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A Note on Cosmic (p, q, r) Strings

Mark G. Jackson
Particle Astrophysics Center, Fermi National Accelerator Laboratory, Batavia IL 60510

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The spectrum of (p, q) bound states of F- and D-strings has a distinctive square-root tension formula that is hoped to be a hallmark of fundamental cosmic strings. We point out that the BPS bound for vortices in $\mathcal{N} = 2$ supersymmetric Abelian-Higgs models also takes the square-root form. In contrast to string theory, the most general supersymmetric field theoretic model allows for (p, q, r) strings, with three classes of strings rather than two. Unfortunately, we find that there do not exist BPS solutions except in the trivial case. The issue of whether there exist non-BPS solutions which may closely resemble the square-root form is left as an open question.

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INTRODUCTION

There has been recent interest in the cosmic rehabilitation of fundamental strings [1]. This has led to the exciting possibility that the properties of a cosmic network made of fundamental strings may have observationally distinct signatures from more mundane solitonic objects. There are two smoking guns. The first is the probability of reconnection which scales as $P \sim g_s^2$ for fundamental strings [2], while it is essentially unity for abelian vortices [3, 4]. Recently it was shown that non-abelian vortices may give rise to a probability $P < 1$ but the velocity dependence differs substantially from the fundamental string case [5].

The second smoking gun [6] is the existence of both F- and D-strings in warped IIB compactifications which form a distinctive spectrum of bound states with tension

$$\mu_{(p,q)} = \sqrt{p^2\,\mu_F^2 + q^2\,\mu_D^2}. \tag{1}$$

A simple consequence of these bound states is the existence of 3-string junctions, with angles dictated by charge conservation and the tension formula (1). Apects of network formation and gravitational lensing of such junctions was studied in [7, 8].

It is rather simple to construct field theories which admit bound states of vortices and the corresponding 3-string junctions. Examples include vortices charged under discrete symmetries [9] and multiple abelian gauge groups [10]. However, none of the field theoretic models studied so far reproduce the stringy spectrum (1). The purpose of this short note is to show that the general Bogomolnyi bound in gauge theories with multiple $U(1)$ gauge groups includes the string spectrum (1). In fact, we shall see that there is a maximum of three different types of supersymmetric vortices, with the tensions bounded by

$$\mu = \sqrt{k_1^2\mu_1^2 + k_2^2\mu_2^2 + k_3^2\mu_3^2} \tag{2}$$

where k_i are integer gauge winding charges. If we choose a field theory without the third type of vortex, then this mimics the IIB string theory spectrum of cosmic strings. Although we find that no nontrivial BPS solutions exist which have this square-root spectrum, it is possible that non-BPS solutions exist which could closely resemble it.

THE BOGOMOLNYI BOUND

In theories with $\mathcal{N} = 2$ supersymmetry, the real D-term and complex F-term are unified into a triplet, transforming under an $SU(2)_R$ R-symmetry. This existence of this triplet is responsible for the three different tensions appearing in (2). Recall that matter lives in a hypermultiplet, consisting of two complex scalars ϕ and $\tilde{\phi}$ transforming in conjugate representations of the gauge group. For a single scalar charged under a single $U(1)$ gauge group, the D- and F-terms in the scalar potential [12] are fixed by $\mathcal{N} = 2$ supersymmetry to be,

$$V = \frac{e^2}{2}(|\phi|^2 - |\tilde{\phi}|^2 - r_3)^2 + \frac{e^2}{2}|2\tilde{\phi}\phi - r_1 - ir_2|^2. \tag{3}$$

Here e^2 is the gauge coupling constant. There are three vacuum expectation values r_1, r_2 and r_3 allowed by supersymmetry (often referred to as a Fayet-Iliopoulos parameters). The $SU(2)_R$ symmetry of this potential can be made manifest by defining the doublet $\omega^T = (\phi, \tilde{\phi}^\dagger)$ and writing

$$V = \frac{e^2}{2}(\omega^\dagger \vec{\sigma}\,\omega - \vec{r})^2 \tag{4}$$

where $\vec{r} = (r_1, r_2, r_3)$ and $\vec{\sigma}$ are the triplet of Pauli matrices.

