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Geometric Transitions, Topological Strings, and Generalized Complex Geometry

Wu-yen Chuang

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GEOMETRIC TRANSITIONS, TOPOLOGICAL STRINGS, AND GENERALIZED COMPLEX GEOMETRY

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Wu-yen Chuang
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I certify that I have read this dissertation and that, in my opinion, it is fully adequate in scope and quality as a dissertation for the degree of Doctor of Philosophy.

(Michael E. Peskin) Principal Adviser

I certify that I have read this dissertation and that, in my opinion, it is fully adequate in scope and quality as a dissertation for the degree of Doctor of Philosophy.

(Shamit Kachru)

I certify that I have read this dissertation and that, in my opinion, it is fully adequate in scope and quality as a dissertation for the degree of Doctor of Philosophy.

(Eva M. Silverstein)

Approved for the University Committee on Graduate Studies.

Abstract

Mirror symmetry is one of the most beautiful symmetries in string theory. It helps us very effectively gain insights into non-perturbative worldsheet instanton effects. It was also shown that the study of mirror symmetry for Calabi-Yau flux compactification leads us to the territory of "Non-Kählerity."

In this thesis we demonstrate how to construct a new class of symplectic non–Kähler and complex non–Kähler string theory vacua via generalized geometric transitions. The class admits a mirror pairing by construction. From a variety of sources, including supergravity analysis and KK reduction on $SU(3)$ structure manifolds, we conclude that string theory connects Calabi-Yau spaces to both complex non-Kähler and symplectic non-Kähler manifolds and the resulting manifolds lie in generalized complex geometry.

We go on to study the topological twisted models on a class of generalized complex geometry, bi-Hermitian geometry, which is the most general target space for $(2,2)$ worldsheet theory with non-trivial H flux turned on. We show that the usual Kähler A and B models are generalized in a natural way.

Since the gauged supergravity is the low energy effective theory for the compactifications on generalized geometries, we study the fate of flux-induced isometry gauging in $N = 2$ IIA and heterotic strings under non-perturbative instanton effects. Interestingly, we find we have protection mechanisms preventing the corrections to the hyper moduli spaces. Besides generalized geometries, we also discuss the possibility of new NS-NS fluxes in a new doubled formalism.

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Chapter 1

Introduction and Overview

The superstring theory is a ten dimensional theory of quantum gravity, which unifies four fundamental interactions in a consistent framework. It has the desirable finiteness property due to the existence of supersymmetry in the theory and the fact that the string length provides a good physical origin of the cutoff scale. Given the observation that our world is four dimensional, we definitely want to find situations within string theory where the nature of the extra dimensions is consistent with real world.

Under this guiding principle Calabi-Yau spaces first arose in the study of superstring as an internal space for heterotic $N = 1$ compactifications, which people believed could lead to realistic $N = 1$ four dimensional models [5]. Besides the phenomenological significance, $N = 2$ type II superstring compactifications on Calabi-Yau spaces are also interesting on its own because of the rich $N = 2$ dynamics and many mathematical impacts, among them mirror symmetry is one such example.

Furthurmore, by incorporating the orientifold planes and turning on the background p-form fluxes, one can get an $N = 1$ model with most of the moduli being stabilized. In addition the Kähler moduli can be stabilized by using more complicated nonperturbative effects [6] [7]. Now we turn our attentions to the subjects which will be covered in this thesis.

1.1 Mirror symmetry

Mirror symmetry is T-duality along the T^3 special Lagrangian fibration in Calabi-Yau spaces [8]. This spacetime picture is intuitive but does not give us much computational power. The power of mirror symmetry comes from the fact that it helps us gain insights into the worldsheet instanton effects in string theory.

This will be best discussed in the worldsheet formalism. The sigma model action of string theory is defined by a map Φ from a compact Riemann surface Σ_g of genus g to the target space X and an action of a two dimensional field theory $S(\Phi, G, B)$. The bosonic part of the action is given by:

$$S = \frac{1}{2\pi\alpha'}\int_{\Sigma_g} d^2\sigma\sqrt{h}(h^{\alpha\beta}G_{ij}(\phi)\partial_\alpha\phi^i\partial_\beta\phi^j + \epsilon^{\alpha\beta}B_{ij}(\phi)\partial_\alpha\phi^i\partial_\beta\phi^j + \cdots) \tag{1.1}$$

where ϕ^i $(i = 1, ..., dim(X))$ and σ^α are local coordinates on Σ_g and X respectively.

In order to have a classical string theory vacuum, this worldsheet theory is required to be conformally invariant. That is to say, the β functions for dilaton, spacetime metric and B field should vanish. The dilaton β function restricts the theory in the critical dimensions while the other two are spacetime Einstein's equation and the equation of motion for B field.

The mirror symmetry essentially is a symmetry between two sigma model with topologically different spacetime targets. It makes the study easier to incorporate worldsheet $(2, 2)$ SUSY into the sigma model. The SUSY will provide for us good control over the system, because the path integral will localize to the fixed loci of the worldsheet fermions, and make A and B twists possible [1]. Without H flux, SUSY will imply Kählerity. The Calabi-Yau condition will be furthur imposed as an anomaly cancelation relation for the B-model. From now on we would like to focus on the CY cases and specialize to topological twisted A and B models in our discussion.

The first explicit computation via the use of mirror symmetry was given in the seminal paper by Candelas, de la Ossa, Green and Parkes [9] where the simplest Calabi-Yau manifold was treated, the quintic $\mathcal{M}$ and mirror quintic $\mathcal{M}'$, which only have one Kähler modulus and complex modulus respectively.

The relevant physical quantities for the topological B-model on the mirror quintic $\mathcal{M}'$ at genus $g = 0$ is a three point function (Yukawa coupling) built out of three $(2, 1)$ form as follows.

$$\int_{\mathcal{M}'} \Omega \wedge b^i \wedge b^j \wedge b^k \Omega_{ijk} \tag{1.2}$$

where $b^i = (b)^i_{\bar{j}} d\bar{z}^{\bar{j}}$ is the unique element in $H^1(\mathcal{M}', TX)$. It is related to the unique

[1] Readers interested in the A and B models are referred to [55] or chapter three.

harmonic $(2,1)$ form by $(b)^i_{\bar{j}} = \frac{1}{2|\Omega|^2}\Omega^{ikl}b_{kl\bar{j}}$. We can furthur parametrize the complex deformation space by a complex variable ψ and obtain

$$\int_{\mathcal{M}'} \Omega \wedge b^i \wedge b^j \wedge b^k \Omega_{ijk} = \int_{\mathcal{M}'} \Omega \wedge \frac{\partial^3 \Omega}{\partial \psi^3} = (\frac{2\pi i}{5})^3 \frac{5\psi^2}{1-\psi^5} \tag{1.3}$$

Since the B-model does not depend on Kähler moduli we can perform the computation at large radius limit. Therefore, this quantity can be computed from the classical geometry. On the other hand, the corresponding three point function in A-model will be subject to the worldsheet instanton effect because A-model depends on Kähler modulus. At $g=0$ the three point function is,

$$\int_{\mathcal{M}} e \wedge e \wedge e + \sum_{d=1}^{\infty} n_d d^3 \frac{q^d}{1-q^d} \tag{1.4}$$

where e is the generator of $H^{1,1}(\mathcal{M})$ in the quintic $\mathcal{M}$ and n_d is the number of degree d rational curves. Moreover there is an exact mirror mapping between ψ and q [9] which enables us to extract n_d to arbitrarily large degree.

In retrospect, mirror symmetry is a perturbative symmetry in spacetime keeping fixed the dilaton, while it is a non-perturbative symmetry on the worldsheet. Due to its quantum non-perturbative feature on the worldsheet, we can simply perform the three point function calculation on the perturbative side and obtain the knowledge of worldsheet instantons on the other side. The complex/symplectic mirror construction in this thesis will mostly rely on the spacetime picture instead of the worldsheet construction. This is because the worldsheet formalism for non-Kähler spaces is not yet well established, unlike the CY case in which we can use gauged linear sigma model approach [27] to produce a large number of examples.

1.2 Topology changing processes

Many other amazing structures also have been studied in the $N=2$ Calabi-Yau compactifications [10]. For instance, it has been suggested that by using topology changing processes, including extremal transitions and flops, one can roll among all the Calabi-Yau spaces with different Hodge numbers. The transition has nice physical realization if we take into account the massless brane states appearing at the conifold points. This property turns into a mathematical statement that all the moduli spaces of Calabi-Yau spaces are connected by the extremal transitions and flops [11].

A stronger version of this conjecture is called "Reid's fantasy," which states that the moduli spaces of all the three folds with trivial canonical bundle are connected too [12]. In the second chapter we will show that in order to realize the transition physically, we need to find a special point in the moduli space, where there is a single curve or three cycle shrinking in the Calabi-Yau space. We will show that we can achieve this transition smoothly, by turning on the appropriate fluxes over a certain cycle. A mirror pairing will then be found by construction. We will compare hints from ten–dimensional supergravity analysis and KK reduction on $SU(3)$ structure manifolds and obtain a picture in which string theory extends Reid's fantasy to connect classes of both complex non-Kähler and symplectic non-Kähler manifolds.

1.3 Generalized complex geometry

The complex/symplectic mirror construction naturally motivates the furthur study on the mirror symmetry for generalized complex geometry, of which $SU(3)$ structure manifold is a special class. Generalized complex geometry (GCG) was first introduced by Hitchin and Gualtieri [13] [14] as a single mathematical framework to unify complex and symplectic geometries by treating the spacetimes B field and the metric on an equal footing. For a short introduction to GCG, please see Appendix A. If we want to study the mirror symmetry from the worldsheet approach, we will find bi-Hermitian geometry[2] unavoidable, for the reason that this is the most general geometry for $(2,2)$ worldsheet supersymmetry. In the third chapter we will study the topological twisted sigma model on bi-Hermitian geometry with H-flux and show that the resulting action consists of a BRST exact term and pullback terms, which only depend on one of the two generalized complex structures and the B-field.

1.4 Gauged supergravities and flux-induced gauging

The most general six dimensional manifold which can result in gauged $N = 2$ supergravity as the low energy effective theory is simply the $SU(3) \times SU(3)$ manifold [44]. The intrinsic torsions in the geometry will determine the gauging of the isometries in the hypermultiplet moduli space. One interesting question to ask is, what is the fate of the flux-induced isometry gauging or the torsion-induced isometry gauging after we take into account the

[2] Also known as Twisted Generalized Kähler Geometry, a class of GCG.

nonperturbative effects in string theory? People have successfully answered the question in IIA and M-theory setting [73] [80]. In the fourth chapter, we will take one step furthur and demonstrate the protection mechanism in $N = 2$ heterotic strings.

1.5 Organization of the thesis

The thesis is organized as follows. The second chapter is based on the work done with Shamit Kachru and Alessandro Tomasiello [1]. In the third chapter I discuss the framework in which one can study the topological twisted models and mirror symmetry on bi-Hermitian geometries [2]. The fourth chapter is about the protection mechanism for the flux-induced isometry gauging in heterotic string theory, based on the paper [3] in collaboration with Peng Gao. In the last chapter we will change gears and discuss the non-geometric fluxes in the doubled geometry [4] [3].

[3]The doubled geometry in fact has some similarity with the generalized geometry.

Chapter 2

Complex/Symplex mirrors

Abstract

We construct a class of symplectic non–Kähler and complex non–Kähler string theory vacua, extending and providing evidence for an earlier suggestion by Polchinski and Strominger. The class admits a mirror pairing by construction. Comparing hints from a variety of sources, including ten–dimensional supergravity and KK reduction on SU(3)–structure manifolds, suggests a picture in which string theory extends Reid's fantasy to connect classes of both complex non-Kähler and symplectic non-Kähler manifolds.

2.1 Complex and symplectic vacua

The study of string theory on Calabi–Yau manifolds has provided both the most popular vacua of the theory, and some of the best tests of theoretical ideas about its dynamics. Most manifolds, of course, are not Calabi–Yau. What is the next simplest class for theorists to explore?

The answer, obviously, depends on what the definition of "simplest" is. However, many leads seem to be pointing to the same suspects. First of all, it has been suggested long ago [15] that type II vacua exist, preserving $\mathcal{N} = 2$ supersymmetry (*the same* as for Calabi–Yau's), on manifolds which are complex and non–Kähler (and enjoy vanishing c_1). Calabi–Yau manifolds are simultaneously complex and symplectic, and mirror symmetry can be viewed as an exchange of these two properties [16]. The same logic seems to suggest that the proposal of [15] should also include symplectic non–Kähler manifolds as mirrors of the complex non–Kähler ones. Attempts at providing mirrors of this type (without using a physical interpretation) have indeed already been made in [17, 18].

In a different direction, complex non–Kähler manifolds have also featured in supersymmetry–preserving vacua of supergravity, already in [19]. More recently the general conditions for preserving $\mathcal{N} = 1$ supersymmetry in supergravity have been reduced to geometrical conditions [20]; in particular, the manifold has to be generalized complex [13]. The most prominent examples of generalized complex manifolds are complex and symplectic manifolds, neither necessarily Kähler. It should also be noted that complex and symplectic manifolds seem to be natural in topological strings.

In this paper we tie these ideas together. We find that vacua of the type described in [15] can be found for a large class of complex non–Kähler manifolds in type IIB and symplectic non–Kähler manifolds in type IIA, and observe that these vacua come in mirror pairs. Although these vacua are not fully amenable to ten–dimensional supergravity analysis for reasons that we will explain (this despite the fact that they preserve $\mathcal{N} = 2$ rather than $\mathcal{N} = 1$ supersymmetry), this is in agreement with the supergravity picture that all (RR) SU(3)–structure IIA vacua are symplectic [21], and all IIB vacua are complex [22, 23, 21], possibly suggesting a deeper structure.

In section 2.2, in an analysis formally identical to [15], we argue for the existence of the new vacua. In section 2.3 we show that the corresponding internal manifolds are not Calabi–Yau but rather complex or symplectic. More specifically, in both theories, they are obtained from a transition that does not preserve the Calabi–Yau property. As evidence for this, we show that the expected physical spectrum agrees with the one obtained on the proposed manifolds. The part of this check that concerns the massless spectrum is straightforward; we can extend it to low-lying massive fields by combining results from geometry [24] and KK reduction on manifolds of SU(3) structure. We actually try in section 2.4 to infer from our class of examples a few properties which should give more control over this kind of KK reduction. Specifically, we suggest that the lightest massive fields should be in correspondence with pseudo–holomorphic curves or pseudo–Special–Lagrangian three–cycles (a notion we will define at the appropriate juncture).

Among the motivations for this paper were also a number of more grandiose questions about the effective potential of string theory. One of the motivations for mathematicians to study the generalized type of transition we consider in this paper is the hope that many moduli spaces actually happen to be submanifolds of a bigger moduli space, not unlike [12] the realization of the various 19–dimensional moduli spaces of algebraic K3's as submanifolds of the 20–dimensional moduli space of abstract K3's. It might be that string theory provides a natural candidate for such a space, at least for the $\mathcal{N} = 2$ theories, whose points would be all SU(3)–structure manifolds (not necessarily complex or symplectic), very possibly augmented by non–geometrical points [25]. We would not call it a moduli space, but rather a configuration space: on it, a potential would be defined, whose zero locus would then be the moduli space of $\mathcal{N} = 2$ supersymmetric string theory vacua, including in particular the complex and symplectic vacua described here. In this context, what this paper is studying is a small neighborhood where the moduli space of $\mathcal{N} = 2$ non-Kähler compactifications meets up with the moduli space of Calabi-Yau compactifications with RR flux, inside this bigger configuration space of manifolds.

2.2 Four–dimensional description of the vacua

We will now adapt the ideas from [15] to our needs. The strategy is as follows. We begin by compactifying the IIB and IIA strings on Calabi-Yau threefolds, and we switch on internal RR fluxes, F_3 in IIB and F_4 in IIA (our eventual interest will be the case where the theories are compactified on mirror manifolds $\mathcal{M}$ and $\mathcal{W}$, and the fluxes are mirror to one another). As also first noted in [15], this will make the four–dimensional $\mathcal{N} = 2$ supergravity gauged; in particular, it will create a potential on the moduli space. This potential has supersymmetric vacua only at points where the Calabi–Yau is singular. However, on those loci of the moduli space new massless brane hypermultiplets have to be taken into account, which will then produce the new vacua.

2.2.1 The singularities we consider

Let us first be more precise about the types of singularities we will consider. In IIB, as we will review shortly, if we switch on F_3 with a non–zero integral along a cycle B_3 of a Calabi-Yau $\mathcal{M}$, a supersymmetric vacuum will exist on a point in moduli space in which only the cycle A_3 conjugate to B_3 under intersection pairing shrinks. It is often the case that several cycles shrink simultaneously, with effects that we will review in the next section, but there are definitely examples in which a single B cycle shrinks. These are the cases we will be interested in. (We will briefly explain in section 2.3.2 how this condition could be relaxed.)

In IIA, switching on F_4 with a non–zero integral on a four–cycle $\tilde{A}_4$ of $\mathcal{W}$ will generate a potential which will be zero only in points in which the quantum–corrected volume of the conjugate two–cycle $\tilde{B}_2$ (the Poincaré dual to F_4) vanishes. This will happen on a wall between two birationally equivalent Calabi–Yau's, connected by a flop of $\tilde{B}_2$. These points will be mirror to the ones we described above for IIB.

The converse is not always true: there can be shrinking three–cycles which are mirror to points in the IIA moduli space in which the quantum volume of the whole Calabi–Yau goes to zero. These walls separate geometrical and Landau–Ginzburg, or, hybrid, phases. One would obtain a vacuum at such a point by switching on F_0 instead of F_4, for instance. The example discussed in [15] (the quintic) is precisely such a case. Since in the end we want to give geometrical interpretations to the vacua we will obtain, we will restrict our attention only to cases in which a curve shrinks in $\mathcal{W}$ – that is, when a flop happens. Although this is not strictly necessary for IIB, keeping mirror symmetry in mind we will restrict our attention to cases in which the stricter IIA condition is valid, not only the IIB one: in the mirror pairs of interest to us, the conifold singularity in $\mathcal{M}$ is mirror to a flop in $\mathcal{W}$. It would be interesting, of course, to find the IIA mirrors to all the other complex non–Kähler manifolds in IIB.

Looking for flops is not too difficult, as there is a general strategy. If the Calabi–Yau $\mathcal{W}$ is realized as hypersurface in a toric manifold V, the "enlarged Kähler moduli space" [11, 26] (or at least, the part of it which comes from pull–back of moduli of V) is a toric manifold W_K itself. The cones of the fan of W_K are described by different triangulations of the cone over the toric polyhedron of V. Each of these cones will be a phase [27]; there will be many non–geometrical phases (Landau–Ginzburg or hybrid). Fortunately, the geometrical ones are characterized as the triangulations of the toric polyhedron of V itself (as opposed to triangulations of the cone over it). This subset of cones gives an open set in W_K which is called the "partially enlarged" Kähler moduli space. This is not the end of the story, however. In many examples, it will happen that a flop between two geometrical phases will involve more than one curve at a time, an effect due to restriction from V to $\mathcal{W}$. Worse still, these curves might have relations, and sometimes there is no quick way to determine this. Even so, we expect that there should be many cases in which a single curve shrinks (or many, but without relations).

Such an example is readily found in the literature [28, 29]: taking $\mathcal{W}$ to be an elliptic fibration over $\mathbb{F}_1$ (a Calabi–Yau whose Hodge numbers are $h^{1,1} = 3$ and $h^{2,1} = 243$), there is a point in moduli space in which a single curve shrinks (see Appendix 2.5 for more details). By counting of multiplets and mirror symmetry, on the mirror $\mathcal{M}$ there will be a single three–cycle which will shrink. This implies that the mirror singularity will be a conifold singularity. Indeed, it is a hypersurface singularity, and as such the shrinking cycle is classified by the so–called Milnor number. This has to be one if there is a single shrinking cycle, and the only hypersurface singularity with Milnor number one is the conifold.

2.2.2 Gauged supergravity analysis

After these generalities, we will now show how turning on fluxes drives the theory to a conifold point in the moduli space; more importantly, we will then show how including the new massless hypermultiplets generates new vacua. We will do this in detail in the IIB theory on $\mathcal{M}$, as its IIA counterpart is then straightforward. The analysis is formally identical to the one in [15] (see also [30, 31]); the differences have been explained in the previous subsection.

As usual, define the symplectic basis of three–cycles A^I, B_J and their Poincaré duals α_I, β^I such that

$$A^I \cdot B_J = \delta^I{}_J \ , \qquad \int_{A^J} \alpha_I = \int_{B_I} \beta^J = \delta_I{}^J \tag{2.1}$$

along with the periods $X^I = \int_{A^I} \Omega$ and $F_I = \int_{B_I} \Omega$. Additionally, the basis is taken so that the cycle of interest described in subsection 2.2.1 is $A = A^1$.

When $X^1 = 0$, the cycle A^1 degenerates to the zero size and $\mathcal{M}$ develops a conifold singularity. By the monodromy argument, the symplectic basis (X^1, F_1) will transform as follows when we circle the discriminant locus in the complex moduli space defined by $X^1 = 0$:

$$X^1 \to X^1 \qquad F_1 \to F_1 + X^1 \ . \tag{2.2}$$

From this we know F_1 near the singularity:

$$F_1 = \text{constant} + \frac{1}{2\pi i} X^1 \ln X^1 + \dots \tag{2.3}$$

The metric on the moduli space can be calculated from the formulae

$$\mathcal{G}_{I\bar{J}} = \partial_I \partial_{\bar{J}} K_V \ , \qquad K_V = -\ln i(\bar{X}^I F_I - X_I \bar{F}^I) \ . \tag{2.4}$$

Therefore we obtain

$$\mathcal{G}_{1\bar{1}} \sim \ln(X^1 \bar{X^1}) \ . \tag{2.5}$$

Now, the internal flux we want to switch on is $F_3 = n_1 \beta^1$. The vectors come from

$$F_5 = F_2^I \wedge \alpha_I - G_{2,I} \wedge \beta^I \ , \tag{2.6}$$

where the F_2^I (G_{2I}) is the electric (magnetic) field strength. The Chern-Simons coupling in the IIB supergravity action is then

$$\epsilon^{ij} \int_{M_4 \times CY} \tilde{F}_5 \wedge H_3^i \wedge B_2^j = n_1 \int_{M_4} F_2^1 \wedge B_2 \tag{2.7}$$

where M_4 is the spacetime. By integration by parts, and since B_2 dualizes to one of the (pseudo)scalars in the universal hypermultiplets, we see that the latter is gauged under the field A^1 whose field strength is $dA^1 = F_2^1$.

