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Michael Puso

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AN IMPROVED LINEAR TETRAHEDRAL ELEMENT FOR PLASTICITY[1]

Michael Puso
Lawrence Livermore National Laboratory
Department of Mechanical Engineering
University of California
Livermore, California 94551
Email: puso@llnl.gov

ABSTRACT

A stabilized, nodally integrated linear tetrahedral is formulated and analyzed. It is well known that linear tetrahedral elements perform poorly in problems with plasticity, nearly incompressible materials, and acute bending. For a variety of reasons, linear tetrahedral elements are preferable to quadratic tetrahedral elements in most nonlinear problems. Whereas, mixed methods work well for linear hexahedral elements, they don't for linear tetrahedrals. On the other hand, automatic mesh generation is typically not feasible for building many 3D hexahedral meshes. A stabilized, nodally integrated linear tetrahedral is developed and shown to perform very well in problems with plasticity, nearly incompressible materials and acute bending. Furthermore, the formulation is analytically and numerically shown to be stable and optimally convergent. The element is demonstrated to perform well in several standard linear and nonlinear benchmarks.

INTRODUCTION

[1] A nodally integrated tetrahedral element was first formulated in [1] and further analyzed in [2]. The element was shown to perform well in several 2D bending problems but it was noted that the formulation was prone to spurious low energy modes. In this work, a stabilization of the nodally integrated tetrahedral is proposed. It is then shown analytically that the proposed scheme is stable and consistent for linear elasticity. As it turns out, the nodal integration provides reduced constraints such that coarse mesh accuracy can be achieved for nearly incompressible materials and plasticity. Finally, example problems demonstrate the improved numerical response of the proposed tetrahedral formulation.

FORMULATION

In what follows, superscript indices in lower case (typically e) refer to element quantities and superscript indices in upper case (typically I) refer to nodal quantities. Here, the standard weak form of linear elasticity is considered

$$a(u,w) = f(w) \quad u,w \in (H^1(\Omega))^3 \tag{1}$$

for the domain $\Omega \subset \mathbb{R}^3$. The discrete problem reads

$$a_h(u_h,w_h) = f(w_h) \tag{2}$$

where u_h w_h are the discrete trial and test functions respectively based on linear tetrahedral interpolation and the discrete bilinear operator $a_h(u_h,w_h)$ is based on average nodal integration. The displacement gradient on the domain of element e (i.e. Ω^e) will be written $\nabla u_h^e = \{\nabla u_h : x \in \Omega^e\}$ and is observed to be constant on Ω^e. Furthermore, strains based on the trial and test functions are respectively written $\varepsilon^e = \nabla_s u_h^e$ and $\delta\varepsilon^e = \nabla_s w^e$ where ∇_s is the symmetric gradient. In order to present the average nodal

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strain formulation, the set S_I is defined to be the group of elements common to node I and N_e and N_n are respectively the total number of elements and nodes defined on Ω. The average nodal gradient ∇u^I and strain ε^I at node I are defined

$$\nabla u_h^I = \frac{1}{V^I}\sum_{e\in S_I}\frac{V^e}{4}\nabla u_h^e$$
$$\varepsilon^I = \frac{1}{2}[\nabla u_h^I + (\nabla u_h^I)^T] \quad \text{or} \quad \varepsilon^I = \frac{1}{V^I}\sum_{e\in S_I}\frac{V^e}{4}\varepsilon^e \tag{3}$$

where $V^e = \mathrm{vol}(\Omega^e)$ and the average nodal volume is given

$$V^I = \sum_{e\in S_I}\frac{V^e}{4} \tag{4}$$

"Virtual" nodal strain quantities analogous to (3) based on the test functions are also defined

$$\nabla w_h^I = \frac{1}{V^I}\sum_{e\in S_I}\frac{V^e}{4}\nabla w^e$$
$$\delta\varepsilon^I = \frac{1}{2}[\nabla w_h^I + (\nabla w_h^I)^T] \quad \text{or} \quad \delta\varepsilon^I = \frac{1}{V^I}\sum_{e\in S_I}\frac{V^e}{4}\delta\varepsilon^e \tag{5}$$

Considering the standard tetrahedral finite element discretization for the equations of motion and applying the preceeding notation, the bilinear form can be written as a sum over elements *or* a sum over nodes as follows

$$a(w_h,u_h) = \sum_{e=1}^{N_e} V^e \delta\varepsilon^e : C\varepsilon^e \tag{6}$$
$$= \sum_{I=1}^{N_n}\sum_{e\in S_I}\frac{V^e}{4}\delta\varepsilon^e : C\varepsilon^e \tag{7}$$

where, again, S_I is the set of elements that are common to node I and C is the material stiffness.

