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# Radiation Diffusion: An Overview of Physical and Numerical Concepts

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# **RADIATION DIFFUSION: AN OVERVIEW OF PHYSICAL AND NUMERICAL CONCEPTS<sup>1</sup>**

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An overview of the physical and mathematical foundations of radiation transport is given. Emphasis is placed on how the diffusion approximation and its transport corrections arise. An overview of the numerical handling of radiation diffusion coupled to matter is also given. Discussions center on partial temperature and grey methods with comments concerning fully implicit methods. In addition finite difference, finite element and Pert representations of the div-grad operator is also discussed

## **1. The “So What?” Question: Why Radiation Transport Matters**

Photons, be they in the radio, optical, X-ray or gamma-ray portion of the radiation spectrum leave their mark across the fabric of the universe in a multitude of ways. At the “smallest” astronomical scales, radiation transport is crucial to understanding the atmospheres of planets. At the largest scales the universe is bathed in an afterglow of its birth called the cosmic background radiation. As is well known, whole fields in astronomy are devoted to the study of different portions of the electromagnetic spectrum.

Photons can act as a signature of some astronomical event. In addition, because of the density and temperatures encountered in many astrophysical applications, photons can effect the movement of a gas or fluid and the movement of the gas or fluid can in turn affect the behavior of the photons. Radiation pressure, changes in spectral shape due to moving fluids, and PdV work done on the radiation field are all important examples of the interaction of matter and photons.

With the advent of large scale computing, the complex system of equations involving radiation transport and fluid dynamics could be solved. Currently, with the introduction of parallel computing it is now possible to model 3D astrophysical phenomena such as supernovae with unprecedented accuracy and with the inclusion of complex physics. In all of these simulations radiation transport remains an exciting but challenging obstacle. In multi-physics codes it tends to dominate CPU time. This is easy to see when one considers that in 3D

dynamic applications the solution of the photon transport problem involves solving a seven dimensional Boltzmann equation<sup>1</sup>. This equation is in general highly non-linear and non-local. In addition, its coupling to the fluid modifies both the fluid equations and the usual hydrodynamic equations. This interaction between radiation and fluids defines the field of radiation hydrodynamics. It is a vast field with excellent references by Mihalas and Mihalas<sup>2</sup>, Pomraning<sup>3</sup>, Bowers and Wilson<sup>4</sup>, and Castor<sup>5</sup>. Besides being very good guides to radiation hydrodynamics they are excellent sources for the field of radiation transport in general.

The challenges of solving the transport problem have led researchers to solving a simpler problem. In many applications the physics allows one to solve the diffusion approximation to the full transport problem. By going to the diffusion limit of the transport equation, the numbers of degrees of freedom in 3D are reduced from seven to five for multi-group and seven to four for Planckian. This approximation is by far the most used approximation to the transport equation. In fact, it makes the radiation problem so tractable that the diffusion approximation is used in regimes where only a transport description is valid. The use of transport corrected diffusion such as flux limiters helps extend the applicability of diffusion. The benefit is of course the cheaper cost of diffusion over transport but at the price of reduced accuracy.

This paper is devoted to discussing the general framework of radiation transport and in particular how diffusion arises from it. In addition a review of the numerical treatments of the diffusion operator and how the coupled radiation material equations are handled is given. Due to the lack of space, the subject of radiation hydrodynamics is not given. Interested readers are urged to read the above listed resources. However, the work presented here should always be thought of in the larger context of a multi-physics code. In addition, subjects not covered here include opacities and scattering. Detailed discussions of these topics can also be found in the above listed references.

## **2. Review of Radiation Transport Concepts**

### ***2.1. Classical and Quantum Properties of the Radiation Field***

The classical manifestation of the radiation field is based on the wave properties of light. A classical description of radiation is consistent with the properties of polarization, diffraction, and refraction. Unfortunately, the fact that

a classical radiator predicts a Rayleigh-Jeans Law for the emission spectra, in violation of the experimental data, means that the wave description of light is not complete. As is well known, the quantum mechanical description of radiation describes a wealth of observed phenomena. The photoelectric effect, where photons which are impinging on a metal surface release electrons above a certain threshold frequency is a well known example that earned Einstein the Nobel Prize. Compton scattering, where the frequency of hard X-ray photons is shifted downwards due to the incoming photons scattering off of stationary electrons and transferring some of their energy and momentum to the electrons, is another example. The observed emission spectra from atoms are a classic example where the quantum mechanical description of radiation explained observations that were previously unexplained. The quantum mechanical treatment of photons and the description of the blackbody spectrum is a famous and singular success that heralded the beginning of a new age. Finally, the beauty behind the quantum mechanical description of radiation culminated with the unification, by Dirac, of the particle and wave descriptions.

### 2.1.1. *The Boltzmann Description of Radiative Transfer*

The standard description of radiative transfer rests on describing the radiation field as a photon gas moving with the speed of light and interacting with a medium via absorption, emission, and scattering. For simplicity, the effects of material motion are ignored here as are the effects of refraction, diffraction, and dispersion. The radiation field is assumed to consist of point particles (photons). Associated with each photon is a frequency  $\nu$ , energy  $h\nu$ , and momentum  $h\nu/c$ . At any time  $t$ , six variables in 3D are required to specify the position of the photon in phase space. There are three position variables and three momentum variables. The three momentum variables are written in terms of the speed of light  $c$  and the photon direction  $\Omega$ . Using the gas analogy, a photon distribution function  $f_\nu(r, \Omega, t)$  is defined such that

$$dn = f_\nu(r, \Omega, t) d^3r d^2\Omega \quad (1)$$

is the number of photons at time  $t$  and position  $r$ , contained in the differential element  $d^3r$ , with frequency  $\nu$ , traveling in the direction  $\Omega$  subtending a solid angle element  $d^2\Omega$ . In the literature, the photon distribution function is

rarely used. Instead, the specific intensity or angular flux is the quantity most used and the object referred to in this paper when the radiation field is mentioned. It is defined by

$$I_\nu(r, \Omega, t) = ch\nu f_\nu(r, \Omega, t) \quad (2)$$

In general,  $I_\nu(r, \Omega, t)$  or  $f_\nu(r, \Omega, t)$  should have an additional index describing the particular polarization state. There are in general four components of the specific intensity, called Stokes parameters, necessary to describe a polarized beam of radiation. One of these components is the radiation specific intensity defined above. The other three components come from the plane of polarization and the ellipticity of the beam. In this paper, it is assumed that the radiation is either unpolarized or the polarization states have been averaged over. For details on the radiative transfer equation for polarized radiation please consult either Chandrasekhar<sup>6</sup> or Pomraning<sup>3</sup>.

Using the specific intensity as the fundamental quantity of interest, a number of physically relevant objects can be defined.

$$\begin{aligned} u &= \int_0^\infty d\nu \int d^2\Omega (h\nu) f_\nu(r, \Omega, t) = \frac{1}{c} \int_0^\infty d\nu \int d^2\Omega I_\nu(r, \Omega, t) \\ F &= \int_0^\infty d\nu \int d^2\Omega (c\Omega) (h\nu) f_\nu(r, \Omega, t) = \int_0^\infty d\nu \int d^2\Omega \Omega I_\nu(r, \Omega, t) \\ M &= \int_0^\infty d\nu \int d^2\Omega \left(\frac{h\nu\Omega}{c}\right) f_\nu(r, \Omega, t) = \frac{1}{c^2} \int_0^\infty d\nu \int d^2\Omega \Omega I_\nu(r, \Omega, t) \\ P &= \int_0^\infty d\nu \int d^2\Omega (c\Omega) \left(\frac{h\nu\Omega}{c}\right) f_\nu(r, \Omega, t) = \frac{1}{c} \int_0^\infty d\nu \int d^2\Omega \Omega \Omega I_\nu(r, \Omega, t) \end{aligned} \quad (3)$$

The quantities  $u$ ,  $F$ ,  $M$ , and  $P$  are respectively the energy density, radiative flux, momentum density, and pressure tensor of the radiation field.

In writing down the equation of radiative transfer, a pedagogic approach is usually taken. By using the analogy of a classical gas, a semi-classical equation can be written down based solely on conservation of photons. Terms due to sink and source terms and scattering are included in a semi-phenomenological fashion.

$$\begin{aligned}
& \frac{1}{c} \frac{\partial I_\nu(r, \Omega, t)}{\partial t} + \Omega \bullet \nabla I_\nu(r, \Omega, t) = j_\nu - \sigma_\nu I_\nu(r, \Omega, t) \\
& + \int d\nu' \int d\Omega' \frac{\nu}{\nu'} \sigma_s(\nu \rightarrow \nu, \Omega \bullet \Omega') I_\nu(r, \Omega, t) \left[ 1 + \frac{c^2 I_\nu(r, \Omega, t)}{2h\nu^3} \right] \\
& - \int d\nu' \int d\Omega' \sigma_s(\nu \rightarrow \nu, \Omega \bullet \Omega') I_\nu(r, \Omega, t) \left[ 1 + \frac{c^2 I_\nu(r, \Omega, t)}{2h\nu'^3} \right]
\end{aligned} \quad (4)$$

The right hand side of equation [4], represents the interaction of the radiation field with matter through three basic mechanisms; (1) emission (2) absorption (3) scattering. The quantity  $j_\nu$  represents the emission of photons from the material and is called the emissivity. The quantity  $\sigma_\nu$  is the absorption coefficient and has dimensions 1/cm. It is related to the opacity  $\kappa_\nu$  and the density  $\rho$  by the simple relation  $\sigma_\nu = \kappa_\nu \rho$ . The quantity  $\sigma_s$  is the scattering kernel. Typically, it will represent the process of Compton scattering<sup>3</sup>. For simplicity, in the rest of this paper scattering is ignored.