The potential is now given by the "electric" formula

$$V = h_{uv} k_I^u k_J^v \bar{X}^I X^J e^{K_V} + (U^{IJ} - 3\bar{X}^I X^J e^{K_V}) \mathcal{P}_I^\alpha \mathcal{P}_J^\alpha \tag{2.8}$$

where

$$U^{IJ} = D_a X^I g^{a\bar{b}} D_{\bar{b}} X^J \tag{2.9}$$

and the $\mathcal{P}^\alpha$ are together the so–called Killing prepotential, or hypermomentum map. In our situation only the flux over B_1 is turned on, and the Killing prepotential is given by

$$\mathcal{P}_1^1 = \mathcal{P}_1^2 = 0\ ; \quad \mathcal{P}_1^3 = -e^{\tilde{K}_H} n_{eI}^{(2)} = -e^{2\phi} n_{eI}^{(2)} \tag{2.10}$$

where ϕ is the dilaton. The potential will then only depend on the period of the dual A^1 cycle, call it X^1:

$$V \sim \frac{(n_1)^2}{\ln X^1 \bar{X}^1}\ . \tag{2.11}$$

The theory will thus be driven to the conifold point where $X^1 = 0$.

This is not the end of the story: at the singular point, one has a new massless hypermultiplet B coming from a brane wrapping the shrinking cycle A^1. The world–volume coupling between the D3-brane and F_5 gives then $\int_{\mathbb{R}\times A^1} A_4 = \int_{\mathbb{R}} A^1$, where $\mathbb{R}$ is the world-line of the resulting light particle in M_4. (The coincidence between the notation for the cycle A^1 and the corresponding vector potential A^1 is rather unfortunate, if standard.)

This means that both the universal and the brane hypermultiplet are charged under the same vector; we can then say that they are all electrically charged and still use the electric formula for the potential (2.8), with the only change being that the Killing prepotential is modified to be

$$\mathcal{P}_1^\alpha = \mathcal{P}_1^\alpha|_{B=0} + B^+ \sigma^\alpha B\ ; \tag{2.12}$$

the black hole hypermultiplet is an $SU(2)$ doublet with components (B_1, B_2). Loci on which the $\mathcal{P}^\alpha$'s are zero are new vacua: it is easy to see that they are given by

$$B = ((e^{\tilde{K}_H} n_1)^{1/2}, 0) = (e^\phi n_1^{1/2}, 0) \ . \tag{2.13}$$

The situation here is similar to [15]: the expectation value of the new brane hypermultiplet is of the order $g_s = e^\phi$. So, as in that paper, the two requirements that g_s is small and that B be small (the expression for the $\mathcal{P}^\alpha$ is a Taylor expansion and will be modified for large B) coincide, and with these choices we can trust these vacua. After the Higgsing the flat direction of the potential, namely, the massless hypermultiplet $\tilde{B}_0$, would be a linear combination of the brane hypermultiplet and the universal hypermultiplet while the other combination would become a massive one $\tilde{B}_m$.

2.2.3 The field theory capturing the transition

It is useful to understand the physics of the transition from a 4d field theory perspective, in a region very close to the transition point on moduli space. While this analysis is in principle a simple limit of the gauged supergravity in the previous subsection, going through it will both provide more intuition and also allow us to infer some additional lessons. In fact, in the IIB theory with n_1 units of RR flux, the theory close to the transition point (focusing on the relevant degrees of freedom) is simply a U(1) gauge theory with two charged hypers, of charges 1 and n_1.

Let us focus on the case $n_1 = 1$ for concreteness. Let us call the $\mathcal{N} = 1$ chiral multiplets in the two hypers $B, \tilde{B}$ and $C, \tilde{C}$. In $\mathcal{N} = 1$ language, this theory has a superpotential

$$W \sim \tilde{B}\varphi B + \tilde{C}\varphi C \tag{2.14}$$

where φ is the neutral chiral multiplet in the $\mathcal{N} = 2$ U(1) vector multiplet. It also has a D-term potential

$$|D|^2 \sim (|\tilde{B}|^2 - |B|^2 + |\tilde{C}|^2 - |C|^2)^2 \ . \tag{2.15}$$

There are two branches of the moduli space of vacua: a Coulomb branch where $\langle\varphi\rangle \neq 0$ and the charged matter fields vanish, and a Higgs branch where $\langle\varphi\rangle = 0$ and the hypers have non-vanishing vevs (consistent with F and D flatness). The first branch has complex dimension one, the second has quaternionic dimension one. These branches meet at the point where all fields have vanishing expectation value.

At this point, the theory has an SU(2) global flavor symmetry. This implies that locally, the hypermultiplet moduli space will take the form $\mathbb{C}^2/\mathbb{Z}_2$ [32]. In fact, the precise geometry of the hypermultiplet moduli space, including quantum corrections, can then be determined by a variety of arguments [32, 33] (another type of argument [34] implies the same result for the case where the hypermultiplets coming from shrinking three–cycles in IIB). The result is the following. Locally, the quaternionic space reduces to a hyperKähler manifold which is an elliptic fibration, with fiber coordinates t, x and a (complex) base coordinate z. Let us denote the Kähler class of the elliptic fiber by λ^2. Then, the metric takes the form

$$ds^2 = \lambda^2 \left(V^{-1}(dt - \mathbf{A}\cdot\mathbf{dy})^2 + V(\mathbf{dy})^2\right) \tag{2.16}$$

where $\mathbf{y}$ is the three-vector with components $(x, \frac{z}{\lambda}, \frac{\bar{z}}{\lambda})$. Here, the function V and the vector of functions $\mathbf{A}$ are given by

$$V = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\left(\frac{1}{\sqrt{(x-n)^2 + \frac{|z|^2}{\lambda^2}}} - \frac{1}{|n|}\right) \ + \ \text{constant} \tag{2.17}$$

and

$$\nabla \times \mathbf{A} \ = \ \nabla V \ . \tag{2.18}$$

This provides us with detailed knowledge of the metric on the hypermultiplet moduli space emanating from the singularity, though it is hard to explicitly map the flat direction to a combination of the universal hypermultiplet and the geometrical parameters of $\mathcal{M}'$ or $\mathcal{W}'$. We shall discuss some qualitative aspects of this map in §3.3. For the reader who is confused by the existence of a Coulomb branch at all, given that e.g. in the IIB picture $F_3 \neq 0$, we note that the Coulomb branch will clearly exist on a locus where $g_s \to 0$ (since the hypermultiplet vevs must vanish). This is consistent with supergravity intuition, since in the 4d Einstein frame, the energetic cost of the RR fluxes vanishes as $g_s \to 0$.

2.3 Geometry of the vacua

We will first of all show that the vacua obtained in the previous section cannot come from a transition to another Calabi–Yau. To this aim, in the next subsection we will review Calabi–Yau extremal transitions. We will then proceed in subsection 2.3.2 to review the less well–known *non*–Calabi–Yau extremal transitions, and then compare them to the vacua we previously found in subsection 2.3.3.

2.3.1 Calabi–Yau extremal transitions

Calabi–Yau extremal transitions sew together moduli spaces for Calabi–Yaus whose Hodge numbers differ; let us quickly review how. For more details on this physically well–studied case, the reader might want to consult [35, 36, 37, 11].

Consider IIB theory on a Calabi–Yau $\mathcal{M}$. (Some of the explanations in this paper are given in the IIB case only, whenever the IIA case would be an obvious enough modification). Suppose that at a particular point in moduli space, $\mathcal{M}$ develops N nodes (conifold points) by shrinking as many three–cycles A_a, $a = 1, \ldots, N$, and that these three–cycles satisfy R relations

$$\sum_{a=1}^{N} r_i^a A_a \ = \ 0, \quad i = 1, \cdots, R \tag{2.19}$$

in H_3. We are *not* using the same notation for the index on the cycles as in section 2.2, as these A_a are not all elements of a basis (as they are linearly dependent). Notice that it is already evident that this case is precisely the one we excluded with the specifications in section 2.2.1. To give a classic example [35], there is a known transition where $\mathcal{M}$ is the quintic, $N = 16$ and $R = 1$. Physically, there will be N brane hypermultiplets B_a becoming massless at this point in moduli space. Vectors come from $h^{2,1}$; since the B_a only span $N - R$ directions in H^3, they will be charged under $N - R$ vectors X^A only, $A = 1, \ldots N - R$. Call the matrix of charges Q^a_A, $A = 1, \ldots, N - R$, $a = 1, \ldots N$.

In this case, when looking for vacua, we will still be setting the Killing prepotential $\mathcal{P}_a$ (which is a simple extension of the one in (2.12)) to zero: the flux is now absent, and the B^2 term now reads

$$\mathcal{P}_A = \sum_a Q^a_A B^+_a \sigma^\alpha B_a \ . \tag{2.20}$$

Notice that we have switched no flux on in this case; crucially, $\mathcal{P} = 0$ now will have an R–dimensional space of solutions, due to the relations.

Let us suppose this new branch is actually the moduli space for a new Calabi–Yau. This new manifold would have $h^{2,1} - (N - R)$ vectors, because all the X^A have been Higgsed; and $h^{1,1} + R$ hypers, because of the N B_a, only R flat directions have survived.

This is exactly the same result one would get from a small resolution of all the N nodes. Indeed, let us call the Calabi–Yau resulting from such a procedure $\mathcal{M}'$, and let us compute its Betti numbers. It is actually simpler to first consider a case in which a single three–cycle undergoes surgery[1], which is the case without relations specified in section 2.2.1; we will go back to the Calabi–Yau case, in which relations are necessary, momentarily.

The result of this single surgery along a three–cycle is that $H^3 \to H^3 - 2$, $H^2 \to H^2$. This might be a bit surprising: one is used to think that an extremal transition replaces a three–cycle by a two–cycle. But this intuition comes from the noncompact case, in which indeed it holds. In the compact case, when we perform a surgery along a three–cycle, we really are also losing its conjugate under Poincaré pairing; and we gain no two–cycle. The difference is illustrated in a low–dimensional analogue in figure 2.1, in which H^2 and H^3 are replaced by H^0 and H^1.

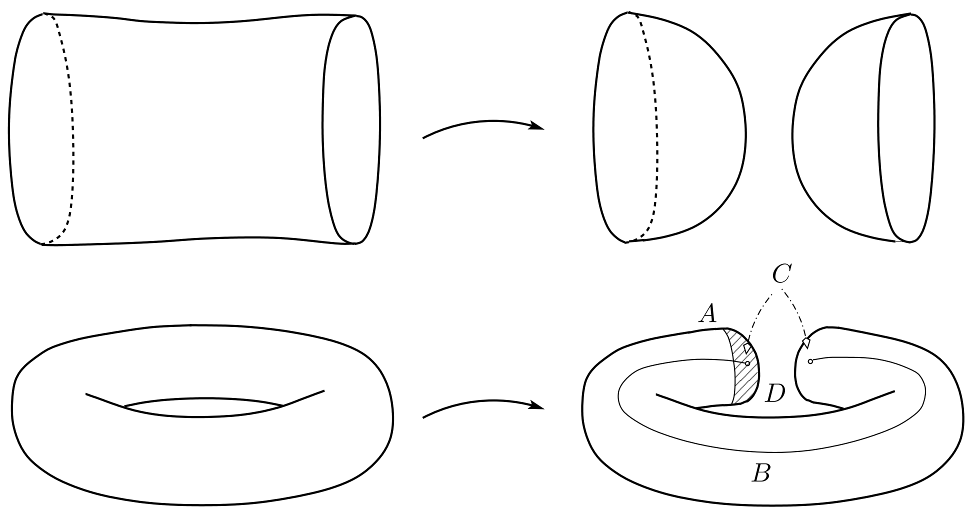


Figure 2.1: Difference between compact and non–compact surgery: in the noncompact case (up), one loses an element in H^1 and one gains an element in H^0 (a connected component). In the compact case (down), one loses an element in H^1 again, but the would–be new element in H^0 is actually trivial, so H^0 remains the same. This figure is meant to help intuition about the conifold transition in dimension 6, where H^0 and H^1 are replaced by H^2 and H^3. We also have depicted various chains on the result of the compact transition, for later use.

Coming back to the Calabi–Yau case of interest in this subsection, let us now consider N shrinking three–cycles with R relations. First of all H^3 only changes by $2(N-R)$, because this is the number of independent cycles we are losing. But this is not the only effect on the homology. A relation can be viewed as a four–chain F whose boundary is $\sum A_a$. After

[1]This is a purely topological computation; in a topological context, an extremal transition is called a surgery, and we will use this term when we want to emphasize we are considering purely the topology of the manifolds involved.

surgery, the boundary of F by definition shrinks to points; hence F becomes a four–cycle in its own right. This gives R new elements in H_4 (or equivalently, in H^2). The change in homology is summarized in Table 2.1, along with the IIA case and, more importantly, in a more general context that we will explain. By comparing with the physical counting above, we find evidence that the new branches of the moduli space correspond to new Calabi–Yau manifolds obtained by extremal transitions.

To summarize, Calabi–Yau extremal transitions are possible without fluxes, but they require relations among the shrinking cycles. This is to be contrasted with the vacua in the previous section, where there are no relations among the shrinking cycles to provide flat directions. Instead, the flux (and resulting gauging) lifts the old Calabi-Yau moduli space (as long as $g_s \neq 0$), but makes up for this by producing a new branch of moduli space (emanating from the conifold point or its mirror).

2.3.2 Non–Calabi–Yau extremal transitions

In this section we will waive the Calabi–Yau condition to reproduce the vacua of the previous section. This is, remember, a case in which cycles shrink without relations. However, we will start with a review of results in the more general case, to put in perspective both the case we will eventually consider and the usual Calabi–Yau case.

We will consider both usual conifold transitions, in which three–cycles are shrunk and replaced by curves, and so–called reverse conifold transitions, in which the converse happens.[2] As a hopefully useful shorthand, we will call the first type a $3 \to 2$ transition and the second a $2 \to 3$. Though the manifolds will no longer be (necessarily) Calabi–Yau, we will still call the initial and final manifold $\mathcal{M}$ and $\mathcal{M}'$ in the $3 \to 2$ case (which is relevant for our IIB picture), and $\mathcal{W}$ and $\mathcal{W}'$ in the $2 \to 3$ case (which is relevant for our IIA picture).

We will first ask whether a $3 \to 2$ transition takes a complex, or symplectic, $\mathcal{M}$ into a complex, or symplectic, $\mathcal{M}'$, and then turn to the same questions about $\mathcal{W}$, $\mathcal{W}'$ for $2 \to 3$ transitions. These questions have to be phrased a bit more precisely, and we will do so case by case.

It is also useful to recall at this point the definitions of symplectic and complex manifolds, which we will do by embedding them in a bigger framework. In both cases, we can

[2]Implicit in the use of the word "conifold" is the assumption that several cycles do not collapse together in a single point of the manifold $\mathcal{M}$. More general cases are also interesting to consider, see for example [37] for the complex case and [18] for the symplectic case.

start with a weaker concept called *G–structure*. By this we mean the possibility of taking the transition functions on the tangent bundle of $\mathcal{M}$ to be in a group G. This is typically accomplished by finding a geometrical object (a tensor, or a spinor) whose stabilizer is precisely G. If we find a two–form J such that $J \wedge J \wedge J$ is nowhere zero, it gives an $\mathrm{Sp}(6,\mathbb{R})$ structure. In presence of a tensor $(1,1)$ tensor (one index up and one down) j such that $j^2 = -1$ (an almost complex structure), we speak of a $\mathrm{Gl}(3,\mathbb{C})$ structure. For us the presence of both will be important; but we also impose a compatibility condition, which says that the tensor $j_m{}^p J_{pn}$ is symmetric and of positive signature. This tensor is then nothing but a Riemannian metric. The triple is an almost hermitian metric: this gives a structure $\mathrm{Sp}(6,\mathbb{R}) \cap \mathrm{Gl}(3,\mathbb{C}) = \mathrm{U}(3)$.

By themselves, these reductions of structure do not give much of a restriction on the manifold. But in all these cases we can now consider an appropriate integrability condition, a differential equation which makes the manifold with the given structure more rigid. In the case of J, we can impose that $dJ = 0$. In this case we say that the manifold is symplectic. For j, a more complicated condition (that we will detail later, when considering SU(3) structures) leads to complex manifolds.

Let us now consider a complex manifold $\mathcal{M}$ (which we will also take to have trivial canonical class $K = 0$). First order complex deformations are parameterized by $H^1(\mathcal{M}, T) = H^{2,1}$. Suppose that for some value of the complex moduli N three–cycles shrink. Replace now these N nodes by small resolutions. The definition of small resolution, just like the one of blowup, can be given locally around the node and then patched without any problem with the rest of the manifold. So the new manifold $\mathcal{M}'$ is still complex. Also, the canonical class K is not modified by the transition because a small resolution does not create a new divisor, only a new curve.[3] Actually, the conjecture that all Calabi–Yau are connected was initially formulated by Reid [12] for all *complex* manifolds (and not only Calabi–Yaus) with $K = 0$, extending ideas by Hirzebruch [38].

If now we consider a symplectic $\mathcal{M}$, the story is different. For one thing, now symplectic moduli are given by $H^2(\mathcal{M}, \mathbb{R})$ [39], so it does not seem promising to look for a point in moduli space where three–cycles shrink. But 2.1 in [17] shows that we can nevertheless shrink a three–cycle symplectically, and replace it by a two–cycle. Whether the resulting

[3]The conditions for $\mathcal{N} = 1$–preserving vacua in ten–dimensional type II supergravity actually only require $c_1 = 0$. The role of this condition is less clear for example in the topological string: for the A model it would seem to unnecessary, as there is no anomaly to cancel; for the B model, it would look like the stronger condition $K = 0$ is required, which means that the canonical bundle should be trivial even holomorphically.

$\mathcal{M}'$ will also be symplectic is not automatic, however. This can be decided using Theorem 2.9 in [17]: the answer is yes precisely when there is at least one relation in homology among the three–cycles.[4]

The case of interest in this paper is actually a blending of the two questions considered so far, whether complex or symplectic properties are preserved. In IIB, we will take a Calabi–Yau $\mathcal{M}$ (which has both properties) and follow it in moduli space to a point at which it develops a conifold singularity. Now we perform a small resolution to obtain a manifold $\mathcal{M}'$ and ask whether this new manifold is still Kähler; this question has been considered also by [42]. As we have seen, the complex property is kept, and the symplectic property is not (though the question in [17] regards more generally symplectic manifolds, disregarding the complex structure, and in particular being more interesting without such a path in complex structure moduli space).

Let us see why $\mathcal{M}'$ cannot be Kähler in our case. A first argument is not too different from an argument given after figure 2.1 to count four–cycles. If the manifold $\mathcal{M}'$ after the transition is Kähler, there will be an element $\omega \in H_4$ dual to the Kähler form. This will have non–zero intersection $\omega \cdot C_a = vol(C_a)$ with all the curves C_a produced by the small resolutions. Before the transition, then, in $\mathcal{M}$, ω will develop a boundary, since the C_a are replaced by three–cycles A_a; more precisely, $\partial\omega = \sum r^a A_a$ for some coefficients r^a. This proves there will have to be at least one relation between the collapsing three-cycles.

We can rephrase this in yet another way. Let us consider the case in which only one nontrivial three–cycle A is shrinking. Since, as remarked earlier (see figure 2.1), in the compact case the curve C created by the transition is trivial in homology, there exists a three–chain B such that $C = \partial B$; then we have, if J is the two–form of the SU(3) structure,

$$0 \neq \int_C J = \int_B dJ \ . \tag{2.21}$$

Hence $dJ \neq 0$: the manifold cannot be symplectic.[5]

Even if a symplectic J fails to exist, there is actually a non–degenerate J compatible with j (since the inclusion $\mathrm{U}(3) \subset \mathrm{Sp}(6,\mathbb{R})$ is a homotopy equivalence, not unlike the way

[4] We should add that the relations must involve all the three–cycles. If there is one three–cycle A which is not involved in any relation, it is possible to resolve symplectically all the other cycles but not A. Examples of this type are found in [41, 42] when $\mathcal{M}$ is Kähler, which is the case of interest to us and to which we will turn shortly. These examples would play in our favor, allowing us to find even more examples of non–Kähler $\mathcal{M}'$, but for simplicity of exposition we will mostly ignore them in the following.

[5] In the mirror picture, a similar argument shows immediately that $d\Omega \neq 0$ on $\mathcal{W}'$, and hence the manifold cannot be complex.

the homotopy equivalence $\mathrm{O}(n) \subset \mathrm{Gl}(n)$ allows one to find a Riemannian metric on any manifold). In other words, the integrable complex structure j can be completed to a U(3) structure (and then to an SU(3) structure, as we will see), though not to a Kähler one.

