\frac{I_o'(kr_{i+1})}{k}\left[r_{i+1}K_o'(kr_i)K_1(kr_{i+1})I_o(kr_o)\right.$$

$$\left.\left.-r_{i+1}I_o'(kr_i)K_1(kr_{i+1})K_o(kr_o)-K_o'(kr_i)\,r_oI_o(kr_o)K_1(kr_o)+I_o'(kr_i)\,r_oK_o(kr_o)K_1(kr_o)\right]\right\}dr_o \tag{85a}$$

if $\nu=m\ell/2=0$, or

$$A_{2\nu} = \frac{1}{K'_\nu(kr_i)I'_\nu(kr_{i+1}) - K'_\nu(kr_{i+1})I'_\nu(kr_i)} \int_{r_o=r_i}^{r_{i+1}} \Bigg\langle \left[I_\nu(kr_o)K'_\nu(kr_{i+1}) - I'_\nu(kr_{i+1})K_\nu(kr_o)\right]\Bigg\{ \frac{K'_\nu(kr_i)}{k}\Big[$$

$$r_o I_{\nu-1}(kr_o) - r_i I_{\nu-1}(kr_i) - \nu \int_{r=r_i}^{r_o} I_{\nu-1}(kr)\,dr\Big] + \frac{I'_\nu(kr_i)}{k}\left[r_o K_{\nu-1}(kr_o) - r_i K_{\nu-1}(kr_i) - \nu \int_{r=r_i}^{r_o} K_{\nu-1}(kr)dr\right] \Bigg\}$$

$$+\left[I_\nu(kr_o)K'_\nu(kr_i) - I'_\nu(kr_i)K_\nu(kr_o)\right]\Bigg\{ \frac{K'_\nu(kr_{i+1})}{k}\left[r_{i+1}I_{\nu-1}(kr_{i+1}) - r_o I_{\nu-1}(kr_o) - \nu \int_{r=r_o}^{r_{i+1}} I_{\nu-1}(kr)dr\right]$$

$$+\frac{I'_\nu(kr_{i+1})}{k}\left[r_{i+1}K_{\nu-1}(kr_{i+1}) - r_o K_{\nu-1}(kr_o) - \nu \int_{r=r_o}^{r_{i+1}} K_{\nu-1}(kr)dr\right] \Bigg\} \Bigg\rangle dr_o \qquad (85b)$$

if $\nu = m\ell/2 > 0$,

and

$$B_2 = \int_{r_o=r_i}^{r_{i+1}} \left(C_{oo}r_o + C_{1o}r_o^2 + C_{2o}r_o^3\right)\Bigg\{ \frac{K'_o(kr_i)}{k}\left[K'_o(kr_{i+1})r_o I_o(kr_o)I_1(kr_o) - r_i K'_o(kr_{i+1})I_1(kr_i)I_o(kr_o)\right.$$

$$-I'_o(kr_{i+1})r_o K_o(kr_o)I_1(kr_o) + r_i I'_o(kr_{i+1})I_1(kr_i)K_o(kr_o)\Big] + \frac{I'_o(kr_i)}{k}\Big[K'_o(kr_{i+1})r_o I_o(kr_o)K_1(kr_o)$$

$$\left. - r_i K'_o(kr_{i+1})K_1(kr_i)I_o(kr_o) - I'_o(kr_{i+1})r_o K_o(kr_o)K_1(kr_o) + r_i I'_o(kr_{i+1})K_1(kr_i)K_o(kr_o)\right]$$

$$+\frac{K'_o(kr_{i+1})}{k}\Big[r_{i+1}K'_o(kr_i)I_1(kr_{i+1})I_o(kr_o) - K'_o(kr_i)r_o I_o(kr_o)I_1(kr_o) - r_{i+1}I'_o(kr_i)I_1(kr_{i+1})K_o(kr_o)$$