The radiative transfer equation is semi-classical in nature. In the kinetic equation, the photons are treated like any other gas. The quantum mechanical effects come through the absorption, emission and scattering terms. Each of these three processes describes at a micro-physical level the quantum mechanical interaction of matter and radiation. In the next section, a simple example of the micro-physical origins of the emission and absorption mechanisms will be given along with consequences of the matter field being in thermodynamic equilibrium.

### 2.1.2. The Einstein Coefficients and the Planck Distribution

In this section, a derivation of the Planck distribution will be given. The derivation presented here (based on the book by Pomraning<sup>3</sup>) elucidates the nature of local thermodynamic equilibrium (LTE) and non-LTE (NLTE) and yields insight into the sink and source terms of the radiative transfer equation. Consider an atom with a number of bound states. Consider two levels with energies  $E_n$  and  $E_m$  and statistical weights  $g_n$  and  $g_m$ . The probability per time that an atom in state  $m$ , exposed to radiation of frequency  $\nu$ , will absorb a photon from the radiation field is given by.

$$P_{mn} = B_{mn} I(\nu_{mn}) d^2 \Omega \quad (5)$$

$$h\nu_{nm} = E_n - E_m \quad (6)$$

$B_{nm}$  is one of the Einstein coefficients. It is a constant of proportionality representing the transition rate at which the presence of the radiation field induces an upward transition in energy.  $I(\nu_{nm})$  is the specific intensity of the radiation field which we will define in a moment and  $d^2\Omega$  is the solid angle subtended by the photon. Besides being absorbed, the atom can also emit a photon. The probability per time that an atom in state  $n$  will emit a photon of frequency  $\nu$  is given by.

$$P_{nm} = [A_{nm} + B_{nm}I(\nu_{nm})]d^2\Omega \quad (7)$$

The term proportional to  $A_{nm}$  is the last Einstein coefficient. It represents the transition rate at which the atom undergoes spontaneous emission. That is, there is a finite probability that an atom in a state  $n$  will emit a photon and undergo a downward transition in energy in the absence of a radiation field. The second term proportional to  $B_{nm}$  represents the effect of stimulated emission. It is the transition rate at which the radiation field induces an atom to undergo a downward transition in energy. That is, the presence of a radiation field itself will enhance the emission process. This stimulated emission is a consequence of the quantum statistics obeyed by bosons. The A and B coefficients are related. Given the Hamiltonian of an atom in a radiation field, time dependent perturbation theory can be used to compute the transition rates for absorption and emission. The fundamental relationship can be derived,

$$A_{nm} / B_{nm} = h(\nu_{nm})^3 / c^2 \quad (8)$$

At this point in the discussion, nothing has been assumed about the absorption and emission processes. The arguments are completely general and are applicable to both systems in and out of thermodynamic equilibrium. If however, thermodynamic equilibrium is assumed, then additional results regarding the Einstein coefficients and the associated radiation field can be derived.

In complete thermodynamic equilibrium, the principle of detailed balance holds. That is, there exists a detailed balance between all absorption and emission processes. Mathematically, this means the probability for emission exactly equals the probability for absorption.



$$N_n [A_{nm} + B_{nm} I(\nu_{nm})] = N_m B_{mn} I(\nu_{nm}) \quad (9)$$

$N_n$  refers to the number density of atoms in state n. Solving for the radiation field yields

$$I(\nu_{nm}) = \frac{A_{nm} / B_{nm}}{[(N_m B_{mn} / N_n B_{nm}) - 1]} \quad (10)$$

Since complete thermodynamic equilibrium has been assumed, the populations are distributed according to the Boltzmann distribution

$$\frac{N_n}{N_m} = \frac{g_n}{g_m} e^{-h\nu_{nm}/kT} \quad (11)$$

Therefore, the Planck distribution is obtained

$$I(\nu_{nm}) = \frac{2h(\nu_{nm})^3 / c^2}{\left[ (B_{mn} / B_{nm}) e^{h\nu_{nm}/kT} - 1 \right]} \quad (12)$$

### 2.1.3. Local Thermodynamic Equilibrium (LTE) and Kirchoff's law

The simple example above proves the point that solving the radiative transfer equation when the matter emits, absorbs, and scatters radiation is complex. This is because, in general, detailed knowledge of the atomic populations and ionization states making up the material must be known if the absorptivities and emissivities are to be calculated. The concept of LTE is a simplifying assumption of the matter that greatly reduces the complexity of trying to solve the radiative transfer problem when radiation-matter interactions are important.

The essential point in establishing LTE in any given material is that the properties of the matter are dominated by atomic collisions which establish thermodynamic equilibrium locally at a space-time point (r,t) and the radiation field does not destroy this equilibrium. Therefore, the main difference between complete thermodynamic equilibrium and LTE is that LTE does not require the radiation field to be Planckian. The implications are that at a given space-time point, only the atomic composition and two thermodynamic quantities (density and temperature) need be specified. The LTE assumption of course assumes very specific features of the states of the atoms and molecules making up the

material. The first is kinetic equilibrium, that is, the electron and ion distributions obey a Maxwellian. The second feature is excitation equilibrium, that is, the population density of the excited states of every species must obey a Boltzmann distribution. Third, ionization equilibrium, that is, the particle densities for neutrals, electrons, and ions obey a Boltzmann like distribution involving ionization potentials. This is the so-called Saha equation<sup>8</sup>. Fourth is the Kirchoff-Planck relation. This is an amazing relation in that it reduces the emissivity to a product of the absorption coefficient and a Planck function whose temperature is characteristic of the local material temperature. Simply put,  $j_\nu = \sigma_\nu B_\nu(T)$ . Therefore, the material-radiation interaction reduces to a study of the absorption mechanisms in a plasma where kinetic, excitation, and ionization equilibrium holds.

When does LTE hold? In order for atomic collisions to dominate over radiative processes, it is clear the plasma must be dense. Griem<sup>7</sup>, has constructed a criterion based on the ratio of radiative to collisional rates in a hydrogen-like atom where the plasma is optically thin enough that photons once emitted are not re-absorbed. The condition for LTE of atomic level  $n$  is

$$1.23 \times 10^{-5} \left( \frac{n}{n-1} \right)^2 \left( \frac{Z^{*7}}{n^{17/2}} \right) \left( \frac{T(eV)}{Z^{*2}(13.6eV)} \right)^{1/2} \left( \frac{1}{\rho} \right) \ll 1 \quad (13)$$

The effective ionization is denoted by  $Z^*$ . Griem's criterion clearly shows that for high density and/or hot plasmas LTE is a good approximation. In this paper, LTE will always be assumed unless otherwise noted.

So far the discussion has focused on the radiation field. In most applications with optically thick matter, the radiation is absorbed, remitted and the material temperature changes. Using the assumption of LTE, every piece of matter acts like a blackbody radiator emitting photons with a Planckian spectrum characteristic of the temperature of the material. Therefore, ignoring scattering, using Kirchoff's law for the emissivity and writing an energy balance relation for the material we have,

$$\frac{1}{c} \frac{\partial I_\nu(r, \Omega, t)}{\partial t} + \Omega \cdot \nabla I_\nu(r, \Omega, t) = \sigma_\nu B_\nu(T) - \sigma_\nu I_\nu(r, \Omega, t) \quad (14)$$

$$\frac{\partial [\rho C_\nu T(r, t)]}{\partial t} = - \int d\nu \sigma_\nu \left[ B_\nu(T) - \int d^2\Omega I_\nu(r, \Omega, t) \right] \quad (15)$$

In writing down the latter equation it is assumed that the opacities are independent of angle and all conduction effects are negligible.

#### 2.1.4. *The Equilibrium Radiation Field*

The isotropic and homogeneous distribution obeyed by a photon gas in complete thermodynamic equilibrium at temperature  $T$ , is the Planck function

$$I_\nu(r, \Omega, t) = B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1} \quad (16)$$

This should not be confused with LTE. The Planck function can be derived in a variety of ways. We have seen one such method in section 2.1.2. Physically, it is a consequence of the photons undergoing three basic processes (1) absorption (2) stimulated emission (3) spontaneous emission and the fact that the photons are bosons. The isotropic nature of the equilibrium radiation field means that the energy density, radiative flux, momentum flux, and pressure tensor quantities can be easily evaluated. Substituting equation into equation yields

$$\begin{aligned} u &= \frac{8\pi^5 k^4}{15h^3 c^3} T^4 \equiv aT^4 \\ F &= M = 0 \\ P &= \frac{aT^4}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 45.7 \text{ Mbar} (T[\text{keV}])^4 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned} \quad (17)$$

As expected for an isotropic gas, the radiative flux and momentum density are zero and the pressure tensor is diagonal. The concept of a temperature for the photon gas is uniquely defined here. That is, for an equilibrium distribution of photons, the temperature is defined by the Planck distribution. One could think of the relationship between energy density and temperature as a defining relation. However, for non-Planckian distributions this is not true. Several definitions of temperature are encountered in multi-group diffusion and transport<sup>8</sup>. The above relations for energy density and pressure give a strong indication of the power of the radiation field to move matter. At  $T=1$  keV, the

pressure exerted by the photons on matter is 45.7 Mbar. In addition, this pressure rapidly rises with temperature. This fact means that for high temperatures, the radiation field can make a substantial or dominant contribution to the overall pressure in a fluid.

#### 2.1.5. *Assumptions of a Kinetic Theory and the Micro-physical Foundations of Radiative Transfer*

The specific form of the radiative transfer equation, whose numerical solution is the focus of this paper, rests on a number of assumptions. The first set of these assumptions is not inherent in the kinetic theory of radiative transfer but rather a simplification. As mentioned previously, polarization effects are ignored. In addition, refraction and dispersion effects are also ignored. A radiative transfer equation incorporating these effects can be readily constructed (Pomraning<sup>3</sup>) but for simplicity they are ignored in this paper. The second class of assumptions is inherent in the semi-classical kinetic approach adopted here. Since photons are treated classically, their wave behavior is ignored. Therefore, the possibility of interference between different photons is ignored. Hence the spread of the photon wave packet is assumed small on the resolution we are interested in. Since the photons are treated like a classical gas, other quantum phenomena such as photon number fluctuation are ignored. Finally, all collision, absorption, and emission processes occur instantaneously.