This is also a good point to make some remarks about the nature of the curve C that we will need later on. The concept of holomorphic curve makes sense even without an integrable complex structure; the definition is still that $(\delta + ij)^m{}_n \partial X^n = 0$, where X is the embedding C in $\mathcal{M}$. For j integrable this is the usual condition that the curve be holomorphic. But this condition makes sense even for an almost complex structure, a fact which is expressed by calling the curve *pseudo*–holomorphic [40]. We will often drop this prefix in the following. In many of the usual manipulations involving calibrated cycles, one never uses integrability properties for the almost complex or symplectic structures on $\mathcal{M}$. For example, it is still true that the restriction of J to C is its volume form vol_C. Exactly in the same way, one can speak of Special Lagrangian submanifolds even without integrability (after having defined an SU(3) structure, which we will in the next section), and sometimes we will qualify them as "pseudo" to signify this.

Let us now consider $2 \to 3$ transitions. It will turn out that the results are just mirror of the ones we gave for $3 \to 2$, but in this case it is probably helpful to review them separately. After all, mirror symmetry for complex–symplectic pairs is not as well established as for Calabi–Yaus, which is one of the motivations of the present work. (Evidence so far includes mathematical insight [16], and, in the slightly more general context of SU(3) structure manifolds, comparisons of four–dimensional theories [43, 44] and direct SYZ computation [45].)

Suppose now we start (in the IIA theory) with a symplectic manifold $\mathcal{W}$ (whose moduli space is, as we said, modeled on $H^2(\mathcal{M}, \mathbb{R})$), and that for some value of the symplectic moduli some curves shrink. Then, it turns out that one can always replace the resulting singularities by some three–cycles, and still get a symplectic manifold (Theorem 2.7, [17]). The trick is that T^*S^3, the deformed conifold, is naturally symplectic, since it is a cotangent bundle. Then [17] proves that this holds even globally: there is no problem in patching together the modifications around each conifold point. One should compare this with the construction used by Hirzebruch and Reid cited above.

It is not automatic that the resulting manifold $\mathcal{W}'$ is complex, even if $\mathcal{W}$ is complex itself. The criterion is that there should be at least one relation in homology between the collapsing curves C_a [46, 47] (see also [48] for an interesting application).[6]

[6]Actually, the criterion also assumes $\mathcal{W}$ to satisfy the $\partial\bar{\partial}$–lemma, to ensure that $H^{2,1} \subset H^3$, which is not

Let us collect the transitions considered so far in a table; we also anticipate in which string theory each transition will be relevant for us. The symmetry among these results is clear; we will not need all of them, though.

	transition	keeps symplectic	keeps complex	Δb_2	Δb_3
IIA	$2 \to 3$	yes [17]	if $\sum r_i^a C_a = 0$ [46, 47]	$N - R$	$2R$
IIB	$3 \to 2$	if $\sum r_i^a B_a = 0$ [17]	yes	R	$2(N - R)$

Table 2.1: The conditions for a transition to send a complex or symplectic conifold to a complex or symplectic manifold.

2.3.3 Vacua versus geometry

We can now apply the results reviewed in the previous subsection to our vacua. Remember that in IIB we have chosen a point in moduli space in which a single three–cycle shrinks, and in IIA one in which a single curve shrinks.

From our assumptions, the singularities affect the manifold only locally (as opposed for example to the IIA case of [15], in which the quantum volume of the whole manifold is shrinking); it is hence natural to assume that the vacua of section 2.2 are still geometrical. Given the experience with the Calabi–Yau case, it is also natural that the brane hypermultiplet B describes a surgery. But then we can use the results of the previous subsection.

In IIB, where we have shrunk a three–cycle, we now know that the manifold obtained by replacing the node with a curve will be naturally complex, but will not be symplectic, since by assumption we do not have any relations. As we have explained, the reason for this is that on the manifold $\mathcal{M}'$ after the transition, there will be a holomorphic curve C which is homologically trivial; and by Stokes, we conclude that the manifold cannot be symplectic.

Summing up, we are proposing that in IIB the vacua we are finding are given by a complex non–symplectic (and hence non–Kähler[7]) manifold. This manifold $\mathcal{M}'$ is defined

always true on complex non–Kähler manifolds; this assumption is trivially valid in the cases we consider, where $\mathcal{W}$ is a Calabi–Yau.

[7] There might actually be, theoretically speaking, a Kähler structure on the manifold which has nothing to do with the surgery. This question is natural mathematically [17], but irrelevant physically: such a Kähler structure would be in some other branch of moduli space, far from the one we are considering, which is connected and close to the original Calabi–Yau by construction.

by a small resolution on the singular point of $\mathcal{M}$, and it has (see table 2.1) Betti numbers

$$b_2(\mathcal{M}') = b_2(\mathcal{M}), \quad b_3(\mathcal{M}') = b_3(\mathcal{M}) - 2 \ . \tag{2.22}$$

In the example described in section 2.2.1, when $\mathcal{M}$ is the mirror of an elliptic fibration over $\mathbb{F}_1$, $\mathcal{M}'$ has $b_2 = 243$, $b_3/2 = 3$.

In IIA, a similar reasoning lets us conjecture that the new vacua correspond to having a symplectic non–complex (and hence non–Kähler) manifold $\mathcal{W}'$, obtained from the original Calabi–Yau $\mathcal{W}$ by replacing the node with a three–cycle. This manifold $\mathcal{W}'$ has

$$b_2(\mathcal{W}') = b_2(\mathcal{W}) - 1, \quad b_3(\mathcal{W}') = b_3(\mathcal{W}) \ . \tag{2.23}$$

In the example from section 2.2.1, when $\mathcal{W}$ is an elliptic fibration over $\mathbb{F}_1$, $\mathcal{W}'$ has $b_2 = 2$, $b_3/2 = 244$.

Notice that these two sets of vacua are mirror by construction: we localize in IIA and in IIB to points which are mirror to each other, and in both cases we add the appropriate brane hypermultiplets to reveal new lines of vacua. What is conjectural is simply the interpretation of the vacua. We now proceed to give evidence for that conjecture.

In the IIB case, the spectrum before the transition is clearly given by $b_3(\mathcal{M})/2-1$ vector multiplets and $b_2(\mathcal{M})+1$ hypermultiplets (the "$+1$" is the universal hypermultiplet). We have seen that the potential generated by F_3 gives mass to one of the vector multiplets, fixing it at a certain point in the complex moduli space. On the other side, the number of massless hypermultiplets remains the same. Indeed, we have added a brane hypermultiplet B; but this combines with the universal hypermultiplet to give only one massless direction, the one given in (2.13).

This is to be compared with the Betti numbers of the proposed $\mathcal{M}'$ from table 2.1: indeed, b_2 remains the same and b_3 changes by 2. Since the manifold is now non–Kähler, we have to be careful in drawing conclusions: "Kähler moduli" a priori do not make sense any more, and though complex moduli are still given by $H^{2,1}$ (by Kodaira–Spencer and $K = 0$), a priori this number is $\neq b_3/2 - 1$, since the manifold is non–Kähler.

However, two circumstances help us. The first is that, by construction, the moduli of the manifolds we have constructed are identified with the moduli of the singular Calabi–Yau on which the small resolution is performed. Then, indeed we can say that there should be $b_3(\mathcal{M}')/2-1+b_2(\mathcal{M}')$ complex geometrical moduli in total (after complexifying the moduli from b_2 with periods of the anti-symmetric tensor field appropriately, and neglecting the scalars arising from periods of RR gauge fields).

A more insightful approach exists, and will also allow us to compare low–lying massive states. Reduction on a general manifold of SU(3) structure (along with a more general class which will not concern us here) has been performed recently in [44]. (Manifolds with $SU(3)$ structures and various differential conditions were also considered from the perspective of supergravity vacua, starting with [49, 50]). We have introduced a U(3) structure in the previous section as the presence on the manifold of both a complex and a symplectic structure with a compatibility condition. The almost complex structure j allows us to define the bundle of $(3,0)$ forms, which is called the canonical bundle as in the integrable case. If this bundle is topologically trivial the structure reduces further to SU(3). The global section Ω of the canonical bundle can actually be used to define the almost complex structure by

$$T^*_{\mathrm{hol}} = \{v_1 \in T^* | v_1 \wedge \bar{\Omega} = 0\} \ . \tag{2.24}$$

The integrability of the almost complex structure is then defined by $(d\Omega)_{2,2} = 0$, something we will not always require.

Let us now review the construction in [44] from our perspective. In general the results of [44] require one to know the spectrum of the Laplacian on the manifold, which is not always at hand; but in our case we have hints for the spectrum, as we will see shortly. We have seen that a U(3) structure, and hence also an SU(3) structure, defines a metric. Let us see it again: since $J \wedge \Omega = 0$, J is of type $(1,1)$, and then a metric can be defined as usual: $g_{i\bar{j}} = -iJ_{i\bar{j}}$.

We can now consider the Laplacian associated to this metric. The suggestion in [44, 43] is to add some low–lying massive eigen–forms to the cohomology. Since $[\Delta, d] = 0$ and $[\Delta, *] = 0$, at a given mass level there will be eigen–forms of different degrees. Suppose for example $\Delta\omega_2 = m^2\omega_2$ for a certain m. Then

$$d\omega_2 \equiv m\beta_3 \tag{2.25}$$

will also satisfy $\Delta\beta_3 = m^2\beta_3$, and similarly for $\alpha_3 \equiv *\beta_3$ and $\omega_4 \equiv *\omega_2$. (The indices denote the degrees of the forms.) We can repeat this trick with several mass levels, even if coincident.

After having added these massive forms to the cohomology, we can use the resulting combined basis to expand $\Omega = X^I\alpha_I + \beta^I F_I$ and $J = t_i\omega_i$, formally as usual but with some of the α's, β's and ω's now being massive. Finally, these expansions for Ω and J can be plugged into certain "universal" expressions for the Kähler prepotential $\mathcal{P}^\alpha$. Without fluxes (we will return on this point later) and with some dilaton factor suppressed, this looks like [44]

$$\mathcal{P}^1 + i\mathcal{P}^2 = \int d(B + iJ) \wedge \Omega, \qquad \mathcal{P}^3 = \int (dC_2 - C_0 dB) \wedge \Omega. \tag{2.26}$$

Since the reader may be confused about the interpretation of the expressions $\int d(B + iJ) \wedge \Omega$ and $\int (dC_2 - C_0 dB) \wedge \Omega$ which appear above (given the ability to integrate by parts), let us pause to give some explanation. Our IIB solutions indeed correspond to complex manifolds, equipped with a preferred closed 3-form which has $d\Omega = 0$. However, the 4d fields which are given a *mass* by the gauging actually include deformations of the geometry which yield $d\Omega \neq 0$, as we discussed above. Therefore, the potential which follows from (2.26) is a nontrivial function on our field space.

Let us try to apply the KK construction just reviewed to the manifold $\mathcal{M}'$. First of all we need some information about its spectrum. We are arguing that $\mathcal{M}'$ is obtained from surgery. In [24], it is found that the spectrum of the Dirac operator changes little, in an appropriate sense, under surgery. If we *assume* that this result goes through after twisting the Dirac operator, we can in particular consider the Dirac operator on bispinors, also known as the signature operator, which has the same spectrum as the Laplacian. All this suggests that for very small B and g_s the spectrum on $\mathcal{M}'$ will be very close to the one on $\mathcal{M}$. Hence there will be an eigenform of the Laplacian ω with a relatively small eigenvalue m (and its partners discussed above), corresponding to the extra harmonic forms generating H^3 before the surgery. By the reasoning above, this will also give eigenforms α, β and $\tilde{\omega}$.

Expanding now $\Omega = X^1\alpha + \Omega_0$, $J = t^1\omega + J_0$, $B = b^1\omega + B_0$ and $C_2 = c^1\omega + C_{20}$ (where Ω_0, J_0, B_0 and C_{20} represent the part of the expansion in cohomology) and using the relation $\int_{\mathcal{M}'} \beta_3 \wedge \alpha_3 = 1$, we get from (2.26):

$$\mathcal{P}^1 + i\mathcal{P}^2 \sim m(b^1 + it^1)X^1, \qquad \mathcal{P}^3 \sim m(c^1 - C_0 b^1)X^1 \ . \tag{2.27}$$

The parameter m measures the non-Kählerness away from the Calabi-Yau manifold $\mathcal{M}$, and should be proportional to the vev of the brane hypermultiplet $\tilde{B}_0$ of §2.2. Clearly the formula is reminiscent of the quadratic dependence on the B hypermultiplet in (2.12). The size of the curve C is measured by t^1. Of course $\tilde{B}_0$ is really a function of the t^1 and universal hypermultiplets. Presumably, it and the massive hyper $\tilde{B}_m$ in section 2.2.2 are different linear combinations of the curve volume and g_s. It is even tempting to map the $\mathcal{M}$ and $\mathcal{M}'$ variables by mapping B directly to $\int_C J = t^1$, and (very reasonably) mapping the dilaton hypermultiplet on $\mathcal{M}$ directly into the one for $\mathcal{M}'$. Indeed, the size of C would then be proportional to g_s (at least when both are small), which is consistent with both being zero at the transition point.

Fixing this would require more detailed knowledge of the map between variables. However, since the formula for the Killing prepotentials has the universal hypermultiplet in it (which can be seen from (2.27), where C_0 is mixed with other hypers and some dilaton factor is omitted in the front), it could have α' corrections. Moreover, (2.26) is only valid in the supergravity regime where all the cycles are large compared with the string length. Hence an exact matching between the Killing prepotentials is lacking.

We can now attempt the following comparison between the spectrum of the vacua and the KK spectrum on the conjectural $\mathcal{M}'$:

- On $\mathcal{M}$, one of the vectors, X^1, is given a mass by the gauging $\int F_3 \wedge \Omega$. On $\mathcal{M}'$, this vector becomes a deformation of Ω which makes it not closed, $\Omega \to \Omega + \alpha$, $\Delta\alpha = m^2\alpha$. In both pictures, the vacuum is at the point $X^1 = 0$. On $\mathcal{M}$, this is because we have fixed the complex modulus at the point in which A^1 shrinks. On $\mathcal{M}'$, the manifold which is natural to propose from table 2.1 is complex, and hence $d\Omega = 0$.

- The remaining vectors are untouched by either gauging and remain massless.

- Both for $\mathcal{M}$ and for $\mathcal{M}'$, there are $b_2 + 1$ massless hypermultiplets.

- From the perspective of the gauged supergravity analysis on $\mathcal{M}$ there is a massive hypermultiplet too: B and the universal hypermultiplet have mixed to give a massless direction, but another combination will be massive. On $\mathcal{M}'$, there is also a massive hypermultiplet: it is some combination of g_s and t^1, which multiplies the massive form ω (with $\Delta\omega = m^2\omega$) in the expansion of J. To determine the precise combination one needs better knowledge of $m(t^1, g_s)$ in (2.27).

Again, this comparison uses the fact that there is a positive eigenvalue of the Laplacian which is much smaller than the rest of the KK tower, and this fact is inspired by the work in [24].

This comparison cannot be made too precise for a number or reasons. One is, as we have already noticed, that it is hard to control the spectrum, and we had to inspire ourselves from work which seemed relevant. Another is that the KK reduction of ten-dimensional supergravity on the manifold $\mathcal{M}'$ will not capture the full effective field theory precisely, as we are close (at small B vevs) to a point where a geometric transition has occurred. Hence, curvatures are large in localized parts of $\mathcal{M}'$, though the bulk of the space can be large and weakly curved. And indeed, we know that ten–dimensional type II supergravities do not allow $\mathcal{N} = 2$ Minkowski vacua from non-Kähler compactification manifolds in a regime where all cycles are large enough to trust supergravity (though inclusion of further ingredients like orientifolds, which are present in string theory, can yield large radius $\mathcal{N} = 2$ Minkowski vacua in this context [51]). The vacua of [15], and our own models, presumably evade this no-go theorem via stringy corrections arising in the region localized around the small resolution. Some of these corrections are captured by the local field theory analysis reviewed in §2.3, which gives us a reasonable knowledge of the hyper moduli space close to the singularity. It should be noted that the family of vacua we have found cannot simply disappear as one increases the expectation values of the B fields and e^{ϕ}: the moduli space of $\mathcal{N} = 2$ vacua is expected to be analytic even for the fully–fledged string theory. However, new terms in the expansion of the $\mathcal{P}^{\alpha}$'s in terms of the B hypermultiplet will deform the line; and large g_s will make the perturbative type II description unreliable.

An issue that deserves separate treatment is the following. Why have we assumed $F_3 = 0$ in (2.26)? It would seem that the integral $\int_B F_3$ cannot simply go away. Usually, in conifold transitions (especially noncompact ones) a flux becomes a brane, as the cycle becomes contractible and surrounds a locus on which, by Gauss' law, there must be a brane. This would be the case if, in figure 2.1, the flux were on A: this would really mean a brane on C. In our case, the flux is on B, on a chain which surrounds nothing. Without sources, and without being non–trivial in cohomology, F_3 has no choice but disappear on $\mathcal{M}'$.

To summarize this section, we have conjectured to which manifolds the vacua found in section 2.2 correspond. In this way, we have also provided explicit symplectic–complex non–Kähler mirror pairs.

2.4 The big picture: a space of geometries

There are a few remarks that can be made about the type of complex and symplectic manifolds that we have just analyzed, and that suggest a more general picture. This is a speculative section, and it should be taken as such.

One of the questions which motivated us is the following. The KK reduction in [44] says that $\int dJ \wedge \Omega$ encodes the gauging of the four–dimensional effective supergravity on $\mathcal{M}'$. Hence in some appropriate sense (to be discussed below), dJ must be integral – one would like $\int dJ \wedge \Omega$ to be expressed in terms of integral combinations of periods of Ω. This is just because the allowed gauge charges in the full string theory form an integral lattice. But from existing discussions, the integral nature of dJ is far from evident. Though one can normalize the massive forms appropriately in such a way that the expression does give an integer, this does not distinguish between several possible values for the gauging: it is just a renormalization, not a quantization.

Without really answering this question, we want to suggest that there must be a natural modification of cohomology that somehow encodes some of the massive eigenvalues of the Laplacian, and that has integrality built in. It will be helpful to refer again to figure 2.1: on $\mathcal{M}'$ (the manifold on the right in the lower line of figure 1), we have depicted a few relevant chains, obviously in a low–dimensional analogy. What used to be called the A cycle is now still a cycle, but trivial in homology, as it is bounded by a four–cycle D. The dual B cycle, from other side, now is no longer a cycle at all, but merely a chain, its boundary being the curve C. This curve has already played a crucial role in showing that $\mathcal{M}'$ cannot be symplectic.

We want to suggest that a special role is played by relative cohomology groups $H_3(\mathcal{M}', C)$ and $H_4(\mathcal{M}', A)$. Remember that relative homology is the hypercohomology of $C_\bullet(C) \xrightarrow{\iota_C} C_\bullet(\mathcal{M}')$, with C_k being chains and the map ι_C being the inclusion. In plain English, chains in $C_k(\mathcal{M}', C)$ are pairs of chains $(c_k, \tilde{c}_{k-1}) \in C_k(\mathcal{M}') \times C_{k-1}(C)$, and homology is given by considering the differential

$$\partial(c_k, \tilde{c}_{k-1}) = (\partial c_k + \iota_C(\tilde{c}_{k-1}), -\partial \tilde{c}_{k-1}) \ . \tag{2.28}$$

So cycles in $H_k(\mathcal{M}', C)$, for example, are ordinary chains which have boundary on C. B is precisely such a chain. A long exact sequence can be used to show that, when C is a curve trivial in $H_2(\mathcal{M}')$ as is our case, $\dim(H_3(\mathcal{M}', C)) = \dim(H_3(\mathcal{M}')) + 1$. So (B, C) and the usual cycles generate $H_3(\mathcal{M}', C)$. Similarly, $\dim(H_4(\mathcal{M}', A)) = \dim(H_4(\mathcal{M}')) + 1$, and the new generator is (D, A).

Similar and dual statements are valid in cohomology. This is defined similarly as for homology: pairs $(\omega_k, \tilde{\omega}_{k-1}) \in \Omega^k(\mathcal{M}') \times \Omega^{k-1}(C)$, with a differential

$$d(\omega_k, \tilde{\omega}_{k-1}) = (d\omega_k, \iota_C^*(\omega_k) - d\tilde{\omega}_{k-1}) \tag{2.29}$$

A non–trivial element of $H^3(\mathcal{M}', C)$ is $(0, \mathrm{vol}_C)$. Since C is a holomorphic curve, $\mathrm{vol}_C = J_{|C} \equiv \iota_C^* J$ and hence this representative is also equivalent to $(dJ, 0)$, using the differential above.

When we deform $\mathcal{M}'$ with the scalar in the massive vector multiplet X^1, the manifold becomes non–complex, as we have shown in the previous section; but one does not require the almost complex structure to be integrable to define an appropriate notion of holomorphic curve. In fact, one might expect then that, when $d\Omega \neq 0$, which corresponds to $\mathcal{M}'$ being non–complex, one can also choose A to be SLag (as we remarked earlier, the definition will not really require that the almost symplectic structure be closed).[8] Definitely, the logic would hold the other way around – if such a SLag A can be found, $\int_A \Omega \neq 0$ and then, again by integration by parts, it follows that $d\Omega \neq 0$.

In our example, we expect the number of units n_1 of F_3 flux present before the transition in the IIB picture, to map to "n_1 units of dJ" on $\mathcal{M}'$. The phrase in quotes has not been precisely defined, but it is reasonable to think that it is defined by some kind of intersection theory in relative homology. We will now try to make this more precise.

As we have seen, the dimension of the relative H_3 can be odd (and it is in our case), so we should not expect a pairing between A and B cycles within the same group. One might try nevertheless to define a pairing between chains in $H_3(\mathcal{M}', C)$ and $H_4(\mathcal{M}', A)$; it would be defined by

$$(B, C) \cdot (D, A) \equiv \#(B \cap A) = \#(C \cap D) \ . \tag{2.30}$$

In fact, if we think of another lower–dimensional analogy, in which both A and C are *one*–dimensional in a three–dimensional manifold, it is easy to see that what we have just defined is a linking number between C and A. Indeed, $\dim(C) + \dim(A) = \dim(\mathcal{M}') - 1$.

