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Generalization of Richardson-Gaudin models to rank-2 algebras*

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Abstract.

A generalization of Richardson-Gaudin models to the rank-2 SO(5) and SO(3,2) algebras is used to describe systems of two kinds of fermions or bosons interacting through a pairing force. They are applied to the proton-neutron isovector pairing model and to the Interacting Boson Model 2, in the transition from vibration to gamma-soft nuclei, respectively. In both cases, the integrals of motion and their eigenvalues are obtained.

1 Introduction

The pairing interaction has been used to describe many properties of strongly correlated many-body quantum systems. In the early sixties, Richardson [1] showed how to exactly solve the pure pairing hamiltonian for fermions and bosons including non-degenerate single-particle orbits. Independently, Cambiaggio, Rivas and Saraceno [2] demonstrated that the pairing hamiltonian was

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integrable, which means that as many integrals of motion as degrees of freedom can be found. In 2001, Dukelsky, Esebbag and Schuck [3] showed how to generalize Richardson's solution making use of analogous work by Gaudin [4] for spin models. Since then, the Richardson-Gaudin (RG) models have been applied to a wide variety of systems in nuclear, condensed-matter and atomic and molecular physics. They have the advantage that they can be used to exactly obtain such physical quantities as energies and occupation probabilities beyond the diagonalization limits, while furthermore serving as tests or initial guesses for other many-body approaches. Some specific examples of applications have been to ultrasmall superconduction grains [5], nuclear superconductivity, Bose-Einstein condensates [6] and to mixed systems involving atoms coupled to molecular dimers in the presence of a Feshbach resonance [7].

All these models are based on several copies of the rank-1 SU(2) and SU(1,1) algebras, each copy representing an orbit. The integrals of motion and their eigenfunctions and eigenvalues depend on a set of parameters (pair energies) that satisfy a system of non-linear coupled equations. In 2002, Asorey, Falcetto and Sierra [8] found a set of L integrals of motion and their corresponding eigenvalues for a system of L copies of a Lie algebra of arbitrary rank r. They depend on L plus r free parameters, and can be written in terms of as many families of spectral parameters as the rank of the algebra. These families of spectral parameters satisfy a system of generalized Richardson equations. Independently, Usveridse [9] found the eigenfunctions by making use of the Gaudin algebras.

In this presentation, we study the rank-2 algebras SO(5) and SO(3,2). For a suitable representation of the generators of the two algebras, we obtain the integrals of motion, their eigenvalues, the associated eigenfunctions and the generalized Richardson equations for the two families of spectral parameters. A specific linear combination of the integrals of motion together with a suitable choice for the free parameters leads us to two hamiltonians that have been used in models of nuclear physics, the proton-neutron isovector pairing model and the Interacting Boson Model-2 in the transition from vibrational to gamma-unstable nuclei. In both cases, the hamiltonans that result permit the inclusion of symmetry breaking terms, isospin symmetry in the SO(5) model and F-spin symmetry in the SO(3,2) model. For both of the models, we obtain the results for states with an arbitrary number of unpaired particles. In the SO(5) case, we report a study of the dependence of the spectral parameters on the isospin symmetry breaking term in the T=0,1 and 2 channels, and present some results for the nucleus ^{64}Ge assuming a ^{40}Ca core and a $pf + g_{9/2}$ valence space. In the SO(3,2) case we report a study of the effect on the spectra of adding a g or an f boson.

2 Integrals of motion and its eigenvalues for a system of L copies of a rank-r Lie Algebra

The RG models are based on several copies of a Lie Algebra. For the i-th copy we can write the generators in the Cartan decomposition. On one hand, this includes the operators that commute with one another (weight operators), which form the so called Cartan subalgebra. The number of these operators (which we will denote by H^a) is the rank of the group. The rest of the operators, E^α, are expressed in such a way that they satisfy the commutation relationship $[H_i^a, E_i^\alpha] = \alpha^a E_i^\alpha$. These operators are called the ladder operators; half of them are raising operators $E^{+\alpha}$, and the other half lowering operators, $E^{-\alpha}$, satisfying $E^{-\alpha} = (E^{+\alpha})^+$. In the SU(2) case, the Cartan or weight operator is J_z and the raising and lowering operators J^+, J^-. The α's are vectors (called roots) that play the role of structure constants. There are as many independent roots as the rank of the algebra, so in our case we only need two roots to define the algebraic quantities.