The fundamental description of photons and their interaction with matter is based on quantum electrodynamics (QED). Is it possible to arrive at a fully self-consistent description of radiative transfer from a fundamental description? The answer is yes. Although a detailed description of the derivation would take us too far a field, suffice it to say that a number of authors have derived kinetic equations for the photons starting with the Hamiltonian of QED (Gelinas and Ott<sup>9</sup>, Cannon<sup>10</sup>, Degl'Innocenti<sup>11</sup>, Graziani<sup>12</sup>). An example of the procedure involves using either the density matrix formalism or the Heisenberg equations of motion to construct dynamic equations for the number operators of the plasma and photon degrees of freedom. A quantum mechanical distribution function, similar to the Wigner distribution can be constructed from products of photon operators. The Wigner distribution is an operator that is a quantum mechanical generalization of the specific intensity. The quantum radiative transfer equation comes from the dynamic evolution of this Wigner function. The plasma degrees of freedom enter naturally into the quantum radiative transfer equation because

of the interaction terms present in the QED Hamiltonian. The classical specific intensity is calculated as the quantum average of the Wigner function. Of course, the quantum radiative transfer equation is more general as it allows one to compute fluctuations of the specific intensity operator. Classically, these fluctuations are negligible. Examples of where such an approach has proved useful are in deriving radiative transfer equations with polarization from first principles, quantum optics and NLTE plasmas.

### 3. The Diffusion Approximation

The radiative transfer equation [4], being in general a six dimensional non-linear integro-differential equation, is not conducive to closed form solutions. In fact it is a current challenge in the astrophysics, atmospheric physics, high energy density physics and nuclear engineering communities to solve it numerically. Therefore, borrowing a page from the kinetic theory community, a method commonly used to simplify the radiative transfer equation is to construct moments of the equation. The first type of moments that can be constructed of the specific intensity are arrived at by multiplying equation [4] by various powers of  $\Omega$ , and integrating both sides of the equation over all solid angle. Defining,

$$\begin{aligned}\mathcal{E}_\nu &= \frac{1}{c} \int d\Omega I_\nu(r, \Omega, t) \\ \phi_\nu &= \int d\Omega \Omega I_\nu(r, \Omega, t) \\ \Pi_\nu &= \frac{1}{c} \int d\Omega \Omega \Omega I_\nu(r, \Omega, t)\end{aligned}\tag{18}$$

These quantities represent the zero, first and second order moments of the specific intensity. Physically, they are the energy density, flux and pressure tensor per frequency. We will refer to  $\mathcal{E}_\nu$  as the spectrum. Using these definitions, the radiative transfer equation can be written as a set of coupled partial differential equations

$$\begin{aligned}
\frac{\partial \varepsilon_\nu(r,t)}{\partial t} + \nabla \cdot \phi_\nu(r,t) &= [4\pi\sigma_\nu B_\nu(T) - c\sigma_\nu \varepsilon_\nu(r,t)] \\
\frac{1}{c} \frac{\partial \phi_\nu(r,t)}{\partial t} + c\nabla \cdot \Pi_\nu(r,t) &= -\sigma_\nu \phi_\nu(r,t) \\
\frac{\partial \Pi_\nu(r,t)}{\partial t} + \text{"Fourth Order Moment"} &\dots\dots
\end{aligned} \tag{19}$$

This set of moment equations exhibit the classic closure problem. That is, zero order moments are coupled to first order moments; first order moments are coupled to second order moments, ad infinitum. The central question in all approximations to the radiative transfer equation is coming up with a suitable closure scheme. Three related topics are examined here; variable Eddington factors, diffusion, and the telegraphers equation.

### 3.1. Variable Eddington Factors

Variable Eddington factors is less a closure scheme than a way of recasting the coupled moment equations. Define a tensor quantity, called the Eddington factor, by

$$\chi_\nu(r,t) = \frac{\int d\Omega \Omega \Omega I_\nu(r,\Omega,t)}{\int d\Omega I_\nu(r,\Omega,t)} = \frac{\Pi_\nu(r,t)}{\varepsilon_\nu(r,t)} \tag{20}$$

Physically, this quantity represents the mean of the tensor  $\Omega\Omega$  over all directions weighted by the specific intensity. Substituting equation [20] into equation [19], yields

$$\begin{aligned}
\frac{\partial \varepsilon_\nu(r,t)}{\partial t} + \nabla \cdot \phi_\nu(r,t) &= [4\pi\sigma_\nu B_\nu(T) - c\sigma_\nu \varepsilon_\nu(r,t)] \\
\frac{1}{c} \frac{\partial \phi_\nu(r,t)}{\partial t} + c\nabla \cdot \chi_\nu \varepsilon_\nu(r,t) &= -\sigma_\nu \phi_\nu(r,t)
\end{aligned} \tag{21}$$

Although the above set of equations is seemingly closed, it is in fact not due to the presence of the Eddington factor. However, by writing the coupled moment equations in this fashion, the Eddington factor provides a degree of freedom to characterize the radiation field. For example, it is possible to solve a transport problem every 10 or 20 cycles or so and construct the Eddington factor. This Eddington factor would then be used in the solution of the coupled moment equations and would be updated when the transport solve was performed. This is essentially what is done in ZEUS-2D<sup>13</sup>.

### 3.2. Multi-Group Diffusion

For an isotropic radiation field, the Eddington factor simplifies considerably to

$$\chi_\nu = \frac{1}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (22)$$

If it is assumed that interactions will matter dominate the flow, that is,,

$$\frac{1}{c} \frac{\partial \phi_\nu(r,t)}{\partial t} \ll -\sigma_\nu \phi_\nu(r,t) \quad (23)$$

Then the flux equation becomes simply a Fick's type law  $\phi_\nu = -\frac{c}{3\sigma_\nu} \nabla \varepsilon_\nu$

and the energy density equation becomes the familiar diffusion equation

$$\frac{\partial \varepsilon_\nu(r,t)}{\partial t} - \nabla \cdot \left( \frac{c}{3\sigma_\nu} \nabla \varepsilon_\nu(r,t) \right) = [4\pi\sigma_\nu B_\nu(T) - c\sigma_\nu \varepsilon_\nu(r,t)] \quad (24)$$

This equation along with the material temperature energy balance equation

$$\frac{\partial [\rho C_\nu T(r,t)]}{\partial t} = - \int d\nu \sigma_\nu [B_\nu(T) - \varepsilon_\nu(r,t)] \quad (25)$$

form the basis for the multi-group diffusion equations. It should be remarked that the above analysis is equivalent to assuming that the specific intensity is nearly isotropic

$$I_\nu(r, \Omega, t) = \frac{c\mathcal{E}_\nu(r, t)}{4\pi} + \frac{3}{4\pi}\Omega \cdot \phi_\nu(r, t) \quad (26)$$

To see how this arises physically, assume LTE and ignore scattering and rearrange the terms in the radiative transfer equation such that

$$I_\nu(r, \Omega, t) = B_\nu(T) - \frac{1}{\sigma_\nu} \left[ \frac{1}{c} \frac{\partial I_\nu(r, \Omega, t)}{\partial t} + \Omega \cdot \nabla I_\nu(r, \Omega, t) \right] \quad (27)$$

Perform an expansion to first order in  $1/\sigma_\nu$  to obtain

$$I_\nu(r, \Omega, t) = B_\nu(T) - \frac{1}{\sigma_\nu} \left[ \frac{1}{c} \frac{\partial B_\nu(T)}{\partial t} + \Omega \cdot \nabla B_\nu(T) \right] \quad (28)$$

Note that an expansion in  $1/\sigma_\nu$  is equivalent to an expansion in  $\Omega$ . Constructing the flux directly from equation [28], yields Fick's Law! That is

$$\phi_\nu(r, t) = \int d\Omega \Omega I_\nu(r, \Omega, t) = -\frac{4\pi}{3\sigma_\nu} \nabla B_\nu(T) \quad (29)$$

Therefore, Fick's Law arises directly from an inverse mean free path expansion or equivalently, an expansion in isotropy.

The multi-group diffusion approximation works well where the radiation field is nearly isotropic yet the medium is transparent enough that photon mean free paths are long and therefore, the radiation temperature is not determined locally but rather is determined by sources that are far away. The multi-group diffusion approximation is frequently applied in astrophysical and inertial confinement fusion (ICF) applications. For example, for an ICF capsule bathed in radiation, the multi-group approximation works very well in predicting capsule performance<sup>14</sup>.

It is clear from the functional form of the Planck function that the above equations forms a set of coupled non-linear integro-differential equations. It is this fact that makes their solution a challenge. As in the transport problem, the



presence of a material temperature couples all groups together and it is this fact that makes their solution a challenge. This paper will discuss several methods for their solution including partial temperatures, grey methods such as Lund-Wilson and source iteration, and full matrix methods.

### 3.3. The Planckian Diffusion Equation

An additional simplification arises if it is assumed that the photon spectrum is Planckian. This will occur in optically thick media where the radiation field is determined by many absorptions and re-emissions. In this case the radiation field rapidly becomes Planckian at a temperature not necessarily that of the material. Assuming the radiation spectrum is Planckian with a characteristic temperature  $T_R$  (i.e.  $\varepsilon_\nu = B_\nu(T_R)$ ), equation [24] can be integrated over frequency to yield

$$\frac{\partial T_R^4}{\partial t} - \nabla \cdot \left( \frac{c}{3\sigma_R(T, T_R)} \nabla T_R^4 \right) = c [\sigma_P(T, T) T^4 - \sigma_P(T, T_R) T_R^4] \quad (30)$$

Where the Planck and Rosseland averaged opacities are defined by

$$\sigma_P(T, T_A) = \frac{\int d\nu \sigma_\nu B_\nu(T_A)}{\left( \frac{acT_A^4}{4\pi} \right)} \quad \sigma_R(T, T_A) = \frac{\int d\nu \frac{1}{\sigma_\nu} \frac{\partial B_\nu(T_A)}{\partial T_A}}{\left( \frac{acT_A^4}{\pi} \right)} \quad (31)$$

The material energy balance equation is just

$$\frac{\partial [\rho C_V T(r, t)]}{\partial t} = -ac [\sigma_P(T, T) T^4 - \sigma_P(T, T_R) T_R^4] \quad (32)$$

The Planck opacity  $\sigma_P$  sets the time scale for local energy exchange between matter and radiation based on emission and absorption. The Rosseland opacity  $\sigma_R$  determines average transport properties of the radiation flow.