This can also be rephrased in relative cohomology. Consider a bump–form δ_A which is concentrated around A and has only components transverse to it, and similarly for C. These can be defined more precisely using tubular neighborhoods and the Thom isomorphism [52]. Since A and C are trivial in homology, we cannot quite say that these bump

[8] The reader should not confuse this potential SLag, which may exist off-shell in the IIB theory, with the pseudo-SLag manifold that exists on $\mathcal{W}'$ where $d\Omega \neq 0$ even on the $\mathcal{N} = 2$ supersymmetric solutions.

forms are the Poincaré duals of A and C. But we can say that $(\delta_A, 0) \in H^3(M, C)$ is the Poincaré dual to the cycle $(D, A) \in H_4(M, A)$, with natural definitions for the pairing between homology and cohomology. δ_A is non–trivial in relative cohomology but trivial in the ordinary cohomology $H^3(M)$, and hence there exists an F_A such that $dF_A = \delta_A$. Then we have

$$\int_{\mathcal{M}'} F_A \wedge \delta_C = \int_C F_A = \int_B dF_A = \#(C \cap D) \equiv L(A, C) \; . \tag{2.31}$$

In other words, in cohomology we have $L(A, C) = \int d^{-1}(\delta_A) \wedge \delta_C$.

Suppose we have now another form $\tilde{\delta}_A$ which can represent the Poincaré dual (in relative cohomology) to (D, A). Then we can use this other form as well to compute the linking, with identical result. This is because $(\delta_A, 0) \sim (\tilde{\delta}_A, 0)$ in $H^3(\mathcal{M}', A)$ means that, by the definition of the differential above, $\delta_A - \tilde{\delta}_A = d\omega_2$ with ω_2 satisfying $\iota_C^* \omega_2 = d\tilde{\omega}_1$ for some form $\tilde{\omega}_1$ on C. Then

$$\int_{\mathcal{M}'} d^{-1}(\delta_A - \tilde{\delta}_A) \wedge \delta_C = \int_{\mathcal{M}'} \omega_2 \wedge \delta_C = \int_C \omega_2 = \int_C d\tilde{\omega}_1 = 0 \tag{2.32}$$

so $L(A, C)$ does not depend on the choice of the Poincaré dual. But now, remember that $(dJ, 0)$ is also a non–trivial element of $H^3(\mathcal{M}', C)$; if we normalize the volume of C to 1, it then has an equally valid claim to be called a Poincaré dual to (D, A). Indeed, $\int_{(B,C)}(dJ, 0) \equiv \int_B dJ = \int_C J = 1 = (D, A) \cdot (B, C)$, and for all other cycles the result is zero. Similar reasonings apply to $d\Omega$. Then we can apply the steps above and conclude that

$$L(A, C) = \int_{\mathcal{M}'} dJ \wedge \Omega \; . \tag{2.33}$$

In doing this we have normalized the volumes of C and A to one; if we reinstall those volumes, we get precisely that $\int dJ \wedge \Omega$ is a linear function of the vectors and hypers with an integral slope.

Another point which seems to be suggesting itself is the relation between homologically trivial Special Lagrangians and holomorphic curves on one side, and massive terms in the expansion of Ω and J on the other. The presence of a holomorphic but trivial curve, as we have already recalled, implies that $dJ \neq 0$: in the previous section we have seen that one actually expects that such curves are in one–to–one correspondence with massive eigenforms of the Laplacian present in the expansion of J (whose coefficients represent massive fields, which vanish in vacuum). We have argued for this relation close to the transition point, and for the $\mathcal{M}'$ that we have constructed, but it might be that this link persists in general. This would mean that inside an arbitrary SU(3) structure manifold, one would have massive fields which are naturally singled out, associated to homologically trivial holomorphic curves.

Similarly, in the IIA on $\mathcal{W}'$, there is a 3-cycle which is (pseudo) Special Lagrangian but homologically trivial. Its presence implies that $d\Omega \neq 0$, in keeping with the fact that the IIA vacua are non-complex.

Reid's fantasy [12] involved the conjecture that by shrinking -1 curves, and then deforming, one may find a connected configuration space of complex threefolds with $K = 0$. Here, we see that it is natural to extend this fantasy to include a mirror conjecture: that the space of symplectic non-complex manifolds with SU(3) structure is similarly connected, perhaps via transitions involving the contraction of (pseudo) Special Lagrangian cycles, followed by small resolutions. The specialization to -1 curves in [12] is probably mirror to the requirement that the SLags be rigid, in the sense that $b^1 = 0$.

In either IIB or IIA, we have seen that (at least close to the transition) there is a natural set of massive fields to include in the low-energy theory, associated with the classes of cycles described above. Allowing these fields to take on expectation values may allow one to move off-shell, filling out a finite–dimensional (but large) configuration space, inside which complex and symplectic manifolds would be zeros of a stringy effective potential. While finding such an $\mathcal{N} = 2$ configuration space together with an appropriate potential to reveal all $\mathcal{N} = 2$ vacua is clearly an ambitious goal, it may also provide a fruitful warm-up problem for the more general question of characterizing the string theory "landscape" of $\mathcal{N} \leq 1$ vacua [53].

In this bigger picture, this paper is a Taylor expansion of the master potential around a corner in which the moduli space of $\mathcal{M}'$ meets the moduli space of compactifications on $\mathcal{M}$ with RR flux.

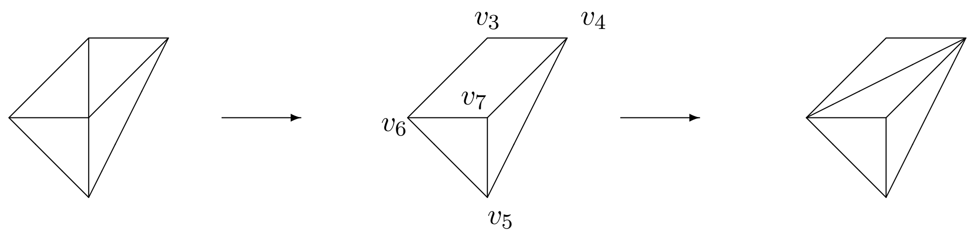
2.5 Details about an example

We will detail here the transition for the example mentioned in section 2.2.1. We will do so on the IIA side, which is the one which involves the strictest assumptions, as explained there.

The Calabi–Yau $\mathcal{W}$ is an elliptic fibration over the Hirzebruch surface $\mathbb{F}_1$. It is convenient to describe it as a hypersurface in a toric manifold V. The fan for the latter is described by the columns of the matrix

$$\begin{array}{c}
\begin{array}{ccccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 \end{array} \\
\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 & 0 \\ 0 & -1 & 2 & 2 & 2 & 2 & 2 \\ -1 & 0 & 3 & 3 & 3 & 3 & 3 \end{bmatrix}
\end{array} .$$

The last five vectors lie in the same plane, determined by the last two coordinates; let us plot the first two coordinates, along with three different triangulations:



The vectors of the fan are indeed the right ones to describe the $\mathbb{F}_1$ base. The fan is further specified by the higher–dimensional cones in the picture, with the first triangulation really describing the elliptic fibration over $\mathbb{F}_1$, the last describing a space related to the first by a flop, and the middle triangulation describing the singular case. (The points have been labeled in the singular case only to avoid cluttering the picture.) We associate as usual a homogeneous coordinate z_i to each of the v_i's, with charge matrix given by the (transposed) kernel of the matrix above:

$$\begin{bmatrix} 0 & 0 & 3 & -2 & 1 & -2 & 0 \\ 6 & 4 & 1 & 0 & 1 & 0 & 0 \\ 3 & 2 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

From the picture we see that the flopped locus in V lies at $z_3 = z_4 = z_6 = z_7 = 0$. One has to check whether this locus intersects the Calabi–Yau only once. This is done by looking at the equation for $\mathcal{W} \subset V$, which for a certain point in the complex moduli space reads $z_1^2 + z_2^3 + z_3^{12} z_4^{18} z_7^6 + z_5^{12} z_6^6 z_7^6 + z_3^{12} z_6^{18} z_7^6 + z_4^6 z_5^{12} z_7^6 = 0$; hence we get the singular locus $z_1^2 + z_2^3 = 0$ on $\mathcal{W}$. Taking into account the $\mathbb{C}^*$ actions, this corresponds to only one point p as desired. To verify that the normal bundle of the shrinking curve has charges $(-1, -1)$, one can identify the combination of the charges that keeps p invariant; this action turns out to be $(1, 1, \lambda, \lambda^{-1}, 1, \lambda^{-1}, \lambda)$, $\lambda \in \mathbb{C}^*$, which is the right one for a conifold point.

Chapter 3

Topological twisted sigma model with H-flux

Abstract

In this section we revisit the topological twisted sigma model with H-flux. We explicitly expand and then twist the worldsheet Lagrangian for bi-Hermitian geometry. we show that the resulting action consists of a BRST exact term and pullback terms, which only depend on one of the two generalized complex structures and the B-field. We then discuss the topological feature of the model.

3.1 Introduction

It is a very convenient and powerful approach to obtain topological field theories by twisting supersymmetric field theory [54]. It was furthur shown that the $N = (2, 2)$ worldsheet sigma model with the Kähler target space admits A and B types of twisting [55]. However the Kähler condition is not crucial to perform the A and B twists. What is really needed is to have $N = (2, 2)$ worldsheet supersymmetry so that $U(1)_V$ and $U(1)_A$ exist.

From the viewpoint of the $N = (2, 2)$ worldsheet supersymmetry algebra the twists are achieved by replacing the 2d Euclideanized spacetime rotation group $U(1)_E$ with the diagonal subgroup of $U(1)_E \times U(1)_R$, where $U(1)_R$ is either $U(1)_V$ or $U(1)_A$ R-symmetry in the $N = (2, 2)$ supersymmetry group.

In 1984 the most general geometric backgrounds for $N = (2, 2)$ supersymmetric sigma models was proposed by Gates, Hull, and Roček [57]. The geometric backgrounds (a.k.a.

bi-Hermitian geometry) consists of a set of data (J_+, J_-, g, H). $J_\pm$ are two different integrable complex structures and the metric g is Hermitian with respect to either one of $J_\pm$. Moreover $J_\pm$ are convariantly constant with respect to the torsional connections $\Gamma \pm g^{-1}H$, where H is a closed 3-form on the manifold. The manifold is apparently non-Kähler due to the presence of the torsions.

Bi-Hermitian geometry started to re-receive new attention after Hitchin introduced the notion of generalized geometry [13] and Gualtieri furthur showed that the geometry is equivalent to a pair of commuting (twisted) generalized complex ((T)GC for short) structures on the manifold M, namely, the twisted generalized Kähler structure [14].

Since the worldsheet theory with bi-Hermitian target has $N = (2, 2)$ supersymmetry, we definitely can consider its topological twisted models. In [58] Kapustin and Li considered such a topological model and showed that on the classical level the topological observables in a given twisted model correspond to the Lie algebroid cohomologies associated with one of the two twisted generalized complex structures. The same problem was also considered by many other authors from Hamiltonian approach or using Batalin-Vilkovisky quantization [59] [60] [61].

Although it is definitely true that the twisted models for bi-Hermitian geometries are topological, the explicit construction of the twisted Lagrangian is lacking. The difficulties of such a calculation lie in that people are so accustomed to using complex geometry that they feel reluctant to perform a calculation which needs to be done in the real coordinate basis with projectors. A priori, we should be able to express the twisted Lagrangian for the generalized geometry as some BRST exact piece plus certain pullback terms which only depend on one of the twisted generalized complex structures.

By the end of the paper we will see that this is indeed true. However since the pullback object is not closed it is not clear that the action is topological. This issue is made clear in [65]. The paper is organized as follows. In Section 3.2 we first review the sigma models with Riemannian and Kähler targets and discuss the properties of the twisted Lagrangian. In Section 3.3 we present the computation of the twisted topological models for bi-Hermitian geometries and express the twisted Lagrangian in the aforementioned way. In section 3.4 we conclude, discuss the limitation of the twisted models, and mention some open questions. Some basics and definitions of the generalized geometry will be presented in the appendix.

3.2 Topological sigma model with Kähler targets

We first recall some basic facts about the worldsheet sigma models with Riemannian or Kähler manifolds as targets. Throughout the whole paper lowercase English letters $a, b, c, ...$ are indices for the real coordinates on the targets, while Greek letter $\mu, \nu, \sigma, ...$ are those for holomorphic coordinates. (And of course $\bar{\mu}, \bar{\nu}, \bar{\sigma}, ...$ for antiholomorphic coordiantes.) Although it has been shown that the off-shell formalism exists even for the bi-Hermitian geometry [62], we will only work in the on-shell supersymmetry formalism to simplify the calculation.

The nonlinear sigma model with a Riemannian manifold M has natural $(1,1)$ worldsheet supersymmetric formalism. The model is governed by an embedding map $\Phi : \Sigma \to M$ where Σ is a Riemann surface. The Lagrangian is

$$L = 2t \int \; d^2z \; d^2\theta \; g_{ab}(\Phi) D_+ \Phi^a D_- \Phi^b \tag{3.1}$$

where

$$D_\pm = \frac{\partial}{\partial \theta^\pm} + i\theta^\pm \left(\frac{\partial}{\partial x^0} \pm \frac{\partial}{\partial x^1}\right) \tag{3.2}$$

$$\Phi^a = \phi^a + \theta^+ \psi^a_+ + \theta^- \psi^a_- + \theta^- \theta^+ F^a \tag{3.3}$$

$$d^2z = \frac{i}{2} dz \wedge d\bar{z} \tag{3.4}$$

Exapnding out (3.1) and then setting $F^a = \Gamma^a_{bc} \psi^b_+ \psi^c_-$ (the on-shell value of F^a) we have

$$\begin{aligned} L = 2t \int d^2z \quad (\quad & \frac{1}{2} g_{ab} \partial_z \phi^a \partial_{\bar{z}} \phi^b + \frac{i}{2} g_{ab} \psi^a_- D_z \psi^b_- \\ & + \frac{i}{2} g_{ab} \psi^a_+ D_{\bar{z}} \psi^b_+ + \frac{1}{4} R_{abcd} \psi^a_+ \psi^b_+ \psi^c_- \psi^d_-) \end{aligned} \tag{3.5}$$

where $D_{\bar{z}} \psi^a_+ = \partial_{\bar{z}} \psi^a_+ + \Gamma^a_{bc} \; \partial_{\bar{z}} \phi^b \; \psi^c_+$ and $D_z \psi^a_- = \partial_z \psi^a_- + \Gamma^a_{bc} \; \partial_z \phi^b \; \psi^c_-$.

If the target space is Kähler the nonlinear sigma model will have an additional $(1,1)$ supersymmetry, turning the theory into $N = (2,2)$ sigma model [56]. The Lagrangian of such a sigma model is written as

$$\begin{aligned} L = 2t \int d^2z \quad (\quad & \frac{1}{2} g_{ab} \partial_z \phi^a \partial_{\bar{z}} \phi^b + i g_{\bar{\mu}\mu} \psi^{\bar{\mu}}_- D_z \psi^\mu_- \\ & + i g_{\bar{\mu}\mu} \psi^{\bar{\mu}}_+ D_{\bar{z}} \psi^\mu_+ + R_{\mu\bar{\mu}\nu\bar{\nu}} \psi^\mu_+ \psi^{\bar{\mu}}_+ \psi^\nu_- \psi^{\bar{\nu}}_-) \end{aligned} \tag{3.6}$$

The detailed supersymmetry transformations are listed as follows [55].

$$
\begin{aligned}
\delta\phi^{\mu} &= i\epsilon_{-}\psi_{+}^{\mu} + i\epsilon_{+}\psi_{-}^{\mu} \\
\delta\phi^{\bar{\mu}} &= i\bar{\epsilon}_{-}\psi_{+}^{\bar{\mu}} + i\bar{\epsilon}_{+}\psi_{-}^{\bar{\mu}} \\
\delta\psi_{+}^{\mu} &= -\bar{\epsilon}_{-}\partial_{z}\phi^{\mu} - i\epsilon_{+}\psi_{-}^{\nu}\Gamma^{\mu}_{\nu\sigma}\psi_{+}^{\sigma} \\
\delta\psi_{+}^{\bar{\mu}} &= -\epsilon_{-}\partial_{z}\phi^{\bar{\mu}} - i\bar{\epsilon}_{+}\psi_{-}^{\bar{\nu}}\Gamma^{\bar{\mu}}_{\bar{\nu}\bar{\sigma}}\psi_{+}^{\bar{\sigma}} \\
\delta\psi_{-}^{\mu} &= -\bar{\epsilon}_{+}\partial_{\bar{z}}\phi^{\mu} - i\epsilon_{-}\psi_{+}^{\nu}\Gamma^{\mu}_{\nu\sigma}\psi_{-}^{\sigma} \\
\delta\psi_{-}^{\bar{\mu}} &= -\epsilon_{+}\partial_{\bar{z}}\phi^{\bar{\mu}} - i\bar{\epsilon}_{-}\psi_{+}^{\bar{\nu}}\Gamma^{\bar{\mu}}_{\bar{\nu}\bar{\sigma}}\psi_{-}^{\bar{\sigma}}
\end{aligned} \tag{3.7}
$$

3.2.1 Kähler A model

An A-twist will turn ψ_{+}^{μ} and $\psi_{-}^{\bar{\mu}}$ into sections of $\Phi^{*}(T^{1,0}X)$ and $\Phi^{*}(T^{0,1}X)$, denoted as χ^{μ} and $\chi^{\bar{\mu}}$. And $\psi_{+}^{\bar{\mu}}$ and ψ_{-}^{μ} become sections of $\Omega_{\Sigma}^{1,0} \otimes \Phi^{*}(T^{0,1}X)$ and $\Omega_{\Sigma}^{0,1} \otimes \Phi^{*}(T^{1,0}X)$, denoted as $\psi_{z}^{\bar{\mu}}$ and $\psi_{\bar{z}}^{\mu}$. In order to get the transformation laws we simply set $\epsilon_{+} = \bar{\epsilon}_{-} = 0$ in (3.7). After A-twist the Lagrangian becomes

$$
\begin{aligned}
L = 2t\int d^2z \quad (\quad & \frac{1}{2}g_{ab}\partial_{z}\phi^{a}\partial_{\bar{z}}\phi^{b} + ig_{\bar{\mu}\mu}\psi_{z}^{\bar{\mu}}D_{\bar{z}}\chi^{\mu} \\
& + ig_{\bar{\mu}\mu}\psi_{\bar{z}}^{\mu}D_{z}\chi^{\bar{\mu}} - R_{\mu\bar{\mu}\nu\bar{\nu}}\psi_{\bar{z}}^{\mu}\psi_{z}^{\bar{\mu}}\chi^{\nu}\chi^{\bar{\nu}})
\end{aligned} \tag{3.8}
$$

The key fact as stated in [55] is that the Lagrangian can be recast into a very suggestive form, which is a BRST exact term plus a pullback term depdending only on the Kähler structure of the target space. Upon deriving this the equatoins of motion of ψ are needed.

$$
L = it\int d^2z\{Q, V_A\} + t\int \Phi^{*}(K) \tag{3.9}
$$

with $V_A = g_{\mu\bar{\nu}}(\psi_{z}^{\bar{\nu}}\partial_{\bar{z}}\phi^{\mu} + \partial_{z}\phi^{\bar{\nu}}\psi_{\bar{z}}^{\mu})$ and $K = -ig_{\mu\bar{\nu}}dz^{\mu}dz^{\bar{\nu}}$. From this expression we realize that the Kähler A model depends only on the cohomology class of K. $\int\Phi^{*}(K)$ also depends on the homotopy class of the mapping Φ, but in the path integral all the homotopy classes will be summed over.

3.2.2 Kähler B model

We also recall some basics about the Kähler B model which will be useful later. The B twist will turn $\psi_{\pm}^{\bar{\mu}}$ into sections of $\Phi^{*}(T^{0,1}X)$, and ψ_{+}^{μ} and ψ_{-}^{μ} into sections of $\Omega_{\Sigma}^{1,0} \otimes \Phi^{*}(T^{0,1}X)$

and $\Omega_\Sigma^{0,1} \otimes \Phi^*(T^{0,1}X)$ respectively. The transformation can be written as

$$\begin{aligned}
\delta\phi^\mu &= 0 \\
\delta\phi^{\bar\mu} &= i\epsilon\eta^{\bar\mu} \\
\delta\eta^{\bar\mu} &= \delta\theta_\mu = 0 \\
\delta\rho^\mu &= -\epsilon d\phi^\mu
\end{aligned} \tag{3.10}$$

where

$$\begin{aligned}
\eta^{\bar\mu} &= \psi_+^{\bar\mu} + \psi_-^{\bar\mu} \\
\theta_\mu &= g_{\mu\bar\mu}(\psi_+^{\bar\mu} - \psi_-^{\bar\mu}) \\
\rho^\mu &= \psi_+^\mu + \psi_-^\mu
\end{aligned} \tag{3.11}$$

After the B twisting the Lagrangian explicitly becomes

$$\begin{aligned}
L = t\int d^2z \quad (\quad & g_{ab}\partial_z\phi^a\partial_{\bar z}\phi^b + ig_{\bar\mu\mu}\eta^{\bar\mu}(D_z\rho_{\bar z}^\mu + D_{\bar z}\rho_z^\mu) \\
& + i\theta_\mu(D_{\bar z}\rho_z^\mu - D_z\rho_{\bar z}^\mu) + R_{\mu\bar\mu\nu\bar\nu}\rho_z^\mu\rho_{\bar z}^\nu\eta^{\bar\mu}\theta_\sigma g^{\sigma\bar\nu})
\end{aligned} \tag{3.12}$$

which can be reexpressed as follows.

$$L = it\int\{Q, V_B\} + tW \tag{3.13}$$

where

$$W = \int_\Sigma(-\theta_\mu D\rho^\mu - \frac{i}{2}R_{\mu\bar\mu\nu\bar\nu}\rho^\mu \wedge \rho^\nu\eta^{\bar\mu}\theta_\sigma g^{\sigma\bar\nu}) \tag{3.14}$$

and the D operator is the exterior derivative on the worldsheet Σ by using the pullback of the Levi-Civita connection on M. The model is topological because it is independent of the complex structure of the worldsheet and the Kähler structure of the target space. However the model do depend on the complex structure, which can be seen from the BRST variations of the fields.