$$+I'_o(kr_i)r_o K_o(kr_o)I_1(kr_o)\Big] + \frac{I'_o(kr_{i+1})}{k}\Big[r_{i+1}K'_o(kr_i)K_1(kr_{i+1})I_o(kr_o) - K'_o(kr_i)r_o I_o(kr_o)K_1(kr_o)$$

$$-r_{i+1}I'_o(kr_i)K_1(kr_{i+1})K_o(kr_o) + I'_o(kr_i)r_o K_o(kr_o)K_1(kr_o)\Big] \Bigg\} dr_o \qquad , \qquad (86)$$

$$C_2 = \int_{r_o=r_i}^{r_{i+1}} \left(C_{o1} r_o + C_{11} r_o^2 + C_{21} r_o^3\right) \Big\langle \left[I_{\ell/2}(kr_o) K'_{\ell/2}(kr_{i+1}) - I'_{\ell/2}(kr_{i+1}) K_{\ell/2}(kr_o) \right] \left\{ \frac{K'_{\ell/2}(kr_i)}{k} \right[$$

$$r_o I_{\ell/2 - 1}(kr_o) - r_i I_{\ell/2 - 1}(kr_i) - \frac{\ell}{2} \int_{r=r_i}^{r_o} I_{\ell/2 - 1}(kr) dr \Bigg] + \frac{I'_{\ell/2}(kr_i)}{k} \Bigg[r_o K_{\ell/2 - 1}(kr_o) - r_i K_{\ell/2 - 1}(kr_i)$$

$$- \frac{\ell}{2} \int_{r=r_i}^{r_o} K_{\ell/2 - 1}(kr) dr \Bigg] \Bigg\} + \left[I_{\ell/2}(kr_o) K'_{\ell/2}(kr_i) - I'_{\ell/2}(kr_i) K_{\ell/2}(kr_o) \right] \left\{ \frac{K'_{\ell/2}(kr_{i+1})}{k} \right[\; r_{i+1} I_{\ell/2 - 1}(kr_{i+1})$$

$$- r_o I_{\ell/2 - 1}(kr_o) - \frac{\ell}{2} \int_{r=r_o}^{r_{i+1}} I_{\ell/2 - 1}(kr) dr \Bigg] + \frac{I'_{\ell/2}(kr_{i+1})}{k} \Bigg[r_{i+1} K_{\ell/2 - 1}(kr_{i+1}) - r_o K_{\ell/2 - 1}(kr_o) - \frac{\ell}{2} \int_{r=r_o}^{r_{i+1}} K_{\ell/2 - 1}(kr) dr \Bigg] \Bigg\} \Big\rangle dr_o \,, \tag{87}$$

$$D_1 = K'_o(kr_i) I'_o(kr_{i+1}) - K'_o(kr_{i+1}) I'_o(kr_i) \qquad , \tag{88}$$

$$D_2 = K'_{\ell/2}(kr_i) I'_{\ell/2}(kr_{i+1}) - K'_{\ell/2}(kr_{i+1}) I'_{\ell/2}(kr_i) \qquad , \tag{89}$$

$$E_2 = \frac{\left[I_o(kr_{i+1})K'_o(kr_{i+1}) - I'_o(kr_{i+1})K_o(kr_{i+1})\right]}{\left[K'_o(kr_i)I'_o(kr_{i+1}) - K'_o(kr_{i+1})I'_o(kr_i)\right]} \left\{ \frac{K'_o(kr_i)}{k}\left[r_{i+1}I_1(kr_{i+1}) - r_i I_1(kr_i)\right] \right.$$

$$\left. + \frac{I'_o(kr_i)}{k}\left[r_{i+1}K_1(kr_{i+1}) - r_i K_1(kr_i)\right] \right\} \quad , \tag{90}$$

and

$$F_2 = \frac{\left[I_o(kr_i)K'_o(kr_i) - I'_o(kr_i)K_o(kr_i)\right]}{\left[K'_o(kr_i)I'_o(kr_{i+1}) - K'_o(kr_{i+1})I'_o(kr_i)\right]} \left\{ \frac{K'_o(kr_{i+1})}{k}\left[r_{i+1}I_1(kr_{i+1}) - r_i I_1(kr_i)\right] \right.$$

$$\left. + \frac{I'_o(kr_{i+1})}{k}\left[r_{i+1}K_1(kr_{i+1}) - r_i K_1(kr_i)\right] \right\} \quad . \tag{91}$$