### 3.4. Some Observations Regarding Diffusion

It is clear that by going to the diffusion approximation, the equations of radiation transport have been fundamentally changed. From a mathematical standpoint the radiative transfer equation is a hyperbolic first order equation requiring for boundary conditions that the initial specific intensity be specified along with for example an incoming value of the specific intensity at a boundary. The diffusion equation however, is a parabolic equation needing two boundary conditions to be specified along with an initial condition. From a physical standpoint, the diffusion approximation is acausal. To see this, consider the multi-group diffusion equation with no sink or source terms

$$\frac{\partial \varepsilon_\nu(r,t)}{\partial t} - \nabla \cdot \left( \frac{c}{3\sigma_\nu} \nabla \varepsilon_\nu(r,t) \right) = 0 \quad (33)$$

This equation describes a signal propagating with a velocity  $V \approx \frac{c}{\sqrt{3\sigma_\nu}}$

which is obviously in violation of the speed of light restriction. It is clear that applying the diffusion approximation to transparent media where a signal can propagate at or near the speed of light is in violation of the approximations used in deriving equation [33] so it is not too surprising that the acausal nature has reared its ugly head. However, all is not lost. Flux limiters, which will be discussed next save the day and allow one to apply the diffusion equation to applications where normally only a transport description based on the radiative transfer equation would do.

Before moving on, a question naturally arises which is; what has been lost by throwing out the time derivative of the flux term? And what happens if it is restored? Consider for simplicity the coupled set of moment equations in a vacuum

$$\begin{aligned} \frac{\partial \varepsilon_\nu(r,t)}{\partial t} + \nabla \cdot \phi_\nu(r,t) &= 0 \\ \frac{1}{c} \frac{\partial \phi_\nu(r,t)}{\partial t} + \frac{c}{3} \nabla \cdot \varepsilon_\nu(r,t) &= 0 \end{aligned} \quad (34)$$

Taking the time derivative of the first equation and using the second equation to close the set yields

$$\frac{\partial^2 \varepsilon_\nu(r, t)}{\partial t^2} - \frac{c^2}{3} \nabla^2 \varepsilon_\nu(r, t) = 0 \quad (35)$$

This is the wave equation that describes ingoing and outgoing waves moving with a velocity of  $c/\sqrt{3}$ . Hence, causality has been restored; however, the finite light speed is too small by a factor of .577. The coupled moment equations in the presence of matter yield the so-called Telegrapher's equation<sup>15</sup>.

### 3.5. Transport Corrected Diffusion: Flux Limiters

In a vacuum, the radiative transfer equation predicts that the energy density propagates with a velocity  $c$  and the flux is given by  $\phi_\nu = c\varepsilon_\nu\Omega_0$ . Where  $\Omega_0$  is the solid angle subtended by the ray. This is the maximum flux allowed physically since it represents photons moving unimpeded. Diffusion predicts a very different behavior. As shown above the flux for the diffusion equation (in the absence of sink and source terms) is  $\phi_\nu = \left( \frac{c}{3\sigma_\nu} \nabla \varepsilon_\nu(r, t) \right)$ . However, it is clear that for very strong gradients and/or when  $\sigma_\nu \rightarrow 0$  the diffusive flux can be greater than the maximum flux allowed.

Jim Wilson first proposed limiting the diffusive flux to correct for the fact that diffusion predicts faster than light flow speeds in transparent media. Define a new flux with a generalized diffusion coefficient which interpolates between the diffusive flux in optically thick media and the transport flux in optically thin media.

$$\begin{aligned}\phi_\nu &= c\Lambda\left(\sigma_\nu, \frac{|\nabla\varepsilon_\nu|}{\varepsilon_\nu}\right)\nabla\varepsilon_\nu \\ \Lambda\left(\sigma_\nu, \frac{|\nabla\varepsilon_\nu|}{\varepsilon_\nu}\right) &= \frac{1}{3\sigma_\nu} \text{ for short photon mean free paths} \quad (36) \\ \Lambda\left(\sigma_\nu, \frac{|\nabla\varepsilon_\nu|}{\varepsilon_\nu}\right) &= \frac{\varepsilon_\nu}{|\nabla\varepsilon_\nu|} \text{ for long photon mean free paths}\end{aligned}$$

The literature<sup>16</sup>, contains a variety of choices for the flux limiter. As an aside, Levermore and Pomraning<sup>17</sup> have derived the flux limiter in a rigorous fashion by performing a Chapman-Enskog like expansion of the transport equation. Their result is not quoted here but interested readers are urged to read their papers.

$$\begin{aligned}\Lambda\left(\sigma_\nu, \frac{|\nabla\varepsilon_\nu|}{\varepsilon_\nu}\right) &= \frac{1}{3\sigma_\nu + \frac{|\nabla\varepsilon_\nu|}{\varepsilon_\nu}} \text{ Wilson sum flux limiter} \\ \Lambda\left(\sigma_\nu, \frac{|\nabla\varepsilon_\nu|}{\varepsilon_\nu}\right) &= \frac{1}{\text{MAX}\left[3\sigma_\nu, \frac{|\nabla\varepsilon_\nu|}{\varepsilon_\nu}\right]} \text{ Maximum flux limiter} \quad (37) \\ \Lambda\left(\sigma_\nu, \frac{|\nabla\varepsilon_\nu|}{\varepsilon_\nu}\right) &= \frac{1}{\left[(3\sigma_\nu)^n + \left(\frac{|\nabla\varepsilon_\nu|}{\varepsilon_\nu}\right)^n\right]^{1/n}} \text{ Larsen flux limiter}\end{aligned}$$

Figure 1 shows a comparison of a variety of choices including the Levermore-Pomraning result  $R = \frac{|\nabla\varepsilon_\nu|}{\sigma_\nu\varepsilon_\nu}$  is plotted on the horizontal axis and  $\Lambda\sigma_\nu$  is plotted on the vertical axis. A slight variation on the sum flux limiter appears in Lund and Wilson and that is, they include a factor of  $\min(1, B_\nu / \varepsilon_\nu)$  in front of the absorption opacity. For hard photons there is no effect but for soft photons the free streaming limit is recovered.

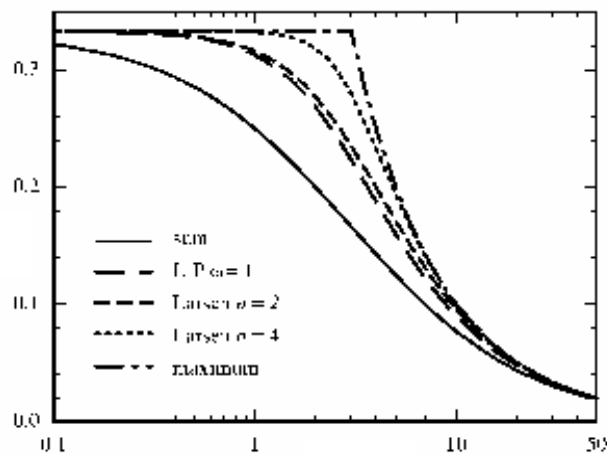


Figure 1. Comparisons of several flux limiter choices (Taken from Olson, Auer, and Hall JQSRT V.56 1996)

Figure 1 shows the Levermore-Pomraning result (long dash-short dash) as derived from the transport equation. Clearly, the Wilson sum flux limiter tends to restrict the flux too much while the MAX flux limiter does not restrict the flux enough. Interestingly, the Larsen flux limiter with  $n=2$  does a very good job of matching the transport solution.

#### 4. Numerical Methods for Diffusion

Complex applications and the nature of the coupled radiation matter equations means that analytic solutions are almost impossible to find. Instead, a numerical approach is sought. The discretization of the multi-group or Planckian diffusion equations means dealing with several issues. The first is time discretization. It is assumed that the coupled radiation and material equations are to be solved implicitly. Explicit time differencing yields a Courant stability condition which severely restricts the time step and therefore is not widely used. There are unconditionally stable explicit schemes for diffusion that have been explored<sup>18</sup>. However, the accuracy of these schemes beyond several Courant time steps is poor. In this paper and in most applications codes, the simple backward Euler difference is used. That is,

$$\frac{\partial A(r, t)}{\partial t} \Rightarrow \frac{(A(r, t_{n+1}) - A(r, t_n))}{\Delta t} \quad \text{where } \Delta t = t_{n+1} - t_n \quad (38)$$

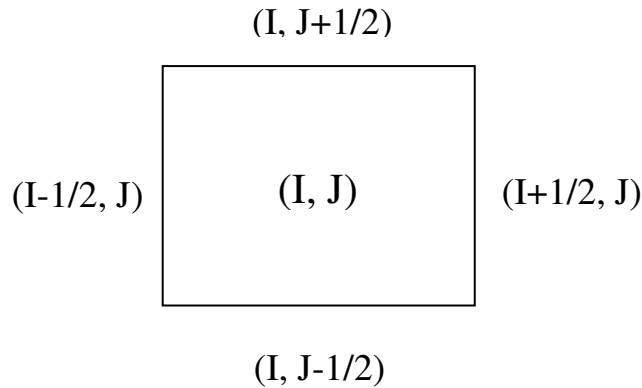
In spite of higher order time differencing schemes such as Crank-Nicholson, the simple first order backward differencing in time is widely used due to its robust behavior in the limit of large time steps. This is due to its non-oscillatory behavior and its ability to recover the static solution as  $\Delta t \rightarrow \infty$ .