3.3 Bi-Hermitian geometry and its topological twisted models

As stated in the introduction the most general $(2,2)$ nonlinear sigma model with H is described in [57], which is also known as "bi-Hermitian geometry." We will simply quote the properties of the geometry, without any derivations of the requirements. With the non-trivial B-field turned on, the worldsheet action is given by

$$L = 2t \int d^2z \; d^2\theta \; (g_{ab}(\Phi) + b_{ab}(\Phi))D_+\Phi^a D_-\Phi^b \tag{3.15}$$

The first set of $(1,1)$ supersymmetry is as usual while the additional $(1,1)$ supersymmetry transformations are given by two different complex structures

$$\begin{aligned} \delta^1\Phi^a &= i\epsilon^1_+ D_+\Phi^a + i\epsilon^1_- D_-\Phi^a \\ \delta^2\Phi^a &= i\epsilon^2_+ D_+\Phi^b J^a_{+b} + i\epsilon^2_- D_-\Phi^b J^a_{-b} \end{aligned} \tag{3.16}$$

where J_+ and J_- are the complex structures seen by the left and right movers respectively. Requiring (3.15) to be invariant under the transformations leads us to the conditions:

$$J^t_\pm g J_\pm = g \qquad \nabla^\pm J_\pm = 0 \tag{3.17}$$

where $\nabla^\pm$ are the covariant derivatives with torsional connections $\Gamma_\pm = \Gamma \pm g^{-1}H$. The first condition implies that the metric is Hermitian with respect to the either one of the complex structures $J_\pm$. And the second condition in (3.17) explicitly becomes

$$J^a_{\pm b,c} = \Gamma^d_{\pm cb} J^a_{\pm d} - \Gamma^a_{\pm cd} J^d_{\pm b}. \tag{3.18}$$

Equation (3.18) will be used when we try to contruct the generalized A/B models in real coordinate basis. Moreover the H field is of type $(2,1)+(1,2)$ with respect to both complex structures $J_\pm$. Expanding (3.15) out and then setting F^a to its on-shell value we have the following worldsheet action in component fields

$$F^a = \Gamma^a_{+bc}\psi^b_+\psi^c_- = -\Gamma^a_{-bc}\psi^b_-\psi^c_+ \tag{3.19}$$

$$\begin{aligned} L &= 2t\int d^2z(\frac{1}{2}(g_{ab}+b_{ab})\partial_z\phi^a\partial_{\bar z}\phi^b + \frac{i}{2}g_{ab}(\psi^a_-\partial_z\psi^b_- + \psi^a_+\partial_{\bar z}\psi^b_+) \\ &+ \frac{i}{2}\psi^a_-\partial_z\phi^b\psi^c_-(\Gamma_{abc}-H_{abc}) + \frac{i}{2}\psi^a_+\partial_{\bar z}\phi^b\psi^c_+(\Gamma_{abc}+H_{abc}) + \frac{1}{4}R_{+abcd}\psi^a_+\psi^b_+\psi^c_-\psi^d_-) \end{aligned} \tag{3.20}$$

where R_{+abcd} is the curvature of the torsional connection Γ^a_{+bc}.

$$R_{\pm abcd} = R_{abcd} \pm \frac{1}{2}(\nabla_d H_{abc} - \nabla_c H_{abd}) + \frac{1}{4}(H^e_{ad}H_{ebc} - H^e_{ac}H_{ebd}) \tag{3.21}$$

Since the theory is of $(2,2)$ type there exist two $U(1)$ R-symmetries for the worldsheet fermions, $U(1)_V$ and $U(1)_A$ [58]. The topological A and B twists will shift the spins of the fermions by the charges of $U(1)_V$ and $U(1)_A$ respectively. The charge assignments are worked out in [58] and [64].

$$\begin{aligned} U(1)_V: &\quad q_V(\bar P_+\psi_+) = -1 \quad q_V(\bar P_-\psi_-) = -1 \\ U(1)_A: &\quad q_A(\bar P_+\psi_+) = -1 \quad q_V(\bar P_-\psi_-) = +1 \end{aligned} \tag{3.22}$$

with the following projectors defined for conveniences.

$$P_\pm = \frac{1}{2}(1+iJ_\pm), \quad \bar P_\pm = \frac{1}{2}(1-iJ_\pm) \tag{3.23}$$

Moreover the $U(1)$ R-symmetry used in the topological twist needs to be non-anomalous. The anomalies are computed by Atiyah-Singer index theorem and the conditions are

$$\begin{aligned} U(1)_V: &\quad c_1(T^{1,0}_-) - c_1(T^{1,0}_+) = 0 \\ U(1)_A: &\quad c_1(T^{1,0}_-) + c_1(T^{1,0}_+) = 0 \end{aligned} \tag{3.24}$$

Using the language of generalized complex geometry we have two commuting twisted generalized complex structures $(\mathcal{J}_1, \mathcal{J}_2)$. $\mathcal{J}_1$ and $\mathcal{J}_2$ are endomorphisms on $TM \oplus T^*M$, which square to -1. Let E_1 and E_2 be the i-eigenbundles of $\mathcal{J}_1$ and $\mathcal{J}_2$. The conditions can be repackaged into

$$\begin{aligned} U(1)_V: &\quad c_1(E_2)=0 \\ U(1)_A: &\quad c_1(E_1)=0 \end{aligned} \tag{3.25}$$

The supersymmetry transformation laws can be derived from (3.16).

$$\begin{aligned} &\delta^1_+\phi=\psi_+ \quad \delta^1_-\phi=\psi_- \quad \delta^2_+\phi=J_+\psi_+ \quad \delta^2_-\phi=J_-\psi_- \\ &\delta^1_+\psi_+=-i\partial_z\phi \quad \delta^1_-\psi_+=F \quad \delta^2_+\psi_+=iJ_+\partial_z\phi \quad \delta^2_-\psi_+=J_-F \\ &\delta^1_+\psi_-=-F \quad \delta^1_-\psi_-=-i\partial_{\bar z} \quad \delta^2_+\psi_-=-J_+F \quad \delta^2_-\psi_-=iJ_-\partial_{\bar z}\phi \end{aligned} \tag{3.26}$$

We can then define the linear combinations of the supersymmetry generators.

$$\begin{aligned} Q_+=\frac{1}{2}(Q^1_++iQ^2_+) &\quad \bar Q_+=\frac{1}{2}(Q^1_+-iQ^2_+) \\ Q_-=\frac{1}{2}(Q^1_-+iQ^2_-) &\quad \bar Q_-=\frac{1}{2}(Q^1_--iQ^2_-) \end{aligned} \tag{3.27}$$

We then express the on-shell variation laws in the following forms

$$\begin{aligned} \delta\phi^a &= i(\epsilon_+(P_+\psi_+)^a+\bar\epsilon_+(\bar P_+\psi_+)^a)+i(\epsilon_-(P_-\psi_-)^a+\bar\epsilon_-(\bar P_-\psi_-)^a) \\ \delta\psi_+ &= -\epsilon_+(\bar P_+\partial_z\phi)^a-\bar\epsilon_+(P_+\partial_z\phi)^a-\Gamma^a_{+bc}\delta\phi^b\psi^c_+ \\ &+ iH^a_{bc}(\epsilon_+(P_+\psi_+)^b+\bar\epsilon_+(\bar P_+\psi_+)^b)\psi^c_+-\frac{i}{2}(\epsilon_+P^a_{+d}+\bar\epsilon_+\bar P^a_{+d})H^d_{bc}\psi^b_+\psi^c_+ \\ \delta\psi_- &= -\epsilon_-(\bar P_-\partial_z\phi)^a-\bar\epsilon_-(P_-\partial_z\phi)^a-\Gamma^a_{-bc}\delta\phi^b\psi^c_- \\ &+ iH^a_{bc}(\epsilon_-(P_-\psi_-)^b+\bar\epsilon_-(\bar P_-\psi_-)^b)\psi^c_--\frac{i}{2}(\epsilon_-P^a_{-d}+\bar\epsilon_-\bar P^a_{-d})H^d_{bc}\psi^b_-\psi^c_- \end{aligned} \tag{3.28}$$

where $\epsilon_\pm$ are the variation parameters of $Q_\pm$.

The BRST operators for the generalized A and B models can be taken as:

$$Q_A=Q_++\bar Q_-, \quad Q_B=\bar Q_++\bar Q_-. \tag{3.29}$$

Before the topological twists we have the worldsheet fermions $P_+\psi_+$, $\bar P_+\psi_+$, $P_-\psi_-$, and $\bar P_-\psi_-$. These fermions are sections of certain bundles. For instance $\bar P_+\psi_+$ is a section of $K^{1/2}\otimes\Phi^*(T^{0,1}_+X)$ where K is the canonical line bundle of the worldsheet (the bundle of $(1,0)$ form.) and $T^{0,1}_+$ is the $(0,1)$ part of the tangent bundle with respect to J_+. After performing topological A-twist, the spins of the fermions will be changed as follows.

$$
\begin{aligned}
(P_+\psi_+)^a &\equiv \chi^a \in \Gamma(\Phi^*(T_+^{1,0}X)) \\
(\bar{P}_+\psi_+)^a &\equiv \chi_z^a \in \Gamma(\Omega_\Sigma^{(1,0)} \otimes \Phi^*(T_+^{0,1}X)) \\
(P_-\psi_-)^a &\equiv \lambda_{\bar{z}}^a \in \Gamma(\Omega_\Sigma^{(0,1)} \otimes \Phi^*(T_-^{1,0}X)) \\
(\bar{P}_-\psi_-)^a &\equiv \lambda^a \in \Gamma(\Phi^*(T_-^{0,1}X))
\end{aligned}
\tag{3.30}
$$

On the other hand the B-twist case can be obtained similarly. For completeness we list the sections in the generalized B-model with the BRST charge $Q_B = \bar{Q}_+ + \bar{Q}_-$.

$$
\begin{aligned}
(P_+\psi_+)^a &\equiv \chi_z^a \in \Gamma(\Omega_\Sigma^{(1,0)} \otimes \Phi^*(T_+^{1,0}X)) \\
(\bar{P}_+\psi_+)^a &\equiv \chi^a \in \Gamma(\Phi^*(T_+^{0,1}X)) \\
(P_-\psi_-)^a &\equiv \lambda_{\bar{z}}^a \in \Gamma(\Omega_\Sigma^{(0,1)} \otimes \Phi^*(T_-^{1,0}X)) \\
(\bar{P}_-\psi_-)^a &\equiv \lambda^a \in \Gamma(\Phi^*(T_-^{0,1}X))
\end{aligned}
\tag{3.31}
$$

3.3.1 Generalized A model

We will use the generalized A-model as our first explicit example. The BRST variation of the fields can be written down by setting the variation of $\bar{Q}_+$ and Q_- in (3.28) to be zero.

$$
\begin{aligned}
\{Q_A, \phi^a\} =& \quad \chi^a + \lambda^a \\
\{Q_A, \chi^a\} =& \quad -i\Gamma^a_{+bc}\lambda^b\chi^c \\
\{Q_A, \lambda^a\} =& \quad -i\Gamma^a_{-bc}\chi^b\lambda^c \\
\{Q_A, \chi_z^a\} =& \quad -i\Gamma^a_{+bc}(\chi^b + \lambda^b)\chi_z^c \\
& \quad -(\bar{P}_+\partial_z\phi)^a + i\bar{P}^a_{+d}H^d_{bc}\chi^b\chi_z^c \\
\{Q_A, \lambda_{\bar{z}}^a\} =& \quad -i\Gamma^a_{-bc}(\chi^b + \lambda^b)\lambda_{\bar{z}}^a \\
& \quad -(P_-\partial_z\phi)^a - iP^a_{-d}H^d_{bc}\lambda^b\lambda_{\bar{z}}^c
\end{aligned}
\tag{3.32}
$$

After the twisting the Lagrangian becomes:

$$\begin{aligned} L \;\; &= 2t\int d^2z(\frac{1}{2}(g_{ab}+b_{ab})\partial_z\phi^a\partial_{\bar z}\phi^b + ig_{ab}(\chi^a_z\partial_{\bar z}\chi^b + \lambda^a_{\bar z}\partial_z\lambda^b) \\ &+ i(\Gamma_{abc}-H_{abc})\chi^a_z\partial_{\bar z}\phi^b\chi^c + i(\Gamma_{abc}+H_{abc})\lambda^a_{\bar z}\partial_z\phi^b\lambda^c + R_{+abcd}\chi^a\chi^b_z\lambda^c_{\bar z}\lambda^d) \end{aligned} \tag{3.33}$$

We mimic the V_A operator in Kähler A model (3.9) by virtue of the projectors.

$$\mathcal{V}_A = g_{ab}(\chi^a_z(P_+\partial_{\bar z}\phi)^b + \lambda^a_{\bar z}(\bar P_-\partial_z\phi)^b) \tag{3.34}$$

The BRST variations of $(P_+\partial_{\bar z}\phi)^b$ and $(\bar P_-\partial_z\phi)^b$ will involve the derivatives of the complex structures and can be re-expressed in terms of $\Gamma_\pm$ and the projectors (3.23) by using (3.18) and $J_\pm = -i(P_\pm - \bar P_\pm)$.

$$\begin{aligned} \{Q_A,(P_+\partial\phi)^b\} = \partial\chi^b + (P_+\partial\lambda)^b \;&+\; \frac{1}{2}\Gamma^d_{+ec}(P_+-\bar P_+)^b_d(\chi^c+\lambda^c)\partial\phi^e \\ &-\; \frac{1}{2}\Gamma^b_{+cd}(\chi^c+\lambda^c)(P_+\partial\phi-\bar P_+\partial\phi)^d \\ \{Q_A,(\bar P_-\partial\phi)^b\} = \partial\lambda^b + (\bar P_-\partial\chi)^b \;&-\; \frac{1}{2}\Gamma^d_{-ec}(P_--\bar P_-)^b_d(\chi^c+\lambda^c)\partial\phi^e \\ &+\; \frac{1}{2}\Gamma^b_{-cd}(\chi^c+\lambda^c)(P_-\partial\phi-\bar P_-\partial\phi)^d \end{aligned} \tag{3.35}$$

Here the ∂ operator could be either ∂_z or $\partial_{\bar z}$. Performing the BRST variations to V by using (3.32) and (3.35) we obtain

$$\begin{aligned} \{Q_A,\mathcal{V}_A\} = ig_{ab}((\bar P_+\partial_z\phi)^a(P_+\partial_{\bar z}\phi)^b + (P_-\partial_{\bar z}\phi)^a(\bar P_-\partial_z\phi)^b) + g_{ab}(\chi^a_z\partial_{\bar z}\chi^b + \lambda^a_{\bar z}\partial_z\lambda^b) \\ +(\Gamma_{abc}+H_{abc})\chi^a_z\partial_{\bar z}\phi^b\chi^c + (\Gamma_{abc}-H_{abc})\lambda^a_{\bar z}\partial_z\phi^b\lambda^c \end{aligned} \tag{3.36}$$

The curvature term will be recovered if we use the equations of motion for χ_z and $\lambda_{\bar z}$. To visualize that the model only depends on one of the generalized complex structure one can use the following identities.

$$\begin{aligned} g(P_\pm\cdot,\bar P_\pm\cdot) = \frac{1}{2}g(\cdot,\cdot) + \frac{i}{2}g(J_\pm\cdot,\cdot) = \frac{1}{2}g(\cdot,\cdot) + \frac{i}{2}\omega_\pm(\cdot,\cdot) \\ g(\bar P_\pm\cdot,P_\pm\cdot) = \frac{1}{2}g(\cdot,\cdot) - \frac{i}{2}g(J_\pm\cdot,\cdot) = \frac{1}{2}g(\cdot,\cdot) - \frac{i}{2}\omega_\pm(\cdot,\cdot) \end{aligned} \tag{3.37}$$

The scalar term in (3.36) becomes

$$g_{ab}((\bar{P}_+\partial_z\phi)^a(P_+\partial_{\bar{z}}\phi)^b + (P_-\partial_{\bar{z}}\phi)^a(\bar{P}_-\partial_z\phi)^b) = 2g_{ab}\partial_z\phi^a\partial_{\bar{z}}\phi^b - i\tilde{\omega}_{ab}\partial_z\phi^a\partial_{\bar{z}}\phi^b \qquad (3.38)$$

where $\tilde{\omega} = \frac{1}{2}(\omega_+ + \omega_-)$ which appear in $\mathcal{J}_2$ in (A.2).

Comparing the twisted action (3.33) and (3.36) we obtain the following suggestive equation, modulo the equations of motion for χ_z and $\lambda_{\bar{z}}$.

$$L = it\int d^2z\ \{Q_A, \mathcal{V}_A\} + t\int \Phi^*(-i\tilde{\omega}) + t\int \Phi^*(b) \qquad (3.39)$$

Apparently the action of the generalized A model depends on one of the generalized complex structures $\mathcal{J}_2$ and the pullback of the spacetime b field. The topological feature of the action will be made clear in the next section.

3.3.2 Generalized B model

The generalized B model has the field contents as listed in (3.31). By projecting out $\epsilon_\pm$ in (3.28) the BRST variations for these fields are similarly obtained.

$$\begin{aligned}
\{Q_B, \phi^a\} &= \chi^a + \lambda^a \\
\{Q_B, \chi^a\} &= -i\Gamma^a_{+bc}\lambda^b\chi^c \\
\{Q_B, \lambda^a\} &= -i\Gamma^a_{-bc}\chi^b\lambda^c \\
\{Q_B, \chi^a_z\} &= -i\Gamma^a_{+bc}(\chi^b + \lambda^b)\chi^c_z \\
&\quad -(P_+\partial_z\phi)^a + iP^a_{+d}H^d_{bc}\chi^b\chi^c_z \\
\{Q_B, \lambda^a_{\bar{z}}\} &= -i\Gamma^a_{-bc}(\chi^b + \lambda^b)\lambda^a_{\bar{z}} \\
&\quad -(P_-\partial_z\phi)^a - iP^a_{-d}H^d_{bc}\lambda^b\lambda^c_{\bar{z}}
\end{aligned} \qquad (3.40)$$

with $Q_B = \bar{Q}_+ + \bar{Q}_-$. Comparing (3.32) and (3.40) we can see that the A and B model variantion laws are simply exchanged if we substitute J_+ by $-J_+$. In generalized B model the operator in the BRST exact term is given by

$$\mathcal{V}_B = g_{ab}(\chi^a_z(\bar{P}_+\partial_{\bar{z}}\phi)^b + \lambda^a_{\bar{z}}(\bar{P}_-\partial_z\phi)^b) \qquad (3.41)$$

The variations of $(\bar{P}_\pm\partial\phi)^b$ are given by

$$\{Q_B,(\bar{P}_\pm\partial\phi)^b\} = (\bar{P}_\pm(\partial\chi+\partial\lambda))^b - \frac{1}{2}\Gamma^d_{\pm ec}(P_\pm-\bar{P}_\pm)^b_d(\chi^c+\lambda^c)\partial\phi^e + \frac{1}{2}\Gamma^b_{-cd}(\chi^c+\lambda^c)(P_\pm\partial\phi-\bar{P}_\pm\partial\phi)^d \quad (3.42)$$

Note that $\bar{P}_+\chi = \chi$ and $\bar{P}_-\lambda = \lambda$. Again the ∂ could be either ∂_z or $\partial\bar{z}$.

The Lagrangian after the twisting is given by

$$L = 2t\int d^2z\big(\frac{1}{2}(g_{ab}+b_{ab})\partial_z\phi^a\partial_{\bar{z}}\phi^b + ig_{ab}(\chi^a_z\partial_{\bar{z}}\chi^b+\lambda^a_{\bar{z}}\partial_z\lambda^b) \quad (3.43)$$
$$+ i(\Gamma_{abc}-H_{abc})\chi^a_z\partial_{\bar{z}}\phi^b\chi^c + i(\Gamma_{abc}+H_{abc})\lambda^a_{\bar{z}}\partial_z\phi^b\lambda^c + R_{+abcd}\chi^a\chi^b_z\lambda^c\lambda^d_{\bar{z}}\big)$$

In order to determine the pullback term we compute $\{Q,\mathcal{V}_B\}$.

$$\{Q_B,\mathcal{V}_B\} = ig_{ab}((P_+\partial_z\phi)^a(\bar{P}_+\partial_{\bar{z}}\phi)^b + (P_-\partial_{\bar{z}}\phi)^a(\bar{P}_-\partial_z\phi)^b) + g_{ab}(\chi^a_z\partial_{\bar{z}}\chi^b+\lambda^a_{\bar{z}}\partial_z\lambda^b)$$
$$+(\Gamma_{abc}+H_{abc})\chi^a_z\partial_{\bar{z}}\phi^b\chi^c + (\Gamma_{abc}-H_{abc})\lambda^a_{\bar{z}}\partial_z\phi^b\lambda^c \quad (3.44)$$

In deriving this we have used the equations of motion of the fermionic fields. Note that (3.36) and (3.44) are almost the same except for the scalar kinetic terms. This will result in the different GCS dependence. Namely,

$$L = it\int d^2z\ \{Q_B,\mathcal{V}_B\} + t\int\Phi^*(i\delta\omega) + t\int\Phi^*(b) \quad (3.45)$$

where $\delta\omega = \frac{1}{2}(\omega_+ - \omega_-)$ appearing in $\mathcal{J}_1$ (A.2). Contrary to the generalized A model, the generalized B model depends on $\mathcal{J}_1$. At first sight the results (3.39) (3.45) seem nice and confirm our original guess. A second thought, however, reveals the issue that neither of $b - i\tilde{\omega}$ and $b + i\delta\omega$ is closed. The consequence of this is that under small coordinate repaprametrization the variation of the pullback will be nonvanishing and proportional to H [66]. One way to solve this issue is to appeal to the GCG [65]. Working in generalized B model, we assume the pure spinor $Ⓢ_1$ associated with TGC structure $\mathcal{J}_1$ can be put into the following form:

$$Ⓢ_1 = exp(b+\beta) \quad (3.46)$$

$$-\bar{\beta} = b \mp i\omega_\pm - \gamma_\pm \quad (3.47)$$

where $d\beta = 0$ and the multiplication in the exponential is the wedge product. A direct but lengthy computation shows, in generalized B model,

$$L = it\int d^2z\ \{Q_B, \mathcal{V}_B + \frac{1}{2}\gamma_{+ab}\chi_z^a\partial_{\bar{z}}\phi^b - \frac{1}{2}\gamma_{-ab}\lambda_{\bar{z}}^a\partial_z\phi^b\} + t\int \Phi^*(\bar{\beta}) \tag{3.48}$$

We refer the interested readers to [65] for more details about this construction. Alternatively one could simply say that without this construction the model is topological in the sense that the worldsheet metric is irrelevant and the puckback term only depends on the homotopy class of the embedding.