Consider now a $U(1)^N$ gauge theory with gauge coupling $e_a^2, a = 1, \ldots, N$. We couple N hypermultiplets ω_i with integer charges $Q^i{}_a$ under the a^{th} gauge group. The covariant derivatives are given by $\mathcal{D}\omega_i = \partial\omega_i - i(\sum_{a=1}^N Q^i{}_a A_a)\omega_i$. The energy functional for static

($\partial_0 = A_0 = 0$) configurations is

$$\mathcal{E} = \sum_{i=1}^{N} |\mathcal{D}\omega_i|^2 + \sum_{a=1}^{N} \frac{1}{2e_a^2} B_a^2 + \frac{e_a^2}{2} (\sum_{i=1}^{N} Q^i{}_a \omega_i^\dagger \, \vec{\sigma} \, \omega_i - \vec{r}_a)^2. \tag{5}$$

with B_a the magnetic field for the $a^{\rm th}$ gauge group. We choose $\det Q \neq 0$ to ensure that in the ground state, defined by $\sum_i Q^i{}_a \omega_i^\dagger \vec{\sigma} \omega_i = \vec{r}_a$, the $U(1)^N$ gauge group is fully broken and the theory exhibits a mass gap.

Lowest energy vortex states may be found by the usual Bogomolnyi method. We search for straight strings, extended in the x^3 direction, by setting $\partial_3 = A_3 = 0$ and writing

$$\mathcal{E} = \sum_{i=1}^{N} |\mathcal{D}_1 \omega_i - i(\vec{m}\cdot\vec{\sigma})\mathcal{D}_2 \omega_i|^2 + \sum_{a=1}^{N} \frac{1}{2e_a^2} \left(\vec{m} B_a - e_a^2 (\sum_{i=1}^{N} Q^i{}_a \omega_i^\dagger \, \vec{\sigma} \, \omega_i - \vec{r}_a) \right)^2 - B_a \vec{m}\cdot\vec{r}_a .$$

The above decomposition holds for any unit vector $\vec{m}$. The last term yields a topological charge when integrated over the plane transverse to the vortex string: $\int d^2x B_a = -2\pi k_a$. Noting that the first two terms are squares, we derive the bound on the tension

$$\mu = \int d^2x \; \mathcal{E} \geq 2\pi \sum_a k_a \vec{m}\cdot\vec{r}_a . \tag{6}$$

This is maximized by choosing $\vec{m}$ parallel to $\sum_a k_a \vec{r}_a$. The resulting bound is the tension formula (2), with $\mu_\alpha = 2\pi \sum_a k_a (\vec{r}_a)_\alpha$ for $\alpha = 1, 2, 3$. In IIB string theory the tension-squared for a string with integer charge vector $k_a = (p, q)$ is expressed as

$$m^2 = \sum\nolimits_{a,b=1,2} (\mathcal{M}^{-1})^{ab} k_a k_b$$

where $\mathcal{M}_{ab}$ is the metric on the IIB auxiliary torus of modular parameter τ. We obtain the same spectrum by defining $(\mathcal{M}^{-1})^{ab} = 2\pi \vec{r}_a \cdot \vec{r}_b$, where now $a, b = 1, 2, 3$.

The bound is saturated by solutions to the equations

$$\begin{aligned} \vec{m} B_a &= e_a^2 \left(\sum_{i=1}^{N} Q^i{}_a \omega_i^\dagger \, \vec{\sigma} \, \omega_i - \vec{r}_a \right) \\ \text{and} \quad \mathcal{D}_1 \omega_i &= i(\vec{m}\cdot\vec{\sigma}) \, \mathcal{D}_2 \omega_i \end{aligned} \tag{7}$$

where, as explained above, $\vec{m}$ is the unit vector parallel to $\sum_a k_a \vec{r}_a$. When all $\vec{r}_a$ lie parallel, for example $\vec{r}_a = (0, 0, r_a)$, these reduce to the usual coupled vortex equations studied in [11]. They have solutions only when the winding n_i of all scalar fields with non-zero expectation value, defined by $n_i = \sum_a Q^i{}_a k_a$ is non-negative. (This is simply the statement that there is no holomorphic vector bundle of negative degree). In this case there is no attractive force between vortices. In contrast, when the $\vec{r}_a$ do not lie parallel and the vortices in different gauge groups are coupled through the scalars ω_i, one may expect bound states to form.

In both the field theoretic and string theoretic contexts, the Bogomolnyi bound is expected to receive corrections at the scale at which the protectorate supersymmetry is broken. In warped IIB compactifications, supersymmetry is broken from 16 supercharges (in the orientifold background) to 4 at the compactification scale, with subsequent low-energy breaking at the TeV scale.