The second issue is how to discretize the diffusion operator. Space limitations prevent a lengthy discussion of this important topic; however, an incomplete survey of methods such as finite difference, finite element, and Pert will be discussed. The discussion will be restricted to quadrilateral meshes in 2D. Generalizations to 3D are straightforward. The third issue is the treatment of radiation-matter coupling. Even in the simplest case of Planckian diffusion, this problem arises as the equations governing radiation diffusion are coupled non-linearly to the material temperature. In multi-group diffusion this manifests itself as the material temperature being coupled to all photon frequencies

#### 4.1. *Spatial Discretization*

##### 4.1.1. *Finite Difference*

The type of spatial discretization performed on the diffusion operator is intimately connected with the mesh type. The simplest mesh often encountered is the static (Eulerian) orthogonal type. Finite difference methods are frequently applied in this case yielding a discrete representation of the diffusion or div-grad operator. Consider for example the 2D zone



Assume the independent variable ( $T_R$  or  $\varepsilon_v$ ) is defined at the zone center. Consider the diffusion operator as the gradient of the flux and integrate it over the cell volume of the (i,j) zone. The following result is obtained.

$$\begin{aligned} \frac{1}{V_{i,j}} \int_{(i,j)} dV \nabla \bullet \Phi &= \frac{1}{V_{i,j}} \int_A dA \bullet \Phi \\ &= \frac{1}{V_{i,j}} \left( A_{i+1/2,j} \Phi_{i+1/2,j} + A_{i,j-1/2} \Phi_{i,j-1/2} \right. \\ &\quad \left. + A_{i-1/2,j} \Phi_{i-1/2,j} + A_{i,j+1/2} \Phi_{i,j+1/2} \right) \end{aligned} \quad (39)$$

Where  $A_{ab}$  and  $\Phi_{ab}$  are the surface area and flux, respectively, at the (a, b) face. The flux at the face is just a function of the difference of the zone centered quantities. For example,

$$\begin{aligned} \Phi_{i+1/2,j} &= \frac{D_{i+1/2,j} (\varepsilon_{i+1,j} - \varepsilon_{i,j})}{\Delta x_{i+1/2,j}} \\ \Phi_{i,j+1/2} &= \frac{D_{i,j+1/2} (\varepsilon_{i,j+1} - \varepsilon_{i,j})}{\Delta y_{i,j+1/2}} \end{aligned} \quad (40)$$

Rearranging terms yields

$$\begin{aligned} &\frac{1}{V_{i,j}} \int_{(i,j)} dV \nabla \bullet \Phi \\ &= \frac{1}{V_{i,j}} \left( \alpha_{i+1/2,j} [\varepsilon_{i+1,j} - \varepsilon_{i,j}] - \alpha_{i-1/2,j} [\varepsilon_{i,j} - \varepsilon_{i-1,j}] \right) \\ &\quad + \frac{1}{V_{i,j}} \left( \alpha_{i,j+1/2} [\varepsilon_{i,j+1} - \varepsilon_{i,j}] - \alpha_{i,j-1/2} [\varepsilon_{i,j} - \varepsilon_{i,j-1}] \right) \end{aligned} \quad (41)$$

$$\text{Where } \alpha_{i+1/2,j} = \frac{A_{i+1/2,j} D_{i+1/2,j}}{\Delta x_{i+1/2,j}} \quad \text{and} \quad \alpha_{i,j+1/2} = \frac{A_{i,j+1/2} D_{i,j+1/2}}{\Delta x_{i,j+1/2}}$$

The time dependent diffusion equation in its simplest form (without sink or source terms) is almost ready to be solved. The last issue is the time stamp associated with the diffusion coefficients. In general these quantities depend on the matter temperature which depends on time. The simplest approach is to evaluate all diffusion coefficients (i.e. opacities) at the old time step value. This means that the discrete representation of the diffusion equation is semi-implicit. That is, the solution at the advance time step can be written

$$\hat{\Phi}(t_{n+1}) = \Delta t \hat{M} \Phi(t_{n+1}) + \Delta t \hat{\Phi}(t_n) \quad (42)$$

The diffusion matrix  $M$  is symmetric and positive definite. In addition, it is an  $M$ -matrix meaning that any given diagonal element is the negative sum of the corresponding off-diagonal elements. This has important consequences in that it implies that the solution vector  $\hat{\Phi}$  is positive. The structure of  $M$  is tri-diagonal with sub and super diagonals representative of the 5-point stencil in equation [41]. The time lag of the diffusion coefficient means that some type of time step control must be enforced. This is to ensure accuracy. Finally, the semi-implicit equation for  $\hat{\Phi}$  can be made implicit simply by wrapping equation [41] in an iteration loop. It is of course required that the diffusion coefficient be continually updated with each iteration. Methods that do this will be discussed at the end.

#### 4.1.2. *Finite Element and Pert Operators*

When the mesh is not aligned with the coordinate system, which can occur for example in radiation-hydrodynamic codes using Lagrangian or ALE (Arbitrary Lagrangian Eulerian) methods, complications can arise when constructing a discrete representation of the diffusion operator. In particular, preserving second order accuracy and positive-definiteness on non-uniform grids continues to be a challenge. In this section, two related methods (finite element and Pert), which have found wide use in codes using complex zoning will be discussed. The discussion will be based on variational methods and is based largely on unpublished notes by R. Tipton<sup>19</sup>.

To begin, the time discretized diffusion equation with no sink or source terms can be derived from a variation of the “action”



$$\Theta = \int dV \left[ \frac{1}{2} \frac{\Phi(t_{n+1})^2}{\Delta t} + \frac{1}{2} D |\nabla \Phi(t_{n+1})|^2 - \frac{\Phi(t_{n+1})\Phi(t_n)}{\Delta t} \right] \quad (43)$$

Changing notation slightly in order to minimize indices, the action in discrete form yields

$$\begin{aligned} \Theta = \frac{1}{2} \sum_a C_a \Phi(a, t_{n+1})^2 - \frac{1}{2} \sum_{a,b} K_{ab} \Phi(a, t_{n+1}) \Phi(b, t_{n+1}) \\ - \sum_a E_a \Phi(a, t_{n+1}) \end{aligned} \quad (44)$$

Where  $a$  denotes node number (i.e.  $a=1,2,3,4$ ) and where

$$C_a = \int_{Aa} dV \frac{1}{\Delta t} \quad \text{and} \quad E_a = \int_a dV \frac{\Phi(a, t_n)}{\Delta t} \quad (45)$$

Note that

$$K_{ab} = \frac{\delta^2 \Theta}{\delta \Phi_a \delta \Phi_b} \quad (46)$$

The  $K$  matrix represents the spatial discretization of the diffusion operator. The choice of this operator determines the discrete form of the diffusion operator.

Consider the continuous form of the ‘action’. The finite element method assumes there exists a set of basis functions such that

$$\Phi(x, y) = \sum_a \varphi_a N_a(x, y) \quad (47)$$

Substituting this representation into the div-grad contribution to the action yields

$$\begin{aligned} \Theta_{Div-grad} &= -\frac{1}{2} \sum_{a,b} K_{ab} \varphi_a \varphi_b \\ K_{ab} &= -\int dV D \nabla N_a \cdot \nabla N_b \end{aligned} \quad (48)$$

Tipton’s application of finite elements requires two additional ingredients. One is the fact that finite elements like to have the unknowns live at the nodes whereas it has been assumed here that the unknowns live at the zone centers. Tipton defines a dual mesh whose nodes live at the zone centers of the regular

mesh. In addition, instead of using quadrilateral basis functions, he splits the quadrilateral into triangles and uses basis functions associated with them. Since there are two ways in 2D of splitting a quadrilateral into triangles, the K matrix is constructed as the result of averaging the two splittings. For a triangular basis function in an r-z cylindrical geometry,

$$K_{ab} = \frac{-2\pi r D s_a \bullet s_b}{2|s_a \times s_b|} \quad (49)$$

Where  $s_a$  is a vector defined on the node opposite node a. If the other two nodes are denoted by b and c, where b and c do not equal a, then  $s_a$  lies normal to the bc leg and its magnitude is equal to the length of bc. Defining  $\theta_{ab}$  as the angle opposite nodes a and b, the K matrix becomes for a single triangular basis function,

$$\begin{aligned} K_{ab} &= -\frac{2\pi r}{2} D \cot(\theta_{ab}) \quad \text{in r - z geometry} \\ K_{ab} &= -\frac{1}{2} D \cot(\theta_{ab}) \quad \text{in planar geometry} \end{aligned} \quad (50)$$

Whereas the finite element described above relies on triangular basis functions, the Pert<sup>20</sup> operator relies on bi-linear quadrilateral elements with two point quadrature. A single point quadrature with bi-linear quadrilateral elements will not work due to the fact that for square or rhomboid meshes, couplings between neighboring zones occurs only through corner couplings and not through faces. Hence, the mesh can develop the so-called checkerboard instability.