3.4 Conclusion and Discussion

In this paper we study the topological twisted models with H-flux. We explicitly expand the $N = (2,2)$ worldsheet action with bi-Hermitian target spaces and twist the action. We found that the generalized twisted models have many similar features to the Kähler twisted models. For example, the action can always be written as a sum of a BRST exact term and some pullback terms, from which the geometric dependence of the topological models can be read off. The generalized A/B model depends only on one of the twisted generalized complex structures $\mathcal{J}_2$/ $\mathcal{J}_1$.

Although it is very powerful to construct interesting examples of topological field theories by "twisting" the spins of the fields, some topological constraints for anomaly cancellations always come with it. Recently people have tried to construct the topological models for generalized geometries by using Batalin-Vilkovisky formalism to get around this limitation[63].

Another advantage of the twisted models is that it makes explicit the studying the mirror symmetry, in this case, of the non-Kähler spaces. The lacking of the non-Kähler examples, however, is a long-standing problem along this direction. Although the "generalized Kähler" examples provided in [1] are not twisted by H-field, it would still be very interesting to study the topological models for those geometries. Another interesting problem is to generalize the usual Kähler quotients to obtain explicit bi-Hermitian examples. We would like to visit these problems in the future.

Chapter 4

Flux-induced isometry gauging in heterotic strings

Abstract

We study the effect of flux-induced isometry gauging of the scalar manifold in $N = 2$ heterotic string compactification with gauge fluxes. We show that a vanishing theorem by Witten provides the protection mechanism. The other ungauged isometries in hyper moduli space could also be protected, depending on the gauge bundle structure. We also discuss the related issue in IIB setting.

4.1 Introduction

It is very difficult to build a fully realistic string model without using flux compactifications [7]. There are by now various sources of evidence suggesting that we should not restrict ourselves to the study of Calabi-Yau spaces as string theory vacua. The study of mirror symmetry for Calabi-Yau flux compactification, for instance, will inevitably lead us to the territory of "Non-Kählerity" [67, 43, 45, 21, 44, 1]

It is also very interesting to study the fate of the well-known IIA/hetrotic string duality if we compactify IIA string on the non-Kähler background. This nonpertubative duality between IIA on $K3$ fibered Calabi-Yau and heterotic string on $K3 \times T^2$ was first studied in [68, 69] and then generalized to the case with fluxes and $SU(3)$-structure manifolds[72, 70]. The effect of gauging induced by torsions in geometry and by various kinds of fluxes in IIA were mapped to the gauge fluxes in heterotic string.

When we turn on the RR or NSNS fluxes in IIA/IIB/heterotic $N = 2$ compactification, supergravity analysis suggests that it will lead to the isometry gauging of the scalar manifold [15]. This means the hypermultiplets become charged under certain vector multiplets. The gauging and the charges are specified by the killing vectors, which are determined by the fluxes turned on. The non-perturbative objects in string theory, D-branes or D-instantons, presumably could destroy the isometries in the hyper moduli space by introducing RR dependence into the action. In [73], the authors showed that the allowed instantons in IIA string setting will not remove the flux-gauged isometries; namely the flux will protect the gauged isometries[1]. However, other isometries are generically lifted by instanton corrections. It is not clear whether the non-perturbative correction still preserves the quaternionic structure. We notice similar arguments are not enough to reach the same conclusion in IIB case, where the shift symmetry of RR scalar C_0 is gauged by the NSNS flux and multiple instanton branes can contribute C_0 dependent corrections to the moduli space metric.

In this paper we study the $N = 2$ gauged supergravity resulting from heterotic string theory compactified on $K3 \times T^2$ with gauge fluxes. The gauging in the supergravity analysis is achieved by turning on the abelian gauge fluxes. The exact matching between the IIA and heterotic flux parameters can be worked out straightforwardly. In $N = 2$ heterotic string compactification, the hyper moduli space could receive α'-correction [10], perturbatively and non-perturbatively. A worldsheet instanton wrapping a holomorphic cycle in K3, for example, could give correction to the hyper moduli space because there are hypermultiplets coming from $H^2(K3)$. However, the isometry gauging is achieved by turning on the abelian gauge fluxes over certain 2 cycle C in K3 [70], which means the gauge bundle restricted to the 2 cycle $V|_C$ is non-trivial. This is precisely the situation where the instanton correction is zero [78].

The paper is organized as follows. In section 4.2 we recall the isometry protection mechanism in IIA setting. In section 4.3 we first review the IIA/heterotic duality and then demonstrate how Witten's vanishing theorem helps protect the gauged isometry in heterotic string. Lastly, discussion and conclusion follow.

[1]See [80] for a similar result in the setting of five-dimensional heterotic M-theory.

4.2 Isometry protection in IIA flux compactificatons

In this section we begin by reviewing the isometry gauging in IIA setting and how NSNS flux protects certain isometries [73]. The protection follows from the tadpole consideration on the world volume of the D-instantons.

First let us consider IIA on a Calabi-Yau M. Each $N = 2$ hypermultiplet contains two complex scalars z^a, $a = 1, \ldots, h^{2,1}$ coming from complex structure moduli of the Calabi-Yau and two scalars φ^α, $\tilde{\varphi}_\alpha$ from expansion of the RR potential C_3 in a particular symplectic marking of $H^3(M)$

$$C_3 = \varphi^\alpha A_\alpha + \tilde{\varphi}_\alpha B^\alpha \ , \alpha = 0, \ldots, h^{2,1} \tag{4.1}$$

φ^0, $\tilde{\varphi}_0$, the dilaton ϕ and the NSNS axion a form the universal hypermultiplet obtained by dualizing the tensor multiplet in four dimensions. In the dimensionally reduced $N = 2$ supergravity theory, these scalars reside on a quarternionic manifold with the metric given by [74, 75]:

$$\begin{aligned} ds^2 &= d\phi^2 + g_{a\bar{b}} dz^a d\bar{z}^{\bar{b}} + \frac{e^{4\phi}}{4} \left[da + \tilde{\varphi}_\alpha d\varphi^\alpha - \varphi^\alpha d\tilde{\varphi}_\alpha\right] \left[da + \tilde{\varphi}_\alpha d\varphi^\alpha - \varphi^\alpha d\tilde{\varphi}_\alpha\right] \\ &\quad - \frac{e^{2\phi}}{2} (\mathrm{Im}\mathcal{M}^{-1})^{\alpha\beta} \left[d\tilde{\varphi}_\alpha + \mathcal{M}_{\alpha\gamma} d\varphi^\gamma\right] \left[d\tilde{\varphi}_\beta + \overline{\mathcal{M}}_{\beta\delta} d\varphi^\delta\right]. \end{aligned}$$

Expanding the background fluxes F_4 and H_3 we get

$$\begin{aligned} F_4 &= \lambda_I \tilde{\omega}^I \\ H_3 &= p^\alpha A_\alpha + q_\alpha B^\alpha \end{aligned} \tag{4.2}$$

where $\tilde{\omega}^i$ a basis for $H^{2,2}(M)$.

We now have the following killing vectors corresponding to the isometries to be gauged[76, 77]:

$$\begin{aligned} (k_F)_I &= -2\lambda_I \partial_a \\ k_H &= (p^\alpha \tilde{\varphi}_\alpha - q_\alpha \varphi^\alpha)\partial_a + p^\alpha \partial_{\varphi^\alpha} + q_\alpha \partial_{\tilde{\varphi}_\alpha} \end{aligned} \tag{4.3}$$

where F_4 and H_3 determine the charges under the Ith vector and the graviphoton fields respectively [2].

[2] Throughout the paper $\alpha\beta\ldots$ and $IJ\ldots$ denote hyper and vector indices respectively.

Due to the absence of 1 and 5-cycles in the Calabi-Yau manifold, the only relevant IIA D-instanton is the D2-instanton wrapping a 3-cycle. Consider an instanton state consisting of E2 branes wrapping a cycle in the homology class expressible as the formal sum

$$\Gamma_{inst} = \sum_i c^i \Gamma_i \, . \tag{4.4}$$

This configuration contributes a φ_i dependence

$$\int_\Gamma C_3 = \sum_i c^i \varphi_i \tag{4.5}$$

to the effective action[3]. Transforming the scalar manifold metric under k_H we find

$$k_H(\int_\Gamma C_3) = \sum_i c^i p_i \ . \tag{4.6}$$

For generic values of c^i, the classical brane action breaks any isometry involving a shift in the value of fields φ_i.

If this were true, it will certainly destroy the consistency of the gauging procedure. However, as noticed in [73] there is a simple mechanism at work which prohibits this from happening. The crucial observation of [73] is that the Bianchi identity for world volume gauge flux reads

$$dF = -H_3 \ . \tag{4.7}$$

On a compact world volume without boundary this requires

$$\sum_i c^i p_i = 0 \ , \tag{4.8}$$

from which it's obvious any physically realized instanton cannot break the gauged isometries.

A more concise way to rephrase the protection mechanism is to recall that H flux induces magnetic charges for the brane gauge field. It implies we can not wrap a D2-instaton over a 3 cycle on which we turned on H flux. This is simply the constraint imposed by Freed-Witten anomaly [79].

[3] Here we dropped the symplectic structure on H^3 and expand in the basis $\{\gamma^i\}$ dual to the homology basis $\{\Gamma_i\}$. We have $C_3 = \varphi_i \gamma^i$ and $H_3 = p_i \gamma^i$.

4.3 Isometry protection in heterotic string

In this section we will review the IIA/heterotic duality with gauge fluxes. We wil provide an exact matching between the flux parameters [69, 72, 70]. Then we show a theorem due to Witten guarantees the protection of gauged isometries.

4.3.1 IIA/heterotic duality

The IIA/heterotic duality was first studied in [68, 69]. Besides the spectrum matching, the conifold transitions in IIA string on CY is mapped to the Higgsing of the charged hypermultiplets in heterotic string. The Higgsing can move the theory around the different moduli space strata with different dimensions. This beautiful phenomenon is not the topic of our paper although the transition in the presence of fluxes is certainly worth further studying.

We will begin by recalling the results in [70, 72]. The anomaly cancellation in the 10d supergravity requires we modify the heterotic H in the following way,

$$H = dB + \omega_{gravity} - \omega_{YM}. \tag{4.9}$$

From this we get a new Bianchi identity:

$$dH = trR \wedge R - trF \wedge F \tag{4.10}$$

where R is the Riemann curvature of the internal manifold and F is the field strength of the Yang-Mills fields.

For heterotic string on $K3 \times T^2$, we will need 24 instanton number to cancel the $\int_{K3} tr(R \wedge R)$ contribution. In earlier literature, people usually studied the gauge bundle with $c_1(V) = 0$. But in fact there exists no obstruction for us to turn on c_1 of the gauge bundle (equivalent to turning on abelian gauge fluxes). It is also possible to turn on c_1 such that it does not contribute to $\int tr(F \wedge F)$. This can be seen as follows. Let us first turn on the following gauge fluxes over the 2 cycles in K3,

$$\int_{\gamma^\alpha} F^I_{gauge} = m^{\alpha I}, \quad I = 0, \cdots, n_V, \quad \alpha = 1, \cdots, 22. \tag{4.11}$$

where I is the index for vector moduli and the zeroth component stands for the graviphton.

These fluxes could contribute to the tadpole condition [70, 71]:

$$\int_{K3} tr(F_{inst} \wedge F_{inst}) + \delta = 24 \tag{4.12}$$

where

$$\delta = \int_{K3} F^I_{gauge} \wedge F^J_{gauge} \eta_{IJ} = m^{\alpha I} m^{\beta J} \rho_{\alpha\beta} \eta_{IJ} \tag{4.13}$$

$\rho_{\alpha\beta}$ is the K3 intersection matrix with signature $(3, 19)$ and η_{IJ} is the invariant tensor on $SO(2, n_V - 1)$. As we will see later, turning on $m^{\alpha I}$ in heterotic is dual to turning on various kinds of fluxes in IIA. So if we start from IIA on K3-fibered CY with fluxes and are interested in finding its heterotic dual with gauge fluxes, we should consider $\delta = 0$ so that the originally balanced tadpole condition will not be disturbed.[4]

Now let us consider the gauging effect of turning on gauge fluxes accroding to (4.11) over 20 2 cycles in an attractive $K3$, following [70]. [5] After expanding ten-dimensional $\mathcal{B}$ filed in terms of the $H^2(K3)$ basis ω_α,

$$\mathcal{B} = B + b^\alpha \omega_\alpha \tag{4.14}$$

the covariant derivative of b^α becomes

$$Db^\alpha = db^\alpha - (\eta_{IJ} m^{\alpha J}) A^I = db^\alpha - m^\alpha_I A^I \tag{4.15}$$

The resulting killing vector is

$$k_I = -m^\alpha_I \partial_{b^\alpha}. \tag{4.16}$$

Recall that b^α in heterotic string corresponds to some φ^α in (4.2). Comparing (4.16) with (4.3), we immediately see that the gauging from the gauge fluxes does not correspond to any H or F in IIA theory. But the effect can be dualized into the gauging coming from the torsions in the geometry; it is dual to IIA on an $SU(3) \times SU(3)$ structure manifold [70]. For a review on $SU(3) \times SU(3)$ structure manifolds in the context of supergravity, see [44].

Note that there are also bundle moduli coming from the gauge bundle. Their number is determined by the dimension of the sheaf cohomology group. In this case the sheaf will

[4] We would have to solve the anomaly cancellation condition from the very beginning if δ does not vanish.

[5] In [72] the gauge fluxes are turned on over the P^1 of the $K3$ in heterotic string, which corresponds to F with support on the base of the $K3$ fibered Calabi-Yau. This flux will charge the axion in heterotic string. The IIA dual of the gauge flux through T^2 fiber in heterotic K3 is unknown.

be the endormophism of the gauge bundle [81]. It is easy to show that the first order deformation of this sheaf is $H^1(K3, E^* \otimes E)$, where E^* is the dual sheaf. The dimension counting, which includes the high order obstruction, can be done by computing the Euler character $\chi(E, E)$,

$$\chi(E,E) = \sum(-1)^i \mathrm{dim} Ext^i(E,E) = \int_X ch(E^*)ch(E)\sqrt{Td(X)} \tag{4.17}$$

Let now X be a K3 surface. If E is a coherent sheaf on X with $rk(E) = r$, $c_1(E) = c_1$, and $c_2(E) = c_2$, the complex dimension of the bundle moduli is given by $2rc_2 - (r-1)c_1^2 - 2(r^2-1)$.

At this moment it is not clear now to charge these bundle moduli under the gauge fields because we know very little about the hyper moduli space. We will revisit this problem in the future. In the next section, we will demonstrate the mechanism which protects the gauged isometries from gauge fluxes in heterotic string.

4.3.2 Witten's vanishing theorem

In this section we will show how Witten's result [78] can protect the gauging in $N = 2$ heterotic theory. In the previous section, the gauging of the b^α results from turning on the gauge flux over the corresponding 2 cycle γ^α. So the worldsheet instanton wrapping γ^α could break this gauging, by a calculation similar to in section 4.2. Namely we can integrate $\mathcal{B}$ over γ^α and find that $k(\int_{\gamma^\alpha} \mathcal{B}) \neq 0$, where k is the killing vector.

In [55], it was shown that the worldsheet instanton correction to the hyper moduli space is given by

$$U_{\gamma^\alpha} = \exp(-\frac{A(\gamma^\alpha)}{2\pi\alpha'} + i\int_{\gamma^\alpha} \mathcal{B})\,\frac{\mathrm{Pfaff}(\bar{\partial}_{V(-1)})}{(\det{}'\bar{\partial}_{\mathcal{O}})^4} \tag{4.18}$$

The exponential factor comes from the classical instanton action while the rest is the one-loop determinant from fluctuations around the classical solution. More precisely, the $\mathrm{Pfaff}(\bar{\partial}_{V(-1)})$ in the numerator comes from one loop determinant of non-zero modes of the left-moving fermions. Three powers of $(\det{}'\bar{\partial}_{\mathcal{O}})$ come from the complex bosons representing the non-compact $\mathbf{R}^4$ directions and the T^2 factor. The remaining one follows by partly canceling the contribution of the normal bundle $(\det\nabla_{\mathcal{O}(-2)})$ in $K3$ against the right moving fermions.

It is in general a very hard problem to compute this quantity. Fortunately the theorem states that U_{γ^α} vanishes if and ony if the gauge bundle V restricted to γ^α is non-trivial[6]. The non-trivial gauge bundle is always the case if we want to gauge the isometry in heterotic string. It is also very likely that the bundle restriction is already non-trivial before turning on the gauge fluxes. In this case, some ungauged isometries are also protected [7]. But we should keep in mind the possibility that the theory can move in the bundle moduli space such that the bundle becomes trivial along some 2 cycle in K3 and then the worldsheet instantons re-appear.

The other potential worry is that the $U(1)$s coming from T^2 and graviphoton do not belong to the $E_8 \times E_8$ bundle in the heterotic string. Turning on their gauge fluxes do not change the bundle restriction $V|_{\gamma^\alpha}$. Therefore, in order to protect the gauged isometries, we need the bundle structure to be non-trivial along the 2 cycles along which we turn on the gauge fluxes. The study of the protection mechanism becomes model dependent; we have to know the bundle structure first before commenting on whether certain gauged isometries are lifted.

Nonetheless, in heterotic string the protection of the flux-induced gauging is still stronger than the IIA case. In IIA case, we have no gauging protection mechanism if we don't turn on H and the ungauged isometries are generically lifted by the quantum effects.

4.4 Discussion and conclusion

In this paper we study the flux-induced isometry gauging in $N = 2$ heterotic string compactified on $K3 \times T^2$ with gauge fluxes. A vanishing theorem by Witten [78] guarantees that the gauging is protected against the worldsheet instanton effect. In heterotic string, the isometry protection can even reach the ungauged ones, which is contrary to IIA. In IIA we can not protect the gauging without H and usually lose the ungauged isometry due to D-instanton effects. However, it is still not clear how to charge the bundle moduli under the vector moduli, which is also contrary to IIA, where any hyper moduli can be charged under the vectors.

In the IIB case, the situation remains obscure since various branes with different dimensions come into play. Especially the $D1$ instanton wraps a 2 cycle and the Freed-Witten

[6] This is equivalent to that the operator $(\bar{\partial}_{V(-1)})$ has a nonempty kernel. For our purpose, V can be taken as the abelian vector bundle where the gauge flux sits and $V(-1) = \mathcal{O}(-1) \otimes V$.

[7] For example, we can embed the K3 spin connection into the gauge group. The bundle will be non-trivial along every 2 cycle in K3.

anomaly argument does not eliminate its existence. In view of the relation between IIB and type I theory, it seems possible that a combination of H flux and the argument discussed here can achieve the protection of gauging in IIB. One could also try to study the closely related problem in $N = 1$ orientifold setting. We leave this for future study.

Chapter 5

Non-geometric fluxes from doubled geometry

5.1 Introduction

Recent arguments from T-duality have suggested the existence of new NS-NS fluxes in string theory [25]. Whereas T-duality acts on R-R fields (and thus fluxes) by shuffling them amongst themselves, the only p-form NS-NS field is the B field. T-duality must then exchange this 3-form flux with some other NS-NS flux. Dualizing along the direction of one $U(1)$ isometry, H flux with a leg along this direction becomes a sort of topological flux coming from the metric, as was formalized in [82] with much previous evidence [51]. Dualizing along two or three directions with H flux has remained mysterious and has been related to, respectively, noncommutative and nonassociative spaces [83]. In other words, from the non-dimensionally reduced point of view, T-dualizing can be very hard. Other dualities are likely to be even harder. Already from S-duality arguments even more new fluxes, this time in the R-R sector, been discovered [84].

In [25], T-duality was studied at the level of the reduced supergravity. There, arguments about the $O(d, d; \mathbb{Z})$ invariance of the superpotential required additional terms whose coefficients were integers related to the H and metric fluxes through duality. On a torus T^d with indices $i, j, k \in \{1, 2, ..., d\}$ and T_i representing the T-duality operation along the i-direction, the formal rule for exchanging the generalized NS-NS fluxes is

$$H_{ijk} \xleftrightarrow{T_i} f^i_{jk} \xleftrightarrow{T_j} Q^{ij}_k \xleftrightarrow{T_k} R^{ijk}, \tag{5.1}$$

where f is the metric flux and Q and R are the new objects.