In Eqs. (86) and (87), the coefficients C_{pm} arise from collecting the coefficients of the Legendre polynomial expansion of Eq. (77b). Specifically,

$$C_{om} = c_{om} - c_{1m} + c_{2m} - \frac{(2c_{1m} - 6c_{2m})r_i}{r_{i+1} - r_i} + \frac{6c_{2m}r_i^2}{(r_{i+1} - r_i)^2} \quad , \tag{92a}$$

$$C_{1m} = \frac{(2c_{1m} - 6c_{2m})}{r_{i+1} - r_i} - \frac{12c_{2m}r_i}{(r_{i+1} - r_i)^2} \quad , \tag{92b}$$

and

$$C_{2m} = \frac{6c_{2m}}{(r_{i+1} - r_i)^2} \quad . \tag{92c}$$

The multiplications in Eqs. (85)-(87) can be carried out to obtain expressions in which the integrations can be attempted term by term. Some of the integrals can then be evaluated explicitly, but in general it is easier to evaluate them numerically, for example by Gaussian quadrature. The expressions so obtained can be used to construct a matrix equation like Eq. (82).

6.0 Transverse Integration for an Equation in *z*

The integration over *r* succeeds in this case, because the integration over θ is carried out first. If Eq. (1) is integrated over θ, the result is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\,{}^{\theta}\overline{\phi}}{\partial r}\right)+\frac{\partial^2\,{}^{\theta}\overline{\phi}}{\partial z^2}-\frac{\Sigma_R}{D}{}^{\theta}\overline{\phi}=\frac{1}{rD}{}^{\theta}\overline{J}'_{\theta}-\frac{{}^{\theta}\overline{S}}{D} \qquad , \qquad (93)$$

where

$${}^{\theta}\overline{\phi}\equiv\frac{1}{(\theta_{j+1}-\theta_j)}\int_{\theta=\theta_j}^{\theta_{j+1}}\phi\,d\theta \qquad , \qquad (94a)$$

$${}^{\theta}\overline{S}\equiv\frac{1}{(\theta_{j+1}-\theta_j)}\int_{\theta=\theta_j}^{\theta_{j+1}}S\,d\theta \qquad , \qquad (94b)$$

and

$${}^{\theta}\overline{J}'_{\theta}\equiv\frac{1}{(\theta_{j+1}-\theta_j)}\left[J_{\theta}\right]_{\theta_j}^{\theta_{j+1}} . \qquad (94c)$$

When Eq. (93) is multiplied by *r* (the weighting function) and integrated from r_i to r_{i+1}, the equation in *z* is obtained straightforwardly:

$$\frac{d^2\,{}^{r\theta}\overline{\phi}}{dz^2}-\frac{\Sigma_R}{D}{}^{r\theta}\overline{\phi}=\frac{1}{D}{}^{r\theta}\overline{J}_r+\frac{1}{D}{}^{r\theta}\overline{J}'_{\theta}-\frac{1}{D}{}^{r\theta}\overline{S} \qquad , \qquad (95)$$

where

$${}^{r\theta}\overline{\phi}\equiv\frac{2}{(r_{i+1}^2-r_i^2)}\int_{r=r_i}^{r_{i+1}}{}^{\theta}\overline{\phi}\,rdr \qquad , \qquad (96a)$$

$${}^{r\theta}\overline{S}\equiv\frac{2}{(r_{i+1}^2-r_i^2)}\int_{r=r_i}^{r_{i+1}}{}^{\theta}\overline{S}\,rdr \qquad , \qquad (96b)$$