Consider the mesh in logical coordinates. Consider a specific zone whose nodes are identified with the iso-parametric coordinates  $(\xi, \eta) = (\pm 1, \pm 1)$ . The bi-linear basis functions are simply

$$\begin{aligned} N_1 &= \frac{1}{4}(1 + \xi)(1 - \eta) & N_2 &= \frac{1}{4}(1 + \xi)(1 + \eta) \\ N_3 &= \frac{1}{4}(1 - \xi)(1 + \eta) & N_4 &= \frac{1}{4}(1 - \xi)(1 - \eta) \end{aligned} \quad (51)$$

The div-grad contribution to the action comes about by substituting equation [51] into equation [48]. The result is

$$\Theta_{Div-grad} = 2\pi \int \frac{d\xi d\eta Dr}{2J} \left[ g_{\xi\xi} \left( \frac{\partial\varphi}{\partial\xi} \right)^2 + g_{\eta\eta} \left( \frac{\partial\varphi}{\partial\eta} \right)^2 + g_{\xi\eta} \left( \frac{\partial\varphi}{\partial\xi} \frac{\partial\varphi}{\partial\eta} \right) \right]$$

The g's are the metric tensor and it represents the coordinate transformation between the logical coordinates and physical coordinates. For example,

$$g_{\xi\eta} = \frac{\partial x}{\partial\xi} \frac{\partial x}{\partial\eta} + \frac{\partial y}{\partial\xi} \frac{\partial y}{\partial\eta} \quad (52)$$

The Pert representation for the diffusion operator comes from a two-point quadrature approximation to the integral in the above equation for  $\Theta_{Div-grad}$ .

This is done by taking the dual mesh quadrilateral and decomposing it into four equal area sections. Therefore, the K matrix becomes

$$\begin{pmatrix} -\frac{4\pi Dr}{A} (g_{\xi\xi} + g_{\eta\eta} + g_{\xi\eta}) & \frac{4\pi Dr}{A} g_{\eta\eta} & \frac{4\pi Dr}{A} g_{\xi\eta} & \frac{4\pi Dr}{A} g_{\xi\xi} \\ \frac{4\pi Dr}{A} g_{\eta\eta} & -\frac{4\pi Dr}{A} (g_{\xi\xi} + g_{\eta\eta} + g_{\xi\eta}) & \frac{4\pi Dr}{A} g_{\xi\xi} & -\frac{4\pi Dr}{A} g_{\xi\eta} \\ \frac{4\pi Dr}{A} g_{\xi\eta} & \frac{4\pi Dr}{A} g_{\xi\xi} & -\frac{4\pi Dr}{A} (g_{\xi\xi} + g_{\eta\eta} + g_{\xi\eta}) & \frac{4\pi Dr}{A} g_{\eta\eta} \\ \frac{4\pi Dr}{A} g_{\xi\xi} & -\frac{4\pi Dr}{A} g_{\xi\eta} & \frac{4\pi Dr}{A} g_{\eta\eta} & -\frac{4\pi Dr}{A} (g_{\xi\xi} + g_{\eta\eta} + g_{\xi\eta}) \end{pmatrix}$$

#### 4.1.3. Local Support Operators

Recently, Jim Morel<sup>25</sup> and co-workers have developed a promising 2D and 3D diffusion discretization scheme based on local support operators. Like the methods discussed above, it yields a sparse matrix representation for the div-grad operator. The method has definite advantages. It is second order accurate on distorted meshes, rigorously treats material discontinuities, and has a symmetric positive definite matrix. The disadvantage of the method is that it requires face center as well as cell center unknowns. There is some subtlety regarding implementing flux limiters into the local support scheme. However, David Miller<sup>26</sup> has done this successfully

#### 4.2. Multi-group and Planckian Diffusion

To summarize, the solution of the material temperature and spectrum needs to be found from the following coupled equations

$$\frac{\partial \varepsilon_\nu(r,t)}{\partial t} - \nabla \cdot \left( \frac{c}{3\sigma_\nu} \nabla \varepsilon_\nu(r,t) \right) = [4\pi\sigma_\nu B_\nu(T) - c\sigma_\nu \varepsilon_\nu(r,t)] \quad (53)$$

$$\frac{\partial [\rho C_\nu T(r,t)]}{\partial t} = - \int d\nu \sigma_\nu [B_\nu(T) - \varepsilon_\nu(r,t)] \quad (54)$$

The inherent difficulty in solving this set of equations is that even though the equation obeyed by the spectrum looks like it is independent of frequency group coupling, the material equation requires knowledge of the spectrum and emission at all frequencies. Hence, the emission function also requires knowledge of the spectrum at all frequencies. Therefore, the equation for the spectrum is inherently non-linear due to an effective group to group coupling. The partial temperature and grey methods are examples of techniques which have found success in dealing with this problem. These are discussed next. Planckian diffusion can be thought of as a subset of multi-group diffusion. Physically of course, Planckian diffusion arises from assuming the spectrum is Planckian and then integrating over all frequency. Numerically, we will see that in the partial temperature scheme it can be merely thought of as a special case.

In all of these methods, heat capacities and opacities are time lagged so even though the solution methods are not fully implicit. In addition, due to this fact, all of the methods considered will require some sort of time step control on the material and/or radiation temperature. As we will see, the partial temperature method requires a very specific type of time step control. In addition, the methods here should be thought of in the wider context of a multi-physics code. Traditionally, physics packages are operator split. Whether this is done first order or second order in time, this fact alone limits the accuracy even in cases where the radiation package is solved fully implicitly. Examples of fully implicit methods will be discussed in section 4.2.3.

The implicit time differencing and the spatial discretization mean that the coupled radiation-material equations will form a system of equations. In general these systems will be non-linear. Due to a variety of techniques, to be discussed next, this non-linear set can be approximated by a system of linear equations. Therefore, linear solvers play an important role in the methods presented here. It

is due to advances in linear solvers and their preconditioners that have made radiation diffusion problems in more complex geometries and in 3D possible. This is due in no small part to scaleable (both in the problem size and parallel sense algorithms<sup>21</sup>).

#### 4.2.1. Multi-group Diffusion: Partial Temperature

The coupled set of equations is differenced implicitly assuming that the heat capacities and opacities are evaluated at the old time step. Ignoring for the moment issues related to spatial discretization and setting all internal and external sources to zero, we obtain

$$\frac{(\varepsilon_v^{n+1} - \varepsilon_v^n)}{c\Delta t} = \nabla \cdot \left( \frac{1}{3\sigma_v} \nabla \varepsilon_v^{n+1} \right) + \sigma_v (4\pi B_v(T^{n+1}) - \varepsilon_v^{n+1}) \quad (55)$$

$$\rho C_v \frac{(T^{n+1} - T^n)}{\Delta t} = - \sum_v \Delta v \sigma_v (4\pi B_v(T^{n+1}) - \varepsilon_v^{n+1}) \quad (56)$$

The summation is taken over the group index and runs from zero to  $N_g$ . Notice that the Planckian diffusion scheme is just a special case of  $v = 1$  where  $B_v(T) \propto T^4$ . Now assume there exist a partial material temperature contribution for each group. That is,

$$T^{n+1} - T^n = \sum_v [\tau_{v+1}^{n+1} - \tau_v^{n+1}] \quad \text{where } \tau_{N_g+1}^{n+1} = T^{n+1} \quad \text{and } \tau_0^{n+1} = T^n$$

$$\frac{\rho C_v}{\Delta t} [\tau_{v+1}^{n+1} - \tau_v^{n+1}] = -(\sigma_v (4\pi B_v(T^{n+1}) - \varepsilon_v^{n+1})) \quad (57)$$

We now approximate the emission or Planck function as if it were coming from each group. That is,

$$B_v(T^{n+1}) \Rightarrow B_v(\tau_{v+1}^{n+1}) \approx B_v(\tau_v^{n+1}) + B'_v(\tau_{v+1}^{n+1}) [\tau_{v+1}^{n+1} - \tau_v^{n+1}] \quad (58)$$

The following linear system results,

$$[\tau_{v+1}^{n+1} - \tau_v^{n+1}] = \frac{(-4\pi\sigma_v B_v(\tau_v^{n+1}) + \sigma_v \varepsilon_v^{n+1})}{\left( \frac{\rho C_v}{\Delta t} + 4\pi\sigma_v B'_v(\tau_{v+1}^{n+1}) \right)} \quad (59)$$

Upon approximating the emission or Planck function as before, the multi-group photon equation becomes

$$\begin{aligned} \frac{(\varepsilon_\nu^{n+1} - \varepsilon_\nu^n)}{c\Delta t} = \nabla \cdot \left( \frac{1}{3\sigma_\nu} \nabla \varepsilon_\nu^{n+1} \right) \\ + \sigma_\nu \left( 4\pi B_\nu(\tau_\nu^{n+1}) + 4\pi B'_\nu(\tau_{\nu+1}^{n+1}) [\tau_{\nu+1}^{n+1} - \tau_\nu^{n+1}] - \varepsilon_\nu^{n+1} \right) \end{aligned} \quad (60)$$

Substituting equation [59] into equation [60] yields a linear implicit solution for  $\varepsilon_\nu^{n+1}$ . This may be solved via a variety of preconditioned linear solvers such as multi-grid preconditioned conjugate gradient<sup>21</sup>.

A word of caution concerning partial temperatures is in order. The method is stable and robust. However, it can suffer in accuracy unless the partial temperature swings for each group from cycle to cycle are limited via a time step control. In Planckian diffusion, since there is just one group, the temperature swing is just the change in matter temperature from cycle to cycle.

#### 4.2.2. Multi-group Diffusion: Iterative Grey Methods

There exists another class of methods that attempt to solve the matter-radiation coupling problem via an iterative procedure. These methods require a grey diffusion accelerator in order to speed-up convergence. We will discuss the fully implicit method of Lund-Wilson<sup>22</sup> and its variant due to Eppley<sup>22</sup>. We will then discuss a semi-implicit method due to Morel, Larsen, and Matzen<sup>22</sup> and its fully implicit extension due to Graziani<sup>22</sup>.