These fluxes appear not only in the superpotential, but also give the low energy gauge group. Reducing on a torus with no flux gives a $U(1)^{2d}$ gauge group, coming from the metric and B field. Denoting the d generators arising from the higher dimensional diffeomorphism invariance by Z_i and the d generators arising from B field gauge invariance by X^i, the gauge algebra is

$$\begin{aligned} [Z_i, Z_j] &= H_{ijk}X^k + f_{ij}^k Z_k \\ [Z_i, X^j] &= -f_{ik}^j X^k + Q_i^{jk} Z_k \\ [X^i, X^j] &= Q_k^{ij} X^k + R^{ijk} Z_k \end{aligned} \tag{5.2}$$

These structure constants must satisfy the Jacobi identity. When Q and R are set to zero, the Jacobi identity becomes the condition that $d^2 = 0$ and $dH = 0$, where the differential operator d is modified by the metric flux.

This f flux dates back to the original Scherk-Schwarz compactifications [85]. From the higher dimensional point of view, f flux gives the structure constants for some globally defined 1-forms, i.e. 1-forms η^i satisfying

$$d\eta^i = -\frac{1}{2} f_{jk}^i \eta^j \wedge \eta^k \tag{5.3}$$

Their existence implies that the compactification manifold is parallelizable, locally the group manifold for the Lie group defined by these structure constants. The integrability of this equation is the Jacobi identity.

We suggest a similar interpretation be given not only to the new non-geometric fluxes, but also to H flux. This requires doubling the dimension of the compactification manifold. In the case of a torus with no flux, this involves simply means taking the product with the dual torus. Flux will be interpreted as nontrivial topology of this manifold exactly as in the Scherk-Schwarz case and complicated flux configurations will admit no good way to separate out a “physical” space time from the dual one.

The motivation comes from the old observation that the momentum and winding modes should be treated more equally, and perhaps even independently. See [86] and references therein. From a string field theory point of view, [87], this seems very natural. States in the string field theory on a torus are labeled by momentum and winding numbers which are the Fourier transforms of coordinates on the physical torus and on the dual torus.

There is further symmetry in the way that flux behaves. In some circumstances, Scherk-Schwarz compactifications on local group manifolds are the same as reductions with "duality twists" in the group of large diffeomorphisms of the torus [88]. A reduction with duality twist means that in going around some cycle, there is a monodromy in a symmetry of the theory. The ubiquitous example of compactifying on a torus bundle over a circle, where the torus undergoes a monodromy $\begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix}$ is a twisted torus compactification that with a realization of a reduction with a geometric duality twist.

When f flux is associated with a duality twist, it is always in the geometric part of the T-duality group. A flux f^i_{jk} indicates either a mixing of the k momentum number into the i momentum number by some monodromy when traversing the j circle or the same with the roles of j and k interchanged. As we will see later, there is sometimes freedom to choose around which circle there is the monodromy and sometimes not. Simultaneously, there is also mixing of the i winding number into the k (or j) winding number, which is sometimes overlooked.

Similarly, H_{ijk} indicates mixing of the j momentum number into the i winding number upon traversing the k direction, or some permutation thereof. In [**?**], it was noted that what we would now call Q flux results in mixing of winding numbers into momentum numbers. Such a monodromy is not geometric, and is related to reductions with duality twists in elements of the T-duality group. This has been studied in several contexts , [90], [91], [92], [93].

Following older approaches to constructing a manifestly T-duality invariant theory, Hull introduced a doubled formalism in [94] for torus bundles. The doubled approach incorporates the above intuition that there should be parity between the treatment of the momentum and winding numbers, or their Fourier transforms. One considers torus bundles over some space-time that locally have fibers of the form of a physical torus times its dual. The transition functions of the torus are allowed to be to be in the entire $O(d, d; \mathbb{Z})$ T-duality group, which acts geometrically on the fiber.

The worldsheet action appears to be a sigma model action without B field. However, one is forced to impose a self-duality constraint on-shell, much as with the five form in IIB supergravity. This suggests that the formalism is not exactly correct for studying the quantum theory. Nonetheless, solving the self-duality constraint shows what happened to the B field. The metric on the doubled torus, subject to a consistency condition, contains exactly the information of usually written as G and B, but in a form that transforms linearly under the T-duality group.

Unlike the B field, which is allowed to be only locally defined, the metric on the doubled fiber is required to be a tensor on the total space of the bundle. Hence, H flux is encoded in the topology of the space. This is not surprising, as Hull's doubled bundles are clearly related to the concept of a correspondence space. See the last chapter of [14] for interesting conjectures on the role of correspondence spaces in duality transformations and an understanding of why H flux should appear in their topology.

Hull restricts his formalism to situations in which nothing is allowed to depend on the fiber directions, thus ruling out reductions with duality twist along fiber directions, i.e. interesting topology of the fiber. We expand Hull's formalism to allow for such situations, making the fiber a local group manifold as in the usual Scherk-Schwarz compactifications. Dependence on coordinates which are doubled is allowed only through globally defined one forms on the doubled manifold, as is the usual ansatz.

When considering fluxes that are completely on the internal space, we discover that they are simply encoding the topology of the doubled space, the natural generalization of Scherk-Schwarz. This provides parity for the NS-NS fields, with the metric and the B field entering on equal footing and likewise with their various fluxes. Moreover, we find that dimensionally reducing the Einstein-Hilbert action on the doubled space reproduces the lower dimensional theory normally found by reducing the entire NS-NS sector with Einstein-Hilbert and Kalb-Ramond field strength terms when the consistency condition is imposed on the doubled metric.

A possible remedy for the problems of Hull's approach is in the older ideas of [86]. Tseytlin uses an action which lacks manifest Lorentz invariance, in fact only having local Lorentz invariance on-shell, but has as the equations of motion the self-duality constraint. One finds by computing graviton scattering amplitudes in this theory the Einstein-Hilbert action, with the constrain on the metric, emerging as the low energy limit of the string theory.

It is surprising that we are able to reproduce the NS-NS part of the dimensionally reduced low energy action. It oughtn't. There is never a regime in which keeping the momentum and winding modes while throwing out the excited states is a good approximation. If the physical circle is large, many KK modes on that circle are much lighter than any winding mode on the dual circle and vice versa. They only become comparable when both circles are at the string scale. That this works remains a mystery.

Nonetheless, it seems clear that this structure forms some topology feature of string field theory. We conjecture that in a formulation of string field theory, the doubled manifold will play the fundamental role. In generic situations, any choice of splitting into physical and dual spaces is bad in the sense that the "physical" picture will not even be locally geometric due to nontrivial dependence on winding numbers.

On a final note, connecting this idea back with the original observation that T-duality may take regular spaces to noncommutative (or nonassociative) ones, we note that Tseytlin observed certain stringy uncertainty principles [95] between a physical coordinate and its dual suggest that noncommutativity plays a role in formulation in which both are treated geometrically. Then one might suspect that fluxes which mix winding modes into momentum modes introduce the noncommutativity into the closed string sector as well. One should note that the noncommutativity of [83] was seen in the K-theory, not for closed strings.

The organization of the paper is as follows. In section 5.2 we briefly review Hull's doubled formalism. Section 5.3 explores an example with the formalism and attempts to use it understand something of the nature of Q flux. We then illustrate the obstruction to using the formalism as is to understand nongeometric fluxes. This motivates our conjecture that the formalism should be expanded to include doubled fibers that are local group manifolds. Section 5.4 explores this idea in the case of a trivial bundle, but with general fiber. We show how this accomodates general NS-NS flux on the fiber and note how the Einstein-Hilbert action on the doubled manifold reproduces the correct low energy theory upon dimensional reduction. This section can be read independently of section 5.3.

5.2 The Doubled formalism

In this section, we will briefly review the setup of [94]. The idea is to allow for general $O(d,d;\mathbb{Z})$ transition functions to have a geometric interpretation. To this end, suppose that the target space M locally looks like a manifold with n freely acting $U(1)$ isometries, i.e. that M has the structure of a T^d bundle. If M had the global structure of a torus bundle, then changing between local patches on the base of the fibration would come with some transition function g, where g takes values in $Gl(d,\mathbb{Z}) \ltimes U(1)^d$. Here the factor of $Gl(d,\mathbb{Z})$ is the group of large diffeomorphisms of the torus.

A fiber bundle is classified by its transition functions, which give an element of $H^1(B,\underline{\mathbf{G}})$

where B is the base of bundle. Here the underline indicates the sheaf of continuous functions into $\mathbf{G}$. When $\mathbf{G}$ has the form $\mathcal{G} \ltimes U(1)^d$ for some discrete group $\mathcal{G}$, we have a surjective map $H^1(B, \underline{\mathbf{G}}) \to H^1(B, \underline{\mathcal{G}}) \cong H^1(B, \mathcal{G})$ because $\mathcal{G}$ is discrete. The element of $H^1(B, \mathcal{G})$ gives the monodromies of the torus fiber around the cycles in the base. While nonabelian H^1 is simply a set and not a group, we still have a trivial element. If and only if the image of the transition functions in $H^1(B, \mathcal{G})$ is trivial is the bundle a principle torus bundle. In that special case, one has the technology of [101].

To allow T-duality transition functions, we should pick $\mathcal{G} = O(d, d; \mathbb{Z})$. The natural action of $O(d, d; \mathbb{Z})$ is on $U(1)^{2d}$, leading us to promote the transition functions to $O(d, d; \mathbb{Z}) \ltimes U(1)^{2d}$. To give this a geometric interpretation, it is natural to choose as the fiber T^{2d} on which this group acts in the obvious way. The interpretation given by Hull is that the fiber is now both the physical torus fiber and its T-dual. This is essentially the generalization of the correspondence space used in [82] and [101].

We would like to have a sort of sigma model with this doubled torus bundle as the target space, but to have this sigma model reduce to the normal sigma model in the case that the transition functions can be chosen not to contain any T-duality transition functions. To this end, we introduce worldsheet fields X^I and Y^μ where $I \in \{1, ..., 2d\}$ is an index on the torus fiber and μ is an index on the base B. The bosonic lagrangian is taken to be

$$\mathcal{L} = \frac{1}{2}\mathcal{H}_{IJ}\partial X^I \wedge *\partial X^J + \partial X^I \wedge *\mathcal{J}_I + \mathcal{L}(Y) \tag{5.4}$$

where $\mathcal{L}(Y)$ is the standard lagrangian for the base. For the moment, we further restrict to the case where the (positive definite) fiber metric $\mathcal{H}$ and source terms $\mathcal{J}$ depend only on the base coordinates. We have taken the world sheet metric to be flat and the Hodge dual is with respect to this metric.

It is further necessary to impose self-duality constraints

$$\partial X^I = L^{IJ} * (\mathcal{H}_{JK}\partial X^K + \mathcal{J}_J) \tag{5.5}$$

to reduce the number of degrees of freedom. The matrix L is chosen to be constant, invertible, and symmetric. For consistency, $L\mathcal{H}$ must square to $\mathbf{1}$. L_{IJ} will denote its inverse. The self-duality constraint implies the equations of motion from the identity $d^2 = 0$. In [94], it is shown that solving the constraint reproduces the familiar sigma model lagrangian.

The lagrangian is invariant under the discrete $Gl(2d,\mathbb{Z})$ preserving the periodic identification of the X coordinates and the self-duality constraint is invariant under an $O(d,d)$ subgroup of $GL(2d)$. Then the entire theory is invariant under an $O(d,d;\mathbb{Z})$ symmetry which we will identity with T-duality. The convention of Hull is that under the T-duality group, $X \to g^{-1}X$ so that $\mathcal{H} \to g^t\mathcal{H}g$ and $\mathcal{J} \to g^t\mathcal{J}$.

One last piece for Hull's formalism is an almost local product structure [57] $\mathcal{R}^I_J$ satisfying $\mathcal{R}^2 = \mathbf{1}$. At each point on the fiber, $\mathcal{R}$ defines two projectors $\Pi = \frac{1}{2}(\mathbf{1}+\mathcal{R})$ and $\tilde{\Pi} = \frac{1}{2}(\mathbf{1}-\mathcal{R})$. For this splitting to extend to a splitting of the coordinates, it must agree with the periodicity of the coordinates and so we take $\mathcal{R} \in Gl(2d,\mathbb{Z})$. Note that this restriction makes $\mathcal{R}$ a local product structure as the analog of the Nijenhuis tensor automatically vanishes. It also means that $\mathcal{R}$ is constant over every patch. If $\mathcal{L}$ is further "pseudo-hermitian" with respect to $\mathcal{R}$, i.e. satisfies

$$\mathcal{L}_{IJ}\mathcal{R}^J_K + \mathcal{L}_{KJ}\mathcal{R}^K_I = 0 \tag{5.6}$$

then the rank of Π and $\tilde{\Pi}$ are each d and define locally a splitting of the fiber into a physical T^d and a dual $\tilde{T}^d$. $\mathcal{R}$ transforms under coordinate transformations on T^{2d} as $\mathcal{R} \to g^{-1}\mathcal{R}g$. Note that $\mathcal{L}$ and $\mathcal{R}$ are preserved by the diagonally embedded $Gl(d)$ subgroup of $Gl(2d)$. Its intersection with $Gl(2d,\mathbb{Z})$ is exactly the subgroup of the T-duality group corresponding to coordinate redefinitions of the physical torus. When convenient, we also split the capital indices $I,J,K \in \{1,...,2d\}$ into lower case indices $i,j,k \in \{1,...,d\}$ with the convention that when I is raised, a raised i is on the torus and a lowered i is on the dual torus, further indicated by a tilde. So that $X^I = (X^i, \tilde{X}_i)$ while $X_I = (X_i, \tilde{X}^i)$.

For convenience, we now work in coordinates where

$$L = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \quad \mathcal{R} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \tag{5.7}$$

In this convention, the physical torus coordinates are the first d coordinates and the dual torus ones the latter. Further, solving the self-duality constraint in these coordinates gives

$$\mathcal{H} = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix} \tag{5.8}$$

with G and B being the metric and B-field on the physical torus, familiar from [97] [98] [99] [102] [86]. Note that $\mathcal{H}$ transforms linearly under $O(d,d;\mathbb{Z})$, while $E = G + B$ transforms with fractional linear transformations.

In this language, a T-duality transformation corresponds to simultaneously acting in every patch on $\mathcal{H}$ and $\mathcal{J}$ with an element of $O(d,d;\mathbb{Z})$ as above while leaving $\mathcal{R}$ fixed or vice versa. Transforming $\mathcal{H}$, $\mathcal{J}$, and $\mathcal{R}$ simultaneously changes nothing. Of course, the theory in the doubled formalism is manifestly T-duality invariant, but the presentation of the theory after solving the constraints will differ.

5.3 A first attempt at geometrizing non-geometric fluxes

Since Q flux seems to be related to T-duality monodromies, it seems reasonable to suspect that in the doubled formalism we can give Q flux a geometric interpretation. What we will see shortly is that this is possible only in special cases where the Q flux is T-dual to normal geometric space-times. We begin by considering a trivial torus bundle with H flux and T-dualizing. This calculation is very similar to one presented in [100].

5.3.1 H flux in the doubled language

Consider a space-time that admits the structure of a T^d bundle, with base B. We will recast the sigma model in the doubled language. The formalism requires that $\mathcal{H}$ and $\mathcal{J}$ depend only on base coordinates[1]. Consequently, we are restricted to considering H flux with only two legs on the torus fiber. If there is anything to be gained from using this approach, it will occur with d=2. After making this restriction, there is one sort of non-geometric flux to be understood, Q_μ^{12}, the flux that is T-dual to H flux with two legs in the fiber directions. For further simplicity, we begin with a principle torus bundle and choice as our representative of the (trivial) class in $H^1(B, Sl(2,\mathbb{Z}))$ the co-cycle that consists only of identity elements.

Cover B with patches U_α where the U_α are a good cover. This means that space-time is covered with patches of the form $U_\alpha \times T^2$ and on each H is exact. We trivialize $H = dB_\alpha$ on each patch.We now double the fiber and put on T^4 the metric

$$\mathcal{H}_\alpha = \begin{pmatrix} \mathbf{1}_{ij} - B_{ik}B^k_j & B^j_i \\ -B^i_j & \mathbf{1}^{ij} \end{pmatrix} \tag{5.9}$$

[1] This is not surprising. Hull's formalism always gives a local geometric description, which is incompatible with the presence of R flux [25]. A B field that depending on fiber coordinates would give H flux entirely on the fiber and thus T-dual to R flux.

$\mathcal{H}$ is required to patch together to form a tensor on the total space of the bundle. Before doubling, B_α differed on different patches by a gauge transformation, which takes the form $B_\alpha = B_\beta - N_{\beta\alpha} dx^1 \wedge dx^2$, $N \in \mathbb{Z}$. On E this takes the form

$$E \to E + N_{\beta\alpha}\mathbf{J} \tag{5.10}$$

with $\mathbf{J}$ the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and so gives a transition function

$$g_{\beta\alpha} = \begin{pmatrix} \mathbf{1} & 0 \\ N_{\beta\alpha}\mathbf{J} & \mathbf{1} \end{pmatrix} \tag{5.11}$$

The co-cycle condition $g_{\alpha\gamma} g_{\gamma\beta} g_{\beta\alpha} = 1$ gives $N_{\alpha\gamma} + N_{\gamma\beta} + N_{\beta\alpha} = 0$, i.e. the $\{N_{\beta\alpha}\}$ define an integral Čech co-cycle in the base. This is $H_{\mu 12}$, which we might also call $\pi_* H$ in this context. Here the map $\pi_* : H^n(M) \to H^{n-2}(B)$ given by integration along the fiber. In the language of [101] it is H_1, as $\Lambda^2(\mathfrak{t}^2) \cong \mathbb{R}$.

We would like for this 1-form to be well-defined with respect to the choice of co-cycle. To begin, consider changing the co-cycle by the action of a geometric 1-chain. That is, we consider a collection $\{h_\alpha\}$ with $h_\alpha \in Sl(n, \mathbb{Z}) \ltimes \mathbb{Z}$, the group of geometric transformations on the torus fiber along with gauge transformations of the B field. These take the form

$$h_\alpha = \begin{pmatrix} A_\alpha & 0 \\ N_\alpha A_\alpha^{-t}\mathbf{J} & A_\alpha^{-t} \end{pmatrix} \tag{5.12}$$

where N_α is the gauge shift in the B field and A_α is an $Sl(2, \mathbb{Z})$ geometric transformation on the torus. Then the new transition functions are

$$g'_{\beta\alpha} = h_\beta g_{\beta\alpha} h_{\beta\alpha}^{-1} = \begin{pmatrix} A_\beta A_\alpha^{-1} & 0 \\ (N_{\beta\alpha} + N_\beta - N_\alpha) A_\beta^{-t}\mathbf{J} A_\alpha^{-1} & A_\beta^{-t} A_\alpha^t \end{pmatrix} \tag{5.13}$$

Using the fact that $A\mathbf{J}A^t = \mathbf{J}$ for every $A \in Sl(2, \mathbb{Z})$, for transition functions of this form, multiplying the bottom left block on the right by the inverse of the top left block allows one to recover the original co-cycle.

This also shows how can recover $\pi_* H$ from the transition functions from a general bundle with only geometric monodromies. Here, the $g_{\beta\alpha}$ are restricted to the form

$$g_{\beta\alpha} = \begin{pmatrix} A_{\beta\alpha} & 0 \\ N_{\beta\alpha} A^{-t}_{\beta\alpha} \mathbf{J} & A^{-t}_{\beta\alpha} \end{pmatrix} \tag{5.14}$$

from which we can recover the 1 cocycle corresponding to H with two legs on the fiber as above.

It is important to note that in order for the procedure to work, we have restricted ourselves to geometric transition functions, in effect considering $H^1(B, Sl(2,\mathbb{Z}) \ltimes \mathbb{Z})$ and so it is not surprising that we were able to identity a copy of $H^1(B,\mathbb{Z})$ sitting inside. What we would really like to do is to consider T-duality transition functions.

5.3.2 T-Dualizing H Flux

First, consider the simple case of transition functions given by (5.11) and perform T-dualities. The matrices giving T-duality along direction 1 and along both directions are

$$g(T^1) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad g(T^{12}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \tag{5.15}$$

Implementing T-duality along the first direction changes these transition functions into the familiar twisted torus type geometric type transition functions, with $A = \begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix}$. The more interesting example is conjugating the transition functions by $g(T^{12})$. When the $g_{\beta\alpha}$ are in the form (5.11), this T-duality rearranges the block 2x2 matrices to give

$$g_{\beta\alpha} = \begin{pmatrix} \mathbf{1} & N\mathbf{J} \\ 0 & \mathbf{1} \end{pmatrix} \tag{5.16}$$

As above, restricting to the $Sl(2,\mathbb{Z}) \ltimes \mathbb{Z}$ subgroup that preserves this form of the transition functions allows us to identity an element of $H^1(B,\mathbb{Z})$. This is one of the non-geometric fluxes, Q^{12}_μ.

5.3.3 Flux, Topology, and Polarization

As seen in §5.3.1, the presence of H flux in a geometric compactification manifests itself as a topological condition on the doubled bundle. The transition functions were forced to be chosen in such a way that $\mathcal{H}$ was a tensor on the total space of the bundle. With that topology, after solving the self-duality constraints, any choice of $\mathcal{H}$ would yielded a physical B field with the correct curvature.

In §5.3.2, we confirmed that turning on Q_μ^{12} determined the same element of $H^1(B, O(d,d;\mathbb{Z}))$ as turning on $H_{\mu 12}$ and so determine the same topology for the doubled bundle. This suggests that while specifying the fluxes specifies the bundle, the converse is not true. Different choices of polarization determine how the flux is interpreted after solving the self-duality constraints.