NON-EXISTENCE OF BPS SOLUTIONS

Now decompose each field ω_i into eigenvectors of $\vec{m}\cdot\vec{\sigma}$, writing

$$\omega_i = \psi_i \, |\vec{m}_+\rangle + \tilde{\psi}_i^\dagger \, |\vec{m}_-\rangle . \tag{8}$$

Then the covariant derivatives become

$$\begin{aligned} \mathcal{D}_z \, \psi_i \;&\equiv\; (\partial_1 - i\partial_2)\psi_i - i \sum_a Q^i{}_a (A_1^a - i A_2^a) \psi_i = 0, \\ \mathcal{D}_{\bar{z}} \, \tilde{\psi}_i^\dagger \;&\equiv\; (\partial_1 + i\partial_2)\tilde{\psi}_i^\dagger - i \sum_a Q^i{}_a (A_1^a + i A_2^a) \tilde{\psi}_i^\dagger = 0. \end{aligned}$$

From this, we see that both ψ_i and $\tilde{\psi}_i^\dagger$ transform with charge $Q^i{}_a$ under the $U(1)_a$ gauge group. Taking the complex conjugate of the second equation, this ensures that both ψ_i and $\tilde{\psi}_i$ are covariantly holomorphically constant, i.e.

$$\mathcal{D}_z \psi_i = \mathcal{D}_z \tilde{\psi}_i = 0 \tag{9}$$

where now ψ_i has charge $Q^i{}_a$ while $\tilde{\psi}_i$ has charge $-Q^i{}_a$. In other words, as the notation suggests, these are the rotated form of ϕ_i and $\tilde{\phi}_i$. We can now look at the first Bogomolnyi equation. Dotting with the unit vector $\vec{m}$ tells us

$$B_a = e_a^2 \left(\sum_{i=1}^{N} Q^i{}_a |\psi_i|^2 - Q^i{}_a |\tilde{\psi}_i|^2 - \vec{m}\cdot\vec{r}_a \right) . \tag{10}$$

Equations (9) and (10) are now in the form of the usual coupled vortex equations described, for example, in Morrison and Plesser [11]. The criterion for the existence of solutions is that for each scalar field ψ_i we can define the winding $n_i = \sum_a Q^i{}_a k_a$, while for each $\tilde{\psi}_i$ we have $\tilde{n}_i = -\sum_a Q^i{}_a k_a$. Clearly $n_i = -\tilde{n}_i$. Solutions to (9) and (10) exist if n_i is non-negative for each ψ_i that gains an expectation value. (If ψ_i has no expectation value for some i then it may remain zero throughout the solution). Similarly, each $\tilde{n}_i$ must be non-negative for each $\tilde{\psi}_i$ which is non-zero. Clearly, since $n_i = -\tilde{n}_i$, either ψ_i or $\tilde{\psi}_i$ is allowed an expectation value, but not both.

There are two further real equations that come from dotting the first equation in (7) with $\vec{l}_\alpha$ where $\vec{l}_\alpha \cdot \vec{m} = 0$, for $\alpha = 1, 2$. We write $\vec{l} = \vec{l}_1 + i\vec{l}_2$. There is an ambiguous phase to the vector $\vec{l}$, associated to rotating the basis $\vec{l}_1$ and $\vec{l}_2$, but we can always pick a basis so that the remaining two real equations combine into the complex equation

$$\sum_{i=1}^{N} Q^i{}_a \tilde{\psi}_i \psi_i = \vec{l} \cdot \vec{r}_a. \tag{11}$$

Thus we see that in order for either ψ_i or $\tilde{\psi}_i$ to be zero, the vector $\sum_a (Q^{-1})^i{}_a \vec{r}_a$ must be perpendicular to $\vec{l}$, making it proportional to $\vec{m}$. Since $\vec{m} \propto \sum_a k_a \vec{r}_a$, this requires k_a to be proportional to $(Q^{-1})^i{}_a$ (and of course k_a must be integer-valued). That is, in order for only n_i to be nonzero we select k_a to be the ith entry of Q^{-1}. There is also the trivial solution when all $\vec{r}_a$ lie parallel, so that each $\vec{r}_a$ is proportional to $\vec{m}$ and thus perpendicular to $\vec{l}$, but this does not produce a square-root spectrum.

While these are the necessary conditions for BPS solutions, we find they are not sufficient. Consider the radially-symmetric field ansatz

$$\omega_i = \begin{pmatrix} \psi_i \\ \tilde{\psi}_i^\dagger \end{pmatrix} = (1 - q_i(\rho)\vec{m}\cdot\vec{\sigma}) \begin{pmatrix} s_i \\ \tilde{s}_i \end{pmatrix} e^{in_i\theta} \tag{12}$$

where we use polar coordinates (ρ, θ) in the (x^1, x^2)-plane. The asymptotic boundary condition is

$$q_i \to 0 \quad \text{as} \quad \rho \to \infty$$

making the vacuum selection at infinity

$$\sum_i Q_a^i \, (s_i^*, \tilde{s}_i^*)\, \vec{\sigma} \begin{pmatrix} s_i \\ \tilde{s}_i \end{pmatrix} = \vec{r}_a. \tag{13}$$