The Lund-Wilson method replaces the set of multi-group diffusion equations with the set

$$\begin{aligned} \frac{(\varepsilon_\nu^{l+1} - \varepsilon_\nu^n)}{c\Delta t} = \nabla \cdot \left( \frac{1}{3\sigma_\nu} \nabla \varepsilon_\nu^{l+1} \right) \\ + \sigma_\nu \left( 4\pi [B_\nu(T^l) + B'_\nu(T^l)(T^{l+1} - T^l)] - \varepsilon_\nu^{l+1} \right) \end{aligned} \quad (61)$$

$$\begin{aligned} \rho C_\nu \frac{(T^{l+1} - T^n)}{\Delta t} = \\ - \sum_\nu \Delta \nu \sigma_\nu \left( 4\pi [B_\nu(T^l) + B'_\nu(T^l)(T^{l+1} - T^l)] - \varepsilon_\nu^{l+1} \right) \end{aligned} \quad (62)$$

The index I is an iteration index where I=0 corresponds to the old time step values. Note that upon convergence,  $T^{I+1} \rightarrow T^I$  and the coupled multi-group equations are recovered. Define the grey coefficients,

$$\begin{aligned}\Sigma_1^I &= 4\pi\Delta t \sum_{\nu} \Delta\nu\sigma_{\nu} B_{\nu}(T^I) \\ \Sigma_2^I &= 4\pi\Delta t \sum_{\nu} \Delta\nu\sigma_{\nu} B'_{\nu}(T^I) \\ E^I &= \frac{1}{c} \sum_{\nu} \Delta\nu\epsilon_{\nu}^I \\ \alpha^I &= \frac{\Delta t \sum_{\nu} \Delta\nu\sigma_{\nu}\epsilon_{\nu}^I}{E^I}\end{aligned}\quad (63)$$

$E^I$  is proportional to an effective radiation temperature raised to the fourth power.  $\alpha^I$  is spectrum averaged opacity. In the matter equation, we make the approximation

$$\Delta t \sum_{\nu} \Delta\nu\sigma_{\nu}\epsilon_{\nu}^{I+1} = \left( \frac{\Delta t \sum_{\nu} \Delta\nu\sigma_{\nu}\epsilon_{\nu}^{I+1}}{E^{I+1}} \right) E^{I+1} \approx \alpha^I E^{I+1} \quad (64)$$

The latter approximation is exact upon convergence. Using the grey coefficients, the, matter equation becomes

$$T^{I+1} - T^I = \frac{\left[ (T^n - T^I) - \frac{[\Sigma_1^I - \alpha^I E^{I+1}]}{\rho C_V} \right]}{\left[ 1 + \frac{\Sigma_2^I}{\rho C_V} \right]} \quad (65)$$

This equation can be solved provided we know  $E^{I+1}$ . This quantity is obtained by summing the multi-group photon equation over frequency. There are two ways of doing this depending on how the diffusion coefficients are averaged.

Consider the diffusion contribution to the multi-group photon equation. Also, we assume a simple 1D finite difference representation of the div-grad operator. We can form an equation for  $E^{I+1}$  by summing the photon multi-group equation over frequency group.

$$\int d\nu \nabla \cdot \left( \frac{1}{3\sigma_\nu} \nabla \varepsilon_\nu^{I+1} \right) \Rightarrow \sum_\nu \Delta \nu D_\nu (i+1/2) [\varepsilon_\nu^{I+1}(i+1) - \varepsilon_\nu^{I+1}(i)] - \sum_\nu \Delta \nu D_\nu (i-1/2) [\varepsilon_\nu^{I+1}(i) - \varepsilon_\nu^{I+1}(i-1)] \quad (66)$$

Lund and Wilson approximate this expression as follows,

$$\begin{aligned} & \sum_\nu \Delta \nu D_\nu (i+1/2) [\varepsilon_\nu^{I+1}(i+1) - \varepsilon_\nu^{I+1}(i)] \\ &= \left( \frac{\sum_\nu \Delta \nu D_\nu (i+1/2) [\varepsilon_\nu^{I+1}(i+1) - \varepsilon_\nu^{I+1}(i)]}{\sum_\nu \Delta \nu [\varepsilon_\nu^{I+1}(i+1) - \varepsilon_\nu^{I+1}(i)]} \right) [E^{I+1}(i+1) - E^{I+1}(i)] \quad (67) \\ &\equiv \gamma^{I+1}(i+1/2) [E^{I+1}(i+1) - E^{I+1}(i)] \\ &\approx \gamma^I(i+1/2) [E^{I+1}(i+1) - E^{I+1}(i)] \end{aligned}$$

The equation for  $E^{I+1}$  becomes,

$$\begin{aligned} E^{I+1} - E^I &= c \Sigma_1^I + c \Sigma_2^I [T^{I+1} - T^I] - c \alpha^I E^{I+1}(i) \\ &+ \gamma^I(i+1/2) [E^{I+1}(i+1) - E^{I+1}(i)] - \gamma^I(i-1/2) [E^{I+1}(i) - E^{I+1}(i-1)] \quad (68) \end{aligned}$$

This is the so-called grey equation. The above equation is a tri-diagonal equation for  $E^{I+1}$  which can be solved via back-substitution methods. In two or three dimensions, the above equation is a matrix which can be solved via preconditioned conjugate gradient methods<sup>21</sup>. In solving the coupled multi-group equations, the following steps must be performed

1. Evaluate the grey coefficients [63] using the last available spectrum
2. Solve the grey equation for  $E^{I+1}$
3. Compute the updated temperature  $T^{I+1}$
4. Knowing  $T^{I+1}$ , compute the updated spectrum  $\varepsilon_\nu^{I+1}$
5. Is the matter temperature converged  $|T^{I+1} - T^I| < \delta$  ?
6. If no, repeat steps 1-5 using the latest spectrum to compute the grey coefficients.



7. If yes, values for matter temperature and spectrum are accepted.  
8.

The method due to Lund and Wilson seems to work well in most cases. It does have one drawback however. The grey averaged diffusion coefficients are not guaranteed to be positive. This can cause havoc for matrix solvers. Eppley's method tries to circumvent this problem by defining new grey averaged diffusion coefficients. In Eppley's scheme, instead of one type of grey diffusion coefficient, he defines two. Namely,

$$\begin{aligned}
& \sum_{\nu} \Delta \nu D_{\nu}(i+1/2) \varepsilon_{\nu}^{I+1}(i+1) \\
&= \left( \frac{\sum_{\nu} \Delta \nu D_{\nu}(i+1/2) \varepsilon_{\nu}^{I+1}(i+1)}{\sum_{\nu} \Delta \nu \varepsilon_{\nu}^{I+1}(i+1)} \right) E^{I+1}(i+1) \\
&\equiv \delta^{I+1}(i+1/2) E^{I+1}(i+1) \approx \delta^I(i+1/2) E^{I+1}(i+1) \quad (69)
\end{aligned}$$

$$\begin{aligned}
& \sum_{\nu} \Delta \nu D_{\nu}(i+1/2) \varepsilon_{\nu}^{I+1}(i) \\
&= \left( \frac{\sum_{\nu} \Delta \nu D_{\nu}(i+1/2) \varepsilon_{\nu}^{I+1}(i)}{\sum_{\nu} \Delta \nu \varepsilon_{\nu}^{I+1}(i)} \right) E^{I+1}(i) \equiv \Delta^{I+1}(i+1/2) E^{I+1}(i) \quad (70) \\
&\approx \Delta^I(i+1/2) E^{I+1}(i)
\end{aligned}$$

The grey equation is now slightly modified from before,

$$\begin{aligned}
& E^{I+1} - E^I = c \Sigma_1^I + c \Sigma_2^I [T^{I+1} - T^I] - c \alpha^I E^{I+1}(i) \\
&+ \delta^I(i+1/2) E^{I+1}(i+1) - \Delta^I(i+1/2) E^{I+1}(i) \\
&- [\delta^I(i-1/2) E^{I+1}(i) - \Delta^I(i-1/2) E^{I+1}(i-1)]
\end{aligned}$$

Eppley's method produces positive definite grey diffusion coefficients. However, the cost is a non-symmetric matrix for the grey equation. In practice, the robustness of Eppley's variant of Lund and Wilson seems to work well. Methods such as GMRES are satisfactory for solving the non-symmetric grey equation. Methods that attempt to solve the grey equation by splitting the matrix into symmetric and asymmetric contributions and including the asymmetric parts in the overall iteration loop do not seem to be robust. Although they have the advantage of allowing the one to use symmetric matrix solver methods, the method occasionally fails to converge.

Even though it has not been explicitly mentioned as such, the solution of the grey equation in the Lund-Wilson and Eppley variant accelerates the iterative process. Without the grey solution step, the number of iterations can grow into the thousands in regimes of the problem where radiation and matter are tightly coupled. This is certainly true in the next method where a grey accelerator is explicitly introduced.

The starting point of the source iteration method of Morel, Larsen, and Matzen<sup>22</sup> is to expand the Planck or emission function, evaluated at the updated temperature, about its value of the previous time steps' temperature. That is

$$B_\nu(T^{n+1}) \approx B_\nu(T^n) + B_\nu(T^n)[T^{n+1} - T^n] \quad (72)$$

This is performed in both the photon and the material energy balance equations. By doing this expansion, the coupled set of multi-group equations is effectively linearized. Substituting the above equation into the material energy balance equation and solving for  $T^{n+1} - T^n$  yields

$$T^{n+1} - T^n = \frac{\sum_\nu \Delta v \sigma_\nu [\varepsilon_\nu^{n+1} - 4\pi B_\nu(T^n)]}{\left[ \frac{\rho C_\nu}{\Delta t} + \sum_\nu \Delta v \sigma_\nu B'_\nu(T^n) \right]} \quad (73)$$

Substituting this expression into the photon equation yields

$$\begin{aligned} -\nabla \cdot \left( \frac{1}{3\sigma_\nu} \nabla \varepsilon_\nu^{n+1} \right) + \left( \frac{1}{c\Delta t} + \sigma_\nu \right) \varepsilon_\nu^{n+1} \\ = Q_\nu^n + \frac{\varepsilon_\nu^n}{c\Delta t} + \eta \chi_\nu \sum_\nu \Delta v \sigma_\nu \varepsilon_\nu^{n+1} \end{aligned} \quad (74)$$