An important algebraic concept is the highest weight state, which defines each irreducible representation. This is a state annihilated by all the raising operators: $E_i^{\alpha_k}|\Lambda_i >= 0 \ \forall \alpha_k$. They have the property of being eigenstates of the Cartan operators.

With these definitions, the L integrals of motion for a system of L copies of a Lie algebra in the rational model are:

$$R_i = \sum_{i' \neq i} \frac{X_i \cdot X_{i'}}{z_{i'} - z_i} + \xi^a F_{ab} h_i^b \ , \tag{1}$$

where $X_i \cdot X_{i'}$ is the scalar product of the generators, F_{ab} is a matrix that defines the algebra [10] and h_i^a are the Cartan operators in the Chevaley basis [10]. These integrals of motion depend on L+r free parameters: $z_i, i = 1...L$ and $\xi^a, a = 1...r$.

For each irreducible representation, labeled by the Dynkin labels [10] of the highest weight state Λ_i, the eigenvalues are:

$$r_i = \xi \cdot F \cdot \Lambda_i + \sum_{i' \neq i} \frac{\Lambda_{i'} \cdot F \cdot \Lambda_i}{z_{i'} - z_i} - \frac{1}{2} \sum_{a=1}^{r} \sum_{\alpha=1}^{M^a} \frac{\Lambda_i |\alpha^a|^2}{E_\alpha^a - z_i} \ , \tag{2}$$

where E_α^a are the 'r' families of spectral parameters that must satisfy the Richardson equations. In our case, there are two families, which we denote by $\{e_\alpha\}$ and $\{\omega_\gamma\}$. As they determine the wave function and all the observables obtained from the r_i, we will study their behavior in terms of the free parameters of the model.

3 The isovector proton-neutron pairing model

3.1 generators, integrals of motion and eigenvalues

A suitable representation for the generators of the i-th copy of the SO(5) algebra in terms of the creation and annihilation operators of protons p_i^+/p_i and neutrons, n_i^+/n_i is:

$$T_i^0 = \frac{1}{2}(p_i^+ p_i + p_{\bar{i}}^+ p_{\bar{i}}) - \frac{1}{2}(n_i^+ n_i + n_{\bar{i}}^+ n_{\bar{i}}) \qquad T_i^+ = \frac{1}{\sqrt{2}}(p_i^+ n_i + p_{\bar{i}}^+ n_{\bar{i}}) \quad (3)$$

$$b_{-1i}^+ = n_i^+ n_{\bar{i}}^+ \qquad b_{0i}^+ = \frac{1}{\sqrt{2}}(n_i^+ p_{\bar{i}}^+ + p_i^+ n_{\bar{i}}^+) \qquad b_{+1i}^+ = p_i^+ p_{\bar{i}}^+ \quad (4)$$

$$H_i = \frac{1}{2}(\hat{N}_i + \hat{N}_{\bar{i}}) - 1 \quad (5)$$

and the corresponding hermitian-conjugates of all the raising operators. Operators (3) form the SU(2) isospin subalgebra. The operators in (4) create a pair of particles in time-reversed states. The two Cartan generators are T_i^0 and H_i, and $\hat{N}$ is the number operator.

Associating each copy i to an orbit in the spherical shell model basis $i \equiv jm$ and choosing a specific linear combination of the integrals of motion, we obtain the hamiltonian [11]

$$H = \sum_j \epsilon_j (N_j + \Delta N_{pj}) + \frac{g}{2} T \cdot T + g \sum_{\mu, jm, j'm'} b_{\mu,jm}^+ b_{\mu,j'm'} \ . \quad (6)$$

The first sum is the single-particle energy term. Note that it includes a term ΔN_{pj}, which modifies the energy of the protons and thus breaks isospin symmetry. The second sum is the isovector pairing interaction. The free parameters are the single-particle energies ϵ_j, the coupling constant g, and the Δ parameter, which as just noted measures the isospin symmetry breaking.