$${}^{r\theta}\overline{J}'_{\theta}\equiv\frac{2}{(r_{i+1}^2-r_i^2)}\int_{r=r_i}^{r_{i+1}}{}^{\theta}\overline{J}'_{\theta}\,rdr \qquad , \qquad (96c)$$

and

$$ {}^{r\theta}\bar{J}_r \equiv \frac{2}{r_{i+1}^2 - r_i^2} \int_{r=r_i}^{r_{i+1}} {}^{\theta}\bar{J}_r \, r dr \quad , \tag{96d} $$

in which

$$ {}^{\theta}\bar{J}_r \equiv \frac{1}{\theta_{j+1} - \theta_j} \int_{\theta_j}^{\theta_{j+1}} J_r \, d\theta \quad . \tag{96e} $$

This problem is readily solved by standard methods. The solutions are hyperbolic sines and cosines.

7.0 Equivalence of Solutions

The actual physical flux distribution in the reactor is unique, so all physically meaningful solutions of the diffusion equation must be equivalent. The equivalence of the solutions obtained in Sections 3.0-6.0 is not apparent.

A proof of the equivalence of the three solutions is beyond the scope of this paper. However, some considerations pertaining to their equivalence are addressed.

In the solution by integral transforms, a double summation is obtained, in which one of the summations is a series of the radial eigenfunctions, which are linear combinations of the Bessel functions $J_{\gamma n}(\lambda_{mn} r)$ and $Y_{\gamma n}(\lambda_{mn} r)$. In the other two solutions, only a single summation is obtained; this is a series of the θ-eigenfunctions, which are cosines. The coefficients of these eigenfunctions are functions of the modified Bessel functions $I_\nu(\lambda_m r)$ and $K_\nu(\lambda_m r)$. A reasonable approach to showing equivalence would be to expand the functions $I_\nu(\lambda_m r)$ and $K_\nu(\lambda_m r)$ in series of $J_\nu(\lambda_m r)$ and $Y_\nu(\lambda_m r)$ in the separation-of-variables and Green's function solutions and try to manipulate those solutions into the form of the integral transform solution.

The separation-of-variables and Green's function solutions are similar in being series of the θ-eigenfunctions, with coefficients dependent on I_ν and K_ν. The key difference is that in the separation-of-variables solution, a set of cumbersome equations (Eqs. (50)) must be solved for the coefficients of I_ν and K_ν, whereas in the Green's function solution a collection of very recalcitrant integrals of modified Bessel functions must be found. Neither problem is readily amenable to solution in closed form. Perhaps the easiest way to show equivalence would be a broad survey of numerical evaluations for a comprehensive parameter set. This would not prove equivalence, but it would justify confidence.

However, any approach to demonstrating equivalence is beyond the scope of the funding contract for the project reported here.

8.0 Summary

The traditional transverse-integration approach has been described for developing nodal methods in reactor physics analysis by the neutron diffusion equation. The method succeeds in the derivation of an ordinary differential equation in z (as demonstrated in Section 6.0). It also succeeds in the derivation of an ODE in r, which the authors have done, but which is not reported here. However, the method fails in the derivation of an ODE in θ.

In lieu of obtaining an ODE in θ by integrating the partial differential diffusion equation in r and θ (already averaged over z) over r, the novel tactic is applied of finding a solution in r and θ directly and then integrating that over r to obtain the solution in θ. The two-dimensional solution is also integrated over θ to obtain a one-dimensional solution in r. The two-dimensional solution is obtained in three different ways (Sections 3.0, 4.0, and 5.1), and the one-dimensional solutions are obtained in Section 5.2. The three solution methods are the method of integral transforms, the method of separation of variables, and the Green's function method.

Although the physical solution is unique, the mathematical solutions obtained in Sections 3.0-5.0 appear quite different. The requirement for equivalence, and some possible paths to demonstrating equivalence, are discussed in Section 7.0.

The Green's function solution, coupled with the solution of the axial equation derived in Section 6.0, will be implemented in the CYNOD code. This code and the numerical results obtained from it will be described in a future report.

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