Where a new set of grey coefficients arises,

$$\begin{aligned}
\eta &= \frac{4\pi \sum_{\nu} \Delta \nu \sigma_{\nu} B'_{\nu}(T^n)}{\frac{\rho C_{\nu}}{\Delta t} + 4\pi \sum_{\nu} \Delta \nu \sigma_{\nu} B'_{\nu}(T^n)} \\
\chi_{\nu} &= \frac{\Delta \nu \sigma_{\nu} B'_{\nu}(T^n)}{\sum_{\nu} \Delta \nu \sigma_{\nu} B'_{\nu}(T^n)} \\
Q_{\nu} &= \sigma_{\nu} B_{\nu}(T^n) - 4\pi \eta \chi_{\nu} \sum_{\nu} \Delta \nu \sigma_{\nu} B_{\nu}(T^n)
\end{aligned} \tag{75}$$

This is a linearized form for the photon multi-group equation. At this point the group to group coupling still exists. The source iteration process involves replacing the above by

$$\begin{aligned}
-\nabla \cdot \left( \frac{1}{3\sigma_{\nu}} \nabla \varepsilon_{\nu}^{I+1} \right) + \left( \frac{1}{c\Delta t} + \sigma_{\nu} \right) \varepsilon_{\nu}^{I+1} \\
= Q_{\nu}^I + \frac{\varepsilon_{\nu}^n}{c\Delta t} + \eta \chi_{\nu} \sum_{\nu} \Delta \nu \sigma_{\nu} \varepsilon_{\nu}^I
\end{aligned} \tag{76}$$

Therefore, when  $\varepsilon_{\nu}^{I+1} \approx \varepsilon_{\nu}^I$ , the above equation converges to the linearized form given by equation [74]. Note that whereas with the Lund-Wilson method we converged on matter temperature, here we converge on the spectrum. In practice, a convergence criteria based on converging the spectrum over all groups is not necessary. It seems to be sufficient to converge on the radiation temperature or the radiation energy density. The benefit is a smaller number of iterations with very little loss of accuracy. Therefore, the procedure is

1. Evaluate the grey coefficients using the old time step values for temperature and spectrum
2. Solve the photon equation [76] via iteration
3. Once  $\varepsilon_{\nu}^{I+1}$  is converged, compute the material temperature

Notice that the grey coefficients are evaluated once, at the start of the cycle, as opposed to Lund-Wilson where they are evaluated at each iteration. . In addition, here the material temperature is evaluated once the spectrum is converged

whereas in Lund-Wilson, it is updated every iteration. In practice the source iteration method works well except the number of iterations to converge the photon equation rises steeply in optically thick regimes. In order to correct this deficiency, the iteration process is accelerated via a grey equation.

In the paper by Morel, Larsen, and Matzen, besides the spectrum, they also define a quantity which is the difference of the spectrum between the exact solution and the latest guess from iterate I. We denote this quantity as,

$$\delta\epsilon_\nu = \epsilon_\nu - \epsilon_\nu^I \quad (77)$$

In terms of this variable, the multi-group photon equation becomes

$$\begin{aligned} -\nabla \cdot \left( \frac{1}{3\sigma_\nu} \nabla \delta\epsilon_\nu^I \right) + \left( \frac{1}{c\Delta t} + \sigma_\nu \right) \delta\epsilon_\nu^I \\ = \eta\chi_\nu \sum_\nu \Delta\nu\sigma_\nu \left[ \delta\epsilon_\nu^I + \epsilon_\nu^I - \epsilon_\nu^I \right] \end{aligned} \quad (78)$$

In this method, the grey equation is derived by assuming that the multi-group spectrum is given by the equilibrium spectrum. The equilibrium spectrum in turn is given by the solution to equation [74] where the gradient term vanishes. That is,

$$\delta\epsilon_\nu|_{eq} = \frac{E^I \chi_\nu \Gamma}{\frac{1}{c\Delta t} + \sigma_\nu} \quad (79)$$

Where the grey coefficient is defined by

$$\Gamma = \left[ \sum_\nu \Delta\nu \left( \frac{\chi_\nu}{\frac{1}{c\Delta t} + \sigma_\nu} \right) \right]^{-1} \quad (80)$$

Substituting this expression into equation [78] and integrating over groups yields,

$$\begin{aligned}
-\nabla \cdot (\langle D \rangle \nabla E^I + \langle \bar{D} \rangle E^I) + \left[ \frac{1}{c\Delta t} + (1-\eta)H \right] E^I \\
= \eta \sum_{\nu} \sigma_{\nu} (\varepsilon_{\nu}^{I+1} - \varepsilon_{\nu}^{I-1})
\end{aligned} \tag{81}$$

The grey coefficients are given by

$$\langle D \rangle = H \sum_{\nu} \Delta \nu \left( \frac{\chi_{\nu}}{\frac{1}{c\Delta t} + \sigma_{\nu}} \right) \left( \frac{1}{3\sigma_{\nu}} \right) \tag{82}$$

$$\langle \bar{D} \rangle = H \sum_{\nu} \Delta \nu \left( \frac{1}{3\sigma_{\nu}} \right) \nabla \left( \frac{\Gamma \chi_{\nu}}{\frac{1}{c\Delta t} + \sigma_{\nu}} \right) \tag{82}$$

$$H = \Gamma \sum_{\nu} \Delta \nu \frac{\chi_{\nu} \sigma_{\nu}}{\frac{1}{c\Delta t} + \sigma_{\nu}}$$

The multi-group solution procedure is similar to the unaccelerated procedure

1. Evaluate the grey coefficients using the old time step information
2. Update the spectrum using the photon multi-group equation
3. Solve the grey equation for the integrated spectrum
4. Correct the spectrum by adding the correction

$$\text{term } \delta \varepsilon_{\nu} \Big|_{eq} = \frac{E^I \chi_{\nu} \Gamma}{\frac{1}{c\Delta t} + \sigma_{\nu}} \text{ to the result from step 2}$$

5. Repeat steps 2-4 until convergence is reached

It would seem that here again the convergence criterion must be based on converging the spectrum in all groups. In practice, however, it turns out to be sufficient to converge on the radiation temperature or the radiation energy density. The number of iterations can be substantially reduced with little loss of accuracy. A more important issue is the grey equation. Unfortunately, the grey

equation, like Eppley, does not have a symmetric matrix. This is due to the  $\langle \tilde{D} \rangle$  term. Without any justification, dropping this term does not seem to harm the acceleration process significantly. This is frequently done in practice. However, this issue needs to be looked at closer.

There exists a variant of the Morel, Larsen, and Matzen method due to Graziani that solves the non-linear coupled multi-group equations. The starting point is to expand the emission or Planck function not around the old time step but rather the old iterate. That is,

$$B_\nu(T^{I+1}) \approx B_\nu(T^I) + B_\nu(T^I)[T^{I+1} - T^I] \quad (83)$$

We go through the same algebra as previously done however, at the end we end up with a slightly different equation for the spectrum.

$$\begin{aligned} -\nabla \cdot \left( \frac{1}{3\sigma_\nu} \nabla \varepsilon_\nu^{I+1} \right) + \left( \frac{1}{c\Delta t} + \sigma_\nu \right) \varepsilon_\nu^{I+1} \\ = \tilde{Q}_\nu^I + \frac{\varepsilon_\nu^n}{c\Delta t} + \eta\chi_\nu \sum_\nu \Delta\nu\sigma_\nu \varepsilon_\nu^I \end{aligned} \quad (84)$$

The grey coefficients are defined by

$$\begin{aligned} \eta &= \frac{4\pi \sum_\nu \Delta\nu\sigma_\nu B'_\nu(T^n)}{\frac{\rho C_\nu}{\Delta t} + 4\pi \sum_\nu \Delta\nu\sigma_\nu B'_\nu(T^n)} \\ \chi_\nu &= \frac{\Delta\nu\sigma_\nu B'_\nu(T^n)}{\sum_\nu \Delta\nu\sigma_\nu B'_\nu(T^n)} \\ \tilde{Q}_\nu &= \sigma_\nu B_\nu(T^n) - 4\pi\eta\chi_\nu \left\{ \frac{\rho C_\nu (T^I - T^n)}{\Delta t} - \sum_\nu \Delta\nu\sigma_\nu B_\nu(T^n) \right\} \end{aligned} \quad (84)$$

Therefore, the ‘‘Q’’ term is slightly modified. The procedure for solving the non-linear multi-group variant follows the same steps as the linear method. However, the convergence criterion is now based on

$$|T^{I+1} - T^I| < \delta$$

The relative merits of solving the non-linear variant of Morel, Larsen, and Matzen has not been investigated. Further work needs to be done.

#### 4.2.3. *Comments on Full Matrix Methods*

With the advent of increased memory and the progress in preconditioned linear solvers, the idea for fully implicit methods has started looking like an attractive alternative the methods discussed above. We give here some reference to a sampling of the research in this area. Rider, Knoll, and Olson<sup>23</sup> used Newton-Krylov methods along with multi-grid preconditioning to 1D and 2D one temperature Planckian diffusion. They showed that the fully implicit method gave increased accuracy over the usual semi-implicit method where opacities and heat capacities were lagged. Simultaneously, building on earlier work using the ODE integrator methods of Axelrod, Dubois, and Rhodes<sup>24</sup>, Brown, Chang, Graziani, and Woodward<sup>24</sup> applied these techniques to 3D multi-group diffusion. Mousseau, Knoll, and Rider<sup>24</sup> extended their earlier work to two temperature Planckian diffusion. Recently, Brown, Shumaker, and Woodward<sup>24</sup> considered fully implicit versus semi-implicit methods where tabulated opacities are used and external sources coming from thermonuclear fusion are included. This latter issue is important as the source term in the radiation equations coming from fusion source terms is a very strong function of temperature and hence places a constraint of time steps and accuracy. Their conclusions were that a fully implicit method can achieve more accurate solutions than semi-implicit methods at a cost comparable to semi-implicit methods. In addition, their method scales very well on parallel machines.

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