The same linear combination of the eigenvalues of the integrals of motion gives the energy eigenvalues of the hamiltonian (6),

$$E = \sum_j \epsilon_j \big[\frac{\nu_j}{2}(2+\Delta) - \Delta \tau_j \big] + \sum_{\alpha=1}^{M} e_\alpha + \frac{\Delta}{2} \sum_{\beta=1}^{M+T_0+t} \omega_\beta + \frac{g}{2} T_0 (T_0 - 1) \ , \quad (7)$$

where ν_j is the seniority of the j shell (the number of unpaired particles) and τ_j the reduced isospin (i.e, the isospin of the unpaired particles).

The first sum gives the single-particle energy contribution of the unpaired particles. The third, as it is proportional to Δ, has to do with the isospin symmetry breaking. As the second sum gives the energy of the paired particles and is a sum over the various pairs, the e_α parameters can be interpreted as pair energies.

3.2 Study of the spectral parameters in terms of isospin symmetry breaking

To show how the spectral parameters behave as a function of the isospin breaking term Δ, we present in figure 1 three solutions of the Richardson equations for

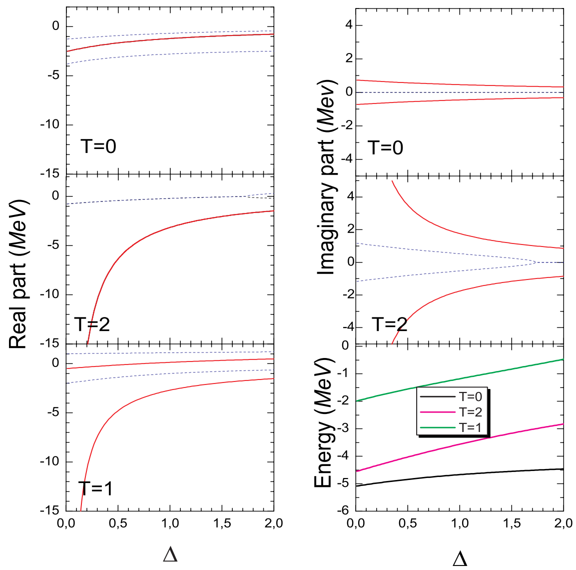

Figure 1. Spectral parameters e_α (blue dashed lines) and ω_γ (red solid lines) for a system of two protons and two neutrons in two shells ($j_0 = 1/2$, $j_1 = 3/2$) with energies $\epsilon_0 = 0$, $\epsilon_1 = 1$, for a coupling constant g=-1. Left panels are the real part and right panels the imaginary part, except in the T=1 case where all parameters are real. Only the lowest energy states of the hamiltonian (6) are represented for each value of T. These energies are plotted in the sixth panel.

a system of two protons and two neutrons in two shells in the seniority zero subspace. The solutions are labeled by the isospin in the $\Delta \to 0$ limit (for $\Delta \neq 0$, isospin is not a good quantum number). It is interesting to remark that in all cases the parameters are either real or form complex-conjugate pairs, so that the sums that appear in the expression of the energy (7) are real. Another important feature is that in the $\Delta \to 0$ limit only M-T of the ω parameters are finite. The rest diverge, but as their contribution to the energy has the form $\sim \Delta \sum_\gamma \omega_\gamma$ the energy remains finite.

3.3 Numerical calculations for ^{64}Ge with a ^{40}Ca core in a $pfg_{9/2}$ valence space

An important feature of RG exactly-solvable models is that they permit calculations beyond the diagonalization limits. As an example, we present numerical results for ^{64}Ge, with a ^{40}Ca core (i.e., 12 valence protons and 12 valence neu-