Generically, the bundle cannot be specified by choices of 1-forms in the base. Our ability to identify $pi_* H$ from the topological data depended on identifying $H^1(B, \mathbb{Z})$ in the transition functions of restricted form. Clearly this will fail for $H^1(B, O(d,d;\mathbb{Z}))$. A detailed understanding of the interpretation of elements of $H^1(B, O(d,d;\mathbb{Z}))$ need not concern us however, as most choices of bundle are inconsistent with string theory, as we will see now.

5.3.4 Troubles with Turning on Multiple Fluxes

Using the doubled formalism, it is possible to give a geometric interpretation to turning on one of the non-geometric fluxes, the Q_μ^{12}. This is T-dual to a standard geometric description, and so not particularly interesting. What would be more interesting is to turn on some combination of fluxes that has no geometric description in the standard formalism, but does in the doubled one. Unfortunately, there is an inherent difficulty in this. We illustrate the general obstruction by consider the simple example of T^3.

Consider turning on the geometric flux $f^1_{23} = N$. One way to think of the resulting space is as a circle bundle, with the 1 direction the fiber and the 2,3 directions as the base. The flux can then be interpreted as the first Chern class of the bundle. Another way to think about it is as a T^2 bundle over S^1 where the torus fiber undergoes a geometric monodromy after goes around the base. In the latter viewpoint, there is a choice. Either the 2 or the 3 direction may be chosen as the base. Upon choosing, say, the 3 direction as the base, the 1,2 fiber undergoes a monodromy $g = \begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix}$ as $x^3 \to x^3 + 1$. We can choose G and B to depend only on x^3 and can use the doubled formalism.

Suppose we also want to turn on $f^2_{13} = M$. These fluxes satisfy the Jacobi identity and traceless conditions, and so there's no obstruction to doing this and the result is purely geometric. Use of the doubled formalism requires that we be able to choose G and B independent of the fiber that we wish to double. Therefore, to double a T^2 will require that there is dependence on only one coordinate. This has a chance to be true only if we can realize both fluxes as monodromies around the 3 direction.

A naive attempt at this is to multiply the monodromy matrices. One issue is that this is not well defined on $H^1(S^1, Sl(2,\mathbb{Z}))$, which as previously mentioned is only a set and has no group structure. There does not seem to be an invariant meaning to this. A second is that the monodromy matrices do not commute. Nonetheless, multiplying them appears to give a valid monodromy matrix

$$g_N g_M = \begin{pmatrix} 1+MN & N \\ M & 1 \end{pmatrix} \tag{5.17}$$

which defines a geometric monodromy. One can use the techniques in [?] to determine the globally defined one forms on this space. Indeed, they do depend only on the x^3 coordinate. However, remembering that

$$d\eta^i = -\frac{1}{2} f^i_{jk} \eta^j \wedge \eta^k \tag{5.18}$$

where η^i are the global one forms, this requires turning on the further flux f^1_{13}, which is disallowed [85]. We expect this to be true of most choices of co-cycle $H^1(B, O(d,d;\mathbb{Z}))$.

Of course it is possible to realize this choice of fluxes geometrically. It is a nilmanifold with the identifications $(x^1, x^2, x^3) \sim (x^1+1, x^2 - Mx^3, x^3) \sim (x^1 - Nx^3, x^2+1, x^3) \sim (x^1, x^2, x^3+1)$. With both fluxes, we were forced to choose the monodromies around the circles that were not x^3.

With a T^2 fiber, we must turn on Q^{12}_μ, $f^1_{\mu 2}$, $f^2_{\mu 1}$, and $H_{\mu 12}$ to create scenario that involves non-geometric flux regardless of choice of polarization. In other words, it cannot be done. It's pretty immediate to see that this problem will plague any attempt to use the doubled torus formalism to study these new fluxes. However, the example just considered suggests what must be done. We should consider the meaning of doubling more general Scherk-Schwarz scenarios, where we allow monodromies around the fiber directions as well.

5.4 The doubled Scherk-Schwarz compactification

The Scherk-Schwarz compactifications were introduced as a way to do consistently dimensionally reductions of gravity to produce massive theories [85]. They are consistent in the sense the solutions to the equations of motion in the reduced theory lift to solutions to the equations of motion of the higher dimensional theory. Performing these types of reductions of type II supergravities in ten dimensions yield in four dimensions supergravities that gauge part of the $O(d,d)$ symmetry that is present in reductions on a d-dimensional torus. For an excellent discussion of the role of Scherk-Schwarz compactifications in string theory, see [96].

In [96] it is noted that there are two types of Scherk-Schwarz reductions: those on twisted tori and those with duality twists. The first refer to reductions on certain quotients of group manifolds. In certain simple circumstances, these correspond to torus bundles, but in general the name twisted torus is somewhat misleading. The second refer to first compactifying d-1 dimensions and then compactifying on a further S^1 with a monodromy in some symmetry group of the theory. When the monodromy is symmetry with a geometric realization, we have a compactification of the first type. However, more general non-geometry symmetries, such as S-, T-, or U-duality transformations are allowed. In F-theory, certain of these have geometric interpretations [103], [88]; typically there are none. Recall however that the goal of the doubled formalism is to give a geometric interpretation of T-duality. Some simple cases of this have been illustrated [94][100]. We will expand on this further and to give new exotic types of compactifications that all have a geometric interpretation in the doubled point of view.

5.4.1 Twisting the doubled torus

As illustrated in §5.3.4, turning on interesting fluxes necessitates a Scherk-Schwarz type compactification where we have monodromies in multiple directions, many of which we wanted to be fiber directions. If traversing a circle carries with it a monodromy, T-duality done along that direction would introduce a monodromy around the dual circle. This has no geometric interpretation and would usually be considered a situation in which the T-duality is not allowed. However, [92] note that from the point of view of the string field theory, there seems to be nothing wrong with allowing non-trivial dependence on the dual circle, i.e. on the winding number. Moreover, the lack of geometric interpretation is promising for understanding the mysterious R flux of [25], in whose presence even D0 branes fail to be defined.

To implement this, consider a doubled bundle that is trivial, but allow the fiber to be a local group manifold of the type used in Scherk-Schwarz. This is a reduction of string theory to 10-d dimensions and so we are considering manifolds of the form $\mathbb{R}^{10-d} \times F$ where F is a local group manifold, i.e. of the form G/Γ for some group G and some discrete cocompact subgroup group Γ[2]. F is parallelizable with T^*F trivialized by a basis of left invariant 1-forms η^I which satisfy a Cartan-Maurer equation

$$d\eta^I + \frac{1}{2} f^I_{JK} \eta^J \wedge \eta^K = 0 \tag{5.19}$$

where the f^I_{JK} are structure constants of the group. The integrability for this equation is the Jacobi identity.

In standard twisted tori reductions, where the local group manifold is the familiar compactification manifold, the left invariant 1-forms are dual to the would-be killing vectors. These vector fields generate the isometries of the group manifold G and so are the Kaluza-Klein vectors. Dimensional reduction gives G as the gauge group. This strongly suggests how, given a set of fluxes, we should chose the form of F.

Expanding on the algebra of [89], [25] gives the gauge algebra that should arise from reducing ten dimensional type II supergravity in the presence of the new fluxes filling out the T-duality multiplet. The structure constants are[3]

$$\begin{aligned} [Z_i, Z_j] &= H_{ijk} X^k + f^k_{ij} Z_k \\ [Z_i, X^j] &= -f^j_{ik} X^k + Q^{jk}_i Z_k \\ [X^i, X^j] &= Q^{ij}_k X^k + R^{ijk} Z_k \end{aligned} \tag{5.20}$$

By considering a reduction with non-geometric $O(d, d; \mathbb{Z})$ duality twist, [92] observed an addition term in the dimensionally reduced gauge group that we would Q flux. There they did not compute the commutator between X_y and X_i, where y is the direction is the direction with the duality twist, leaving out a term in the algebra. Its existence is implicit

[2]Being cocompact means that G/Γ is compact, which is necessary for a mass gap. Not all groups admit such subgroups and they are disallowed in the Scherk-Schwarz reductions. At least for nilpotent groups, such a subgroup is admitted if and only if the structure constants can be chosen to be rational[104]. String theory knows about this restriction as the structure constants are related to fluxes that are integrally quantized.

[3]There is some issue of normalization between the numbers appearing in the gauge algebra and what one would like to call flux. In this section, we will use H, f, Q, and R to denote the numbers appearing the structure constants, as per [25].

from duality arguments.

This is not quite symmetry algebra for the theory, but this is not important for our application. See [96] for a discussion of how the gauge algebra listed above is related to the (field dependent) symmetry algebra in the geometric case. Setting Q and R to zero gives the familiar algebra from Scherk-Schwarz compactifications.

Using the conventions on indices from §5.2 and setting $X_I = (Z_i, X^i)$, we can rewrite this algebra compactly as

$$[X_I, X_J] = f^K_{IJ} X_K \tag{5.21}$$

which gives for the dual 1-form basis

$$d\eta^I = -\frac{1}{2} f^I_{JK} \eta^J \wedge \eta^K \tag{5.22}$$

In the physical-dual coordinates given by a polarization, this is

$$\begin{aligned} d\eta^i &= -\frac{1}{2}\left(f^i_{jk}\eta^j \wedge \eta^k - Q^{ij}_k \eta^k \wedge \eta_j + R^{ijk}\eta_j \wedge \eta^k\right) \\ d\eta_i &= -\frac{1}{2}\left(f^j_{ik}\eta^k \wedge \eta_j + Q^{jk}_i \eta_j \wedge \eta_k + H_{ijk}\eta^j \wedge \eta^k\right) \end{aligned} \tag{5.23}$$

Luckily, this form of the algebra is exactly what is required to allow one to be sloppy about the placement of indices. The appearance of f^i_{jk} and Q^{ij}_k in both of the places where the configuration of indices would allow it means that one doesn't have to worry whether, for example, f^i_{jk} comes from f^I_{JK} with all the indices "physical" or if it is from f^J_{IK} with both J and K "dual" indices.

From the dimensionally reduced point of view, this comes as no surprise. We are supposed to be gauging some subgroup of $O(d,d)$ and these conditions on f and Q, along with the total antisymmetry of R and H are required for the gauge group, in the adjoint representation, to be a subgroup of $O(d,d)$[4] From the higher dimensional perspective, it is a condition that we must impose on the topology and is a sort of self-duality constraint.

As a concluding remark, note how this construction gives a nice geometric interpretation to the NS-NS Bianchi identities found in [25]. The Scherk-Schwarz compactifications had long given this status to the f flux and we have now extended it democratically not only to H, but to all the fluxes that fill out the T-duality multiplet.

[4]This requirement that we look at the adjoint representation is nontrivial. $U(1)^2 \not\subset O(1,1) \cong Sl(2)$, but in the adjoint representation, $U(1)^2$ is just the trivial group.

5.4.2 Dimensional Reduction

As all of our arguments have been guided by the dimensionally supergravity, we should see how this doubled construction connects with the reduced gravity theory. We continue to ignore the RR-sector, which remains mysterious in our picture, and supersymmetry. To this end, note that the doubled lagrangian (5.4) looks very much like the bosonic string lagrangian. As noted, the doubled action is misleading. That we must impose the self-duality constraint indicates that this is not the correct action. In fact, the constraint is some how more fundamental than this action and presumably comes from the correct formulation of the theory. Nonetheless, we will take this action seriously and write down a "low energy" effective theory.

In the NS-NS sector, (5.4) looks a standard lagrangian on the base and only a metric on the doubled torus. In the case of a standard geometric compactification, the H flux on the base is not the same as the H flux on the total space restricted to the base. For understanding how to pick out the gauge invariant field strength, see [89] or [96]. For a mathematical discussion of the same idea in the case of a principal torus bundle, see [101].

We propose to adapt the familiar ansatz to the doubled twisted torus with, in Einstein frame, a metric of the form[5]

$$ds^2 = e^{2\alpha\varphi} g_{\mu\nu} dx^\mu dx^\nu + e^{2\beta\varphi} \mathcal{H}_{IJ} \nu^I \nu^J \tag{5.24}$$

where as usual the 1-forms ν^I take the form

$$\nu^I = \eta^I - A^I = \eta^I - A^I_\mu dx^\mu \tag{5.25}$$

Here $g_{\mu\nu}$ is a metric on $\mathbb{R}^{10-d}$ or whatever other manifold to which we are compactifying, say B, and $\mathcal{H}$ is the metric on F. $\mathcal{H}$ has determinant one As is standard, we demand that g, $\mathcal{H}$, and A depend only on B, with the only dependence on F arising in the η^Is. This is the most general left-invariant metric.

Consider reducing to 10-d dimensions a term in the action

$$S = \int R_{10+d} * 1 \tag{5.26}$$

[5] For details of these results, including the definitions of α and β, see [96] and earlier references [85], [89], [105]. Here we are simply comparing the reduced Einstein-Hilbert action in eq. 4.7 to the $O(d,d)$ covariant Scherk-Schwarz reduction in eq. 5.29 from Hull and Reid and noting the remarkable fact that they are essentially the same.

where R_{10+d} is the 10+d dimensional Ricci scalar and $*$ is the 10+d dimensional Hodge dual. The resulting 10-d dimensional term in the Lagrangian is

$$\begin{aligned}\mathcal{L}_{10-d} = R_{10-d} * 1 - \frac{1}{2} * d\varphi \wedge \varphi - \frac{1}{2}\mathcal{H}^{IJ}\mathcal{H}^{KL} * D\mathcal{H}_{IJ} \wedge D\mathcal{H}_{KL} \\ - \frac{1}{2}e^{2(\beta-\alpha)\varphi}\mathcal{H}_{IJ} * F^I \wedge F^J - \frac{1}{2}e^{2(\beta-\alpha)\varphi}(\mathcal{H}_{IJ}\mathcal{H}^{KL}\mathcal{H}^{MN} f^I_{KM} f^J_{LN} \\ + 2\mathcal{H}^{IJ} f^K_{IL} f^L_{JK}) * 1\end{aligned} \tag{5.27}$$

where $\mathcal{H}^{IJ}$ is the inverse of $\mathcal{H}_{IJ}$, $*$ is the 10-d dimensional Hodge dual,

$$D\mathcal{H}_{IJ} = d\mathcal{H}_{IJ} + \mathcal{H}_{IK} f^K_{JL} A^L + \mathcal{H}_{JK} f^K_{IL} A^L \tag{5.28}$$

and

$$F^I = dA^I + \frac{1}{2} f^I_{JK} A^J \wedge A^k \tag{5.29}$$

is the field strength of vector fields A^I. The reduction of this term gives the correct gauge group. However, it does more. In the case where Q and R flux are set to zero and $\mathcal{H}$ is written in the form (5.8), this is exactly the dimensionally reduced action modulo the term coming from H flux on the base. From the $O(d, d)$ covariant form of this action given in [96], it's clear the from T-duality that this equality should hold in general.

The necessary condition for the two actions to agree was the restricted form of $\mathcal{H}$. Recall that $\mathcal{H}$ is not allowed to be any symmetric matrix, but is required to satisfy $L\mathcal{H}L = \mathcal{H}$. This arose as a consistency condition for imposing the self-duality constraint. When this is satisfied, the two actions coincide. Relating $\mathcal{H}$ to G and B involves solving the self-duality constraint by choosing some variables to be the physical ones and solving for the others in terms of them. This choice relied on the polarization and so was somewhat arbitrary. Of course it works regardless of polarization, and so the we note that the fundamental constraint is that raising the indices on $\mathcal{H}$ with the metric L inverts $\mathcal{H}$.

5.4.3 Graviton Scattering and the Low Energy Effective Action

As stressed before, the doubled formalism can't be exactly right. The imposition of the self-duality constraint after solving for the equations of motion tells us that we have incorrectly chosen the action. A solution is in the work of Tseytlin, [86], which takes a slightly different approach to writing a manifestly T-duality covariant CFT. There, the action is such that self-duality constraint is the equation of motion. The price on pays is a loss of manifest

Lorentz invariance. In fact, local Lorentz invariance holds only on-shell.

These papers calculate a term in the low energy effective action of this doubled theory. In particular, they consider an example with no B field, so that the doubled metric is set to be

$$\mathcal{H} = \begin{pmatrix} G & 0 \\ 0 & G^{-1} \end{pmatrix} \tag{5.30}$$

From a calculation of three graviton scattering amplitudes, it is concluded that the action contains terms whose natural off shell generalization is

$$\mathcal{L} = \sqrt{G}\sqrt{G^{-1}}(R(G) + R(G^{-1}))d^{2d}x = R(\mathcal{H}) * 1 \tag{5.31}$$

This is further evidence that (5.26) is the correct low energy effective action.

5.5 Conclusion

In this paper we have attempted to understand the origins of the non-geometric fluxes found in [25]. We have argued that they arise in an intrinsically stringy way, involving nontrivial mixing between momentum and winding modes of the closed string. Previous attempts at understanding reductions with duality twists and simple T-duality experiments on T^3 with H flux indicated that non-geometric fluxes are related to T-duality type monodromies or transition functions. Hull's doubled formalism suggest a way to geometrize such backgrounds. We have extended this line of thought to its natural conclusion.

Compactifications with all the new NS-NS fluxes turned on become compactifications on local group manifolds of twice the usual dimension. Putting the ordinary Einstein-Hilbert action on these twisted double tori dimensionally reduces to the entire NS-NS sector following the familiar Scherk-Schwarz reductions. This approach unifies the metric and the B field and turns all the flux into topological data.

The doubled formalism involves a self-duality constraint to be imposed after varying the action, like the self-duality constraint of type IIA supergravity. This requires a consistency condition on the doubled metric and must also be imposed on the low energy action. Because of this, it is not clear how to quantize this picture.

At the expense of local Lorentz invariance, Tseytlin's formalism may be used, which appears a better candidate for quantization. Graviton scattering amplitudes computed in Tseytlin's original papers confirm the Einstein-Hilbert action as the low energy limit of the doubled approach. This remains somewhat mysterious, as the truncation to winding and momentum modes while ignoring all other excited states is never a good approximation.

While not exactly correct, this approach is probing some underlying topological structure in string field theory. It would be interesting to see how to extend this to the other sectors of string theory and to incorporate supersymmetry.

Appendix A

Generalized complex geometry

In the appendix we give a short summary of the definitions of (twisted) generalized complex structure (GC or TGC for short). Let M be an even dimensional manifold and H be a closed 3-form on M. The twisted Dorfman backet $\circ$ is defined as a binary operation on the sections of $TM \oplus T^*M$.

$$(X \oplus \zeta) \circ (Y \oplus \eta) = [X, Y] \oplus (\mathcal{L}_X \eta - \imath_Y d\zeta + \imath_Y \imath_X H) \tag{A.1}$$

where $X, Y \in \Gamma(TM)$ and $\zeta, \eta \in \Gamma(T^*M)$. The bundle $TM \oplus T^*M$ has a metric h with (n, n) signature defined by an inner product for the sections in $TM \oplus T^*M$.

Definition A TGC-structure on M is an endomorphism $\mathcal{J}$ on $TM \oplus T^*M$ such that
(1) $\mathcal{J}^2 = -1$
(2) $h(\cdot, \cdot) = h(\mathcal{J}\cdot, \mathcal{J}\cdot)$
(3) The i-eigenbundle of $\mathcal{J}$ is closed (or involutive) with respect to the twisted Dorfman bracket. This condition is equivalent to an integrability condition for the (T)GC-structure.

Setting $H = 0$ the word "twisted" is dropped everywhere and we will get the definitions for Dorfman brackets and GC-structures.

Definition (Twisted) generalized Kähler structure consists of two commuting (T)GC-structures $\mathcal{J}_1$ and $\mathcal{J}_2$ such that $\mathcal{G} = -\mathcal{J}_1\mathcal{J}_2$ is a positive definite metric on $TM \oplus T^*M$.

A (twisted) generalized Kähler structure is physically relevant because it has been shown that the structure is equivalent to the bi-Hermitian geometry [14]. The two (twisted) commuting generalized complex structures $\mathcal{J}_1$ and $\mathcal{J}_2$ can be expressed in terms of the data of the bi-Hermitian geometry, namely, (J_+, J_-, g, H).

$$\mathcal{J}_1 = \begin{pmatrix} \tilde{J} & -\alpha \\ \delta\omega & -\tilde{J}^t \end{pmatrix}, \qquad \mathcal{J}_2 = \begin{pmatrix} \delta J & -\beta \\ \tilde{\omega} & -\delta J^t \end{pmatrix} \tag{A.2}$$

where

$$\begin{aligned} \tilde{J} &= \frac{1}{2}(J_+ + J_-), \quad \beta = \frac{1}{2}(\omega_+^{-1} + \omega_-^{-1}), \quad \tilde{\omega} = \frac{1}{2}(\omega_+ + \omega_-), \\ \delta J &= \frac{1}{2}(J_+ - J_-), \quad \alpha = \frac{1}{2}(\omega_+^{-1} - \omega_-^{-1}), \quad \delta\omega = \frac{1}{2}(\omega_+ - \omega_-). \end{aligned} \tag{A.3}$$

$$\omega_\pm(\cdot,\cdot) = g(J_\pm\cdot,\cdot) \tag{A.4}$$

The H is preserved by $J_\pm$ in the sense that the following constraints are satisfied and moreover it is of $(2,1)+(1,2)$ type with respect to both $J_\pm$.

$$H(X,Y,Z) = H(J_\pm X, J_\pm Y, Z) + H(J_\pm X, Y, J_\pm Z) + H(X, J_\pm Y, J_\pm Z) \tag{A.5}$$

$$H(J_\pm X, J_\pm Y, J_\pm Z) = H(J_\pm X, Y, Z) + H(X, J_\pm Y, Z) + H(X, Y, J_\pm Z) \tag{A.6}$$

The following identity is useful in deriving equations.

$$H(X,Y,Z) = \mp d\omega_\pm(J_\pm X, J_\pm Y, J_\pm Z) \tag{A.7}$$

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