The ansatz for the gauge potential is

$$A_\mu^a = \frac{\epsilon_{\mu\nu} x^\nu}{\rho^2} A_a(\rho), \qquad A_a = -k_a + \rho f_a(\rho) \tag{14}$$

where we similarly require $f_a \to 0$ as $\rho \to \infty$. The field strength is given by the simple expression

$$B^a = \partial_x A_y^a - \partial_y A_x^a = \frac{1}{\rho}\frac{\partial A_a}{\partial \rho}. \tag{15}$$

To determine the long-distance behavior of the fields, we insert the ansatz into the equations (7) and expand to linear order in q_i and f_a to obtain

$$\left(\tfrac{f_a}{\rho} + f_a'\right) = -2e_a^2 \textstyle\sum_i Q_a^i (|s_i|^2 + |\tilde{s}_i|^2) q_i, \tag{16}$$
$$\left(\vec{m}\cdot\vec{\sigma}\partial_\rho - \textstyle\sum_a Q_a^i f_a\right)(1 - q_i(\rho)\vec{m}\cdot\vec{\sigma}) \begin{pmatrix} s_i \\ \tilde{s}_i \end{pmatrix} = 0.$$

The second equation can be solved (to linear order!) to give

$$q_i' = -\sum_a Q^i{}_a f_a. \tag{17}$$

Differentiation of these first-order equations then produces the modified Bessel equations

$$\begin{aligned} f_a'' + \frac{1}{\rho} f_a' - \frac{1}{\rho^2} f_a - \sum_b L_{ab}^2 f_b &= 0, \\ q_i'' + \frac{1}{\rho} q_i' - \sum_j M_{ij}^2 q_j &= 0 \end{aligned}$$

where the mass-squared matrices are given by

$$\begin{aligned} L_{ab}^2 &= 2e_a^2 \sum_i (|s_i|^2 + |\tilde{s}_i|^2) Q^i{}_a Q^i{}_b, \\ M_{ij}^2 &= 2(|s_i|^2 + |\tilde{s}_i|^2) \sum_a e_a^2 Q^i{}_a Q^j{}_a. \end{aligned}$$

As should be expected from a BPS solution, the gauge and matter mass-squared matrices L_{ab}^2 and M_{ij}^2 have identical eigenvalues, which can be seen by acting with $Q^i{}_a$ as a similarity transformation. Denoting the mass eigenvalues as λ_A (so that the mass-squared eigenvalues are λ_A^2), the solution to (16) and (17) is then given by

$$\begin{aligned} f_a &= \sum_A S_{aA} C_A \lambda_A K_1(\lambda_A \rho), \\ q_i &= \sum_{a,A} Q^i{}_a S_{aA} C_A K_0(\lambda_A \rho) \end{aligned} \tag{18}$$

where S_{aA} is the diagonalization matrix for L_{ab}^2 and the coefficients C_A cannot be determined in the linear approximation and would have to be fixed from numerical comparison to the non-linear solution. We would expect that only gauge fields charged under the given k_a and the matter field ω_i with nonzero winding n_i should attain a profile, but from (18) we see that the $Q^i{}_a$ mix the matter and gauge fields into a basis such that non-charged fields are excited. To prevent this it must be that the L_{ab}^2 and M_{ij}^2 are diagonal, making the C_A proportional to k_a. This gives us a total of 6 constraints (3 each from setting the off-diagonal components of a symmetric matrix to zero). These are precisely enough constraints to set the off-diagonal components of $Q^i{}_a$ to zero, which now makes the interaction trivial. The acceptable BPS windings are then simply of the form $k_a = (p, 0, 0), (0, q, 0)$ or $(0, 0, r)$, which will not display any distinctive square-root behavior.

The situation is very reminiscint of supersymmetric quantum mechanics, with the doublet $\omega_i = (\psi_i, \tilde{\psi}_i^\dagger)$ taking the place of the bosonic and fermionic components of the wavefunction. The ground state (BPS) solution (when it exists) is given by first-order solutions which allow one component or the other, but not both. It is still likely that non-BPS solutions exist which will contain both components, but it will have energy greater than the BPS bound.

CONCLUSION

We have shown the BPS spectrum for supersymmetric vortices exhibits the same square-root cosmic string spectrum as superstring theory, including not just two but three types of vortices. Unfortunately no BPS solutions exist which actually exhibit this square-root spectrum. It is still likely that non-BPS solutions exist which would have an energy higher than the BPS bound, but which might approximate the square-root BPS spectrum for a certain choice of parameters. It would be interesting to make a full analysis of these solutions.

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