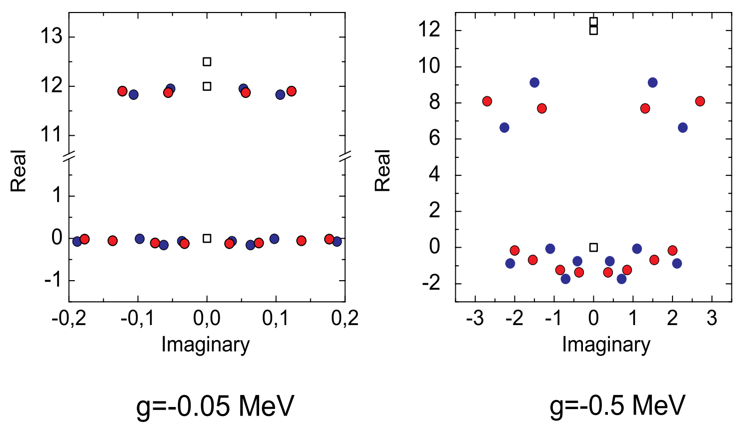

Figure 2. Complex-plane representation of the pair energies e (blue circles) and ω parameters (red circles). Left panels correspond to g=-0.05 MeV and right ones to g=-0.5 MeV. The values of the single -particle energies are $\varepsilon_{f_{7/2}} = 0.00$ MeV, $\varepsilon_{p_{3/2}} = 6.00$ MeV, $\varepsilon_{f_{5/2}} = 6.25$ MeV , $\varepsilon_{p_{1/2}} = 7.1$ MeV, $\varepsilon_{g_{9/2}} = 9.60$ MeV

trons) for a valence space built from the orbits of the pf shell and the $g_{9/2}$ orbit.

In figure 2, we plot the spectral parameters in the isospin symmetric limit for two values of the coupling constant, g=-0.05 MeV (weak coupling) and g=-0.5 MeV (strong coupling), for the ground state (T=0). Blue circles represent the pair energies e_α, red circles the ω_γ parameters and the squares are twice the single-particle energies of the first three orbits. In the weak coupling limit, there are as many e_α parameters as the degeneracy of the first two shells with a real part roughly twice the single-particle energy. Physically it means that particles are filling orbits as in a non-interacting system. As the interaction increases, the real part of the e parameters decreases, and therefore the energy also decreases: correlations make the system more bound. They also expand in the complex plane. The ω parameters are always intertwined with them.

As was mentioned these spectral parameters determine not only the energies but also the wave functions and such physical observables such as occupation numbers. Such calculations can be found in [11].

4 The Interacting Boson Model-2

The Interacting Boson Model (IBM), developed by A. Arima and F. Iachello [12], describes the quadrupole-quadrupole excitations of even-even nuclei in terms of a system of s and d bosons that microscopically represent fermion pairs.

It has proven to be very powerful in the prediction of properties of many nuclei. The IBM-2 version [13] distinguishes between proton pairs (π bosons) and neutron pairs (ν bosons), and thus introduces a new quantum number, F spin, which is similar to isospin but for bosons rather of fermions. It has been found that F-spin SU(2) symmetry is approximately preserved in nuclei.

The most general hamiltonian of this model has the form:

$$H = \epsilon_\pi N_{d\pi} + \epsilon_\nu N_{d\nu} + V_{QQ} + M \ , \tag{8}$$

where the first two terms are the single-particle energies, and V_{QQ} is the quadrupole-quadrupole interaction, which has the form:

$$V_{QQ} = \kappa_{\pi\pi} Q_\pi^2 + \kappa_{\pi\nu} Q_\pi Q_\nu + \kappa_{\nu\nu} Q_\nu^2 \ . \tag{9}$$

The quadrupole operator, Q_ρ, depends on a free parameter χ_ρ,

$$Q_\rho(\chi_\rho) = d_\rho^+ s_\rho + d_\rho^+ s_\rho + \chi_\rho \left[d_\rho^+ \tilde{d}_\rho \right] \ . \tag{10}$$

In the F-spin symmetry limit, the IBM-2 hamiltonian has three dynamical symmetries, SU(3), O(6) and U(5), that describe axially symmetric deformed nuclei, gamma-unstable nuclei, and vibrational nuclei, respectively. In the latter two cases, the value of the parameter χ_ρ is zero, both for the π and ν bosons.

The term M in (8) is called the Majorana interaction, and usually has the form $M = \zeta[F_{max}(F_{max}+1) - F^2]$. In the F-spin symmetry limit it is found that states with $F = F_{max}$, called maximally symmetric states (SS), are the lowest in energy. These states are the only ones that appear in the IBM1 model, and they are completely symmetric under the interchange of π and ν bosons. The Majorana term splits the energy of the states with $F < F_{max}$, the so called mixed symmetry states (MSS).

4.1 SO(3,2): Generators and Hamiltonian

The generators that close the SO(3,2) commutation relationships for each l-shell in terms of the creation/annihilation operators of bosons (l_ρ^+ / l_ρ), $\rho = \pi, \nu$ are:

$$F_l^0 = \frac{1}{2}(N_{l\pi} - N_{l\nu}) \qquad F_l^+ = l_\pi^+ \cdot l_\nu \qquad F_l^- = l_\nu^+ \cdot l_\pi \tag{11}$$

$$b_{-1l}^+ = (-1)^{l/2} l_\nu^+ \cdot l_\nu^+ \qquad b_{0l}^+ = (-1)^{l/2} l_\pi^+ \cdot l_\nu^+ \qquad b_{0l}^+ = (-1)^{l/2} l_\pi^+ \cdot l_\pi^+ \tag{12}$$

$$b_{-1l}, \quad b_{0l}, \quad b_{1l} \qquad H_l^2 = \frac{1}{2}(\hat{N}_{\pi l} + \hat{N}_{\nu l} + \Omega_l) \tag{13}$$

As in SO(5), the first three close the F-spin SU(2) subalgebra. Furthermore, $b_{\mu l}^+$ creates a pair of bosons in time reversal states, and H_l^2 together with F_l^0 form the Cartan subalgebra.

By taking a particular linear combination of the integrals of motion (1), we obtain the hamiltonian [14]:

$$H = \sum_l \epsilon_l (N_{\pi l} + N_{\nu l} + \Delta N_{\pi l}) - \frac{g}{4} \sum_{l<l'} \sum_{L=|l'-l|}^{l'+l} (-1)^L$$
$$(Q^L_{l_\nu l'_\nu} + Q^L_{l_\pi l'_\pi}) \cdot (Q^L_{l'_\nu l_\nu} + Q^L_{l'_\pi l_\pi}) \;, \tag{14}$$

with

$$Q^L_{l_\rho l'_\rho} = (l^+_\rho \tilde{l'}_\rho - (-1)^{l+(l+l')/2} l'^+_\rho \tilde{l}_\rho)^L \;. \tag{15}$$

In the case of l=0,2,this hamiltonian takes the form

$$H = \epsilon_d (N_{\pi d} + N_{\nu d} + \Delta N_{\pi d}) - \frac{g}{4}(Q_\pi + Q_\nu)^2 \;, \tag{16}$$

where Q_ρ has the same form as the IBM-2 quadrupole operator (10), but with $\chi_\rho = 0$. Thus, while (16) is an IBM-2 hamiltonian, it is not the most general one. It is limited to describe the transition from U(5) to O(6). Furthermore, it has several other specific features, namely that the quadrupole-quadrupole interaction that enters is an F-spin scalar and that it contains a term proportional to Δ that when active breaks F-spin symmetry. Lastly, it does not contain a specific Majorana term. However, in the F-spin symmetry limit ($\Delta \to 0$) a Majorana term can be added while maintaining the exact solvability of the model.

The key point of this class of RG models is that we can add other bosons degrees of freedom and still solve the problem exactly, going beyond what is possible with diagonalization methods. In particular we can include a g-boson (l=4), which has been included, for example, to explain some intruder states in some nuclei such as ^{192}Os [**?**] or an f-boson (l=3), which has been introduced to explain negative parity bands in some isotopes.

4.2 Energy levels. The effect of adding a g or an f boson

In order to study the effect on the spectra of adding other bosons degrees of freedom, we plot in figure 3 the spectra of the s-d hamiltonian (16) (upper panel) and the sdg hamiltonian in (14) (botton panel), as a function of the coupling constant g, for a system of 10 π bosons and 10 ν bosons. The calculations were carried out in the F-spin symmetry limit, for $\Delta \to 0$. Each level is obtained by giving different values to the seniorities, and they are multiplets of angular momentum, specified by the labels. Solid lines represent SS and dotted and circled lines represent MSS with $F = F_{max} - 1$, which as we noted earlier can be shifted in energy by adding a Majorana term. It can be seen that in the sd case, the g=0 limit corresponds to a vibrational (U(5)) spectrum, whereas with increasing g a transition to O(6) takes place. In the sdg case, many more possible values for J^P appear. In the figure thicker solid and circled lines correspond to SS and MSS, respectively, with unpaired g bosons. Most of the states that also

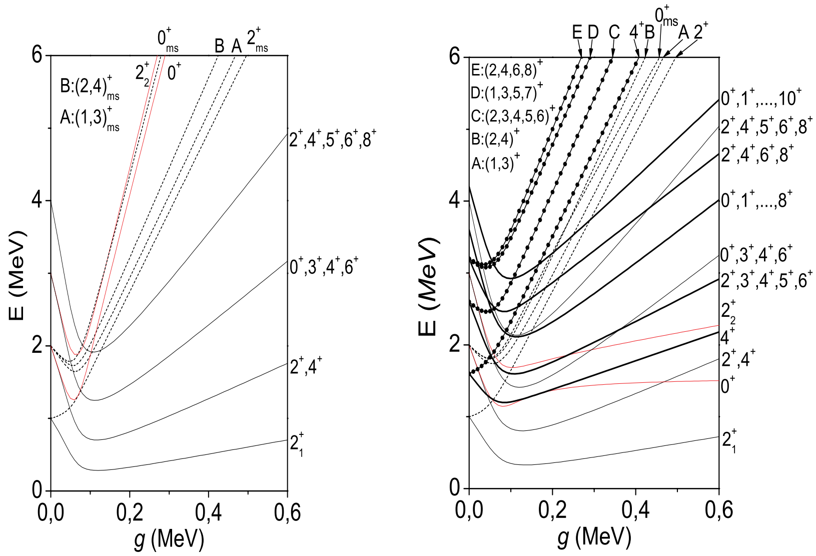

Figure 3. Energy levels for a system of 10 π bosons and 10 ν bosons of the sdg hamiltonian (14) (bottom panel) compared to those of the sd hamiltonian (16) (upper panel), as a function of the coupling constant g. The values of the single particle energies are $\epsilon_s = 0\ MeV$, $\epsilon_d = 1\ MeV$ and $\epsilon_g = 1.6\ MeV$. Except for $J^P = 2^+$, only the first excited state of each seniority is shown. Only those mixed symmetry states up to total seniority two are presented. The meaning of the different types of lines is explained in the text.

appear in the sd case are relatively unaffected by the addition of the new boson degree of freedom, except in some cases such as the first 0^+ state or the second 2_2^+ state, plotted in red. For these states, in the sd case the energy goes up with the coupling constant g, while in the sdg case the energy flattens out.

5 Summary and Conclusions

We have generalized the exactly solvable Richardson-Gaudin models to two algebras of rank-2, SO(5) and SO(3,2). We have obtained the integrals of motion and its eigenvalues, which are written in terms of two families of spectral parameters that must satisfy a system of non-linear coupled equations, the so-called Richardson equations. By choosing appropriate linear combinations of the integrals of motion, we have derived two pairing hamiltonians for two kinds of particles, fermions in the SO(5) case and bosons in the SO(3,2) case.

The SO(5) algebra gives rise to an isovector proton-neutron pairing model, but which includes a one-body term that breaks isospin symmetry. We have studied the dependence of the spectral parameters as a function of this term. We have also presented some numerical results for ^{64}Ge with a ^{40}Ca core within the valence shells $fpg_{9/2}$, which could not have been done using numerical diagonalization methods.

One possible representation of the SO(3,2) algebra gives rise to a specific version of the Interacting Boson Model-2 in the transition region between the dynamical symmetries U(5) and O(6). The hamiltonian we obtain does not have a Majorana term, but in the F-spin symmetry limit we can add one without affecting its exact solvability. This model involves only two copies of the Lie algebra, one with $l = 0$ (the s boson) and one with $l = 2$ (the d boson). Other boson degrees of freedom have also been introduced in extended versions of the IBM-2 to explain intruder states (an $l = 4$ g boson) or negative-parity bands (an $l = 3$ f boson) that appear in nuclei. We have studied the effect of taking the g boson into account within an exactly solvable SO(3,2) context by comparing the energy levels for an sd hamiltonian with an sdg hamiltonian for both maximal symmetry states and mixed symmetry states.

The models that we have developed are not limited, however, to proton-neutron fermion or boson models of nuclei. Any physical problem involving two species of particles in which pairing is dominant can be modelled in this way. In particular, the bosonic case could be applied to problems involving a mixture of ^{97}Rb atoms in the hyperfine states $|F = 1, M_f = 1\rangle$, $|F = 1, M_f = -1\rangle$.

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