The following modified, nodal strain definition of the bilinear form is proposed

$$a_h(w_h,u_h) = \sum_{I=1}^{N_n} V^I\delta\varepsilon^I : C\varepsilon^I + \sum_{I=1}^{N_n}\sum_{e\in S_I}\alpha^{e,I}\frac{V^e}{4}(\delta\varepsilon^I - \delta\varepsilon^e):\tilde{C}(\varepsilon^I-\varepsilon^e) \tag{8}$$

where $\alpha^{e,I}$ is a stabilization parameter that can potentially depend on element e and node I and $\tilde{C}$ could be an alternate material stiffness. Of course $C = \tilde{C}$ would be the natural choice but given a nearly incompressible material with Poisson's ration $\nu = 0.4999$, better results may be obtained by letting Poisson's ratio be $\nu = 0.4$ in $\tilde{C}$. The average nodal strain formulation proposed in [1] and analyzed in [2] is recovered when $\alpha^{e,I} = 0$. In fact, results from aforementioned references are quite good despite the absence of the stabilization term. Nonetheless, the eigenalysis in the RESULTS Section demonstrate the necessity of the stabilization term.

STABILITY

The strain energy energy from (8) is written

$$a_h(u_h,u_h) = \frac{1}{2}\sum_{I=1}^{N_n} V^I\varepsilon^I : C\varepsilon^I + \frac{1}{2}\sum_{I=1}^{N_n}\sum_{e\in S_I}\alpha^{e,I}\frac{V^e}{4}(\varepsilon^I - \varepsilon^e):\tilde{C}(\varepsilon^I-\varepsilon^e) \tag{9}$$

With $\alpha^{e,I} = 0$, a spurious zero energy mode can arise in the event of an infinite domain with a regular lattice of points where the displacement oscillates such that strains ε^e are positive to the left of every node and negative to the right of every node thus producing an average strain $\varepsilon^I = 0$ at every node and thus a zero energy mode. For a finite domain, the zero energy mode cannot propagate since the nodes on the boundaries have strain contributions from only one side thus precluding the zero energy mode. Nonetheless, the energy of this above described mode can be very small and stability in the H_1 norm is not ensured. With the stabilization, the energy in (9) reduces to the following when $\varepsilon^I = 0$

$$a_h(u_h,u_h) = \frac{1}{2}\sum_{I=1}^{N_n}\sum_{e\in S_I}\alpha^{e,I}\frac{V^e}{4}\varepsilon^e : C\varepsilon^e \tag{10}$$

From Korn's lemma (10) is always non zero for any displacement field modulo rigid body modes such that

$$a_h(u_h,u_h) \geq \alpha\|u_h\|_1^2 \tag{11}$$

where α is a mesh independent constant. Consequently, the form (9) has no spurious zero energy modes and is thus V coercive. In the event that $\alpha^{e,I} = 0$, the discrete bilinear form may be greater than zero for non-trivial displacements but α in (11) will depend on h precluding it from the following convergence analysis.

CONVERGENCE

In this section, optimal convergence in the H^1 (energy norm) is proved analytically for the proposed discrete bilinear form in (8). Repeated indices will imply summation unless otherwise specified and the following notation will be used throughout for norms and semi-norms

$$\|y\|_1 = \left(\int_\Omega (y_i y_i + y_{i,j} y_{i,j})\, d\Omega\right)^{1/2} \quad |y|_1 = \left(\int_\Omega y_{i,j} y_{i,j}\, d\Omega\right)^{1/2} \tag{12}$$

and the Frobenius norms will be used for matrices where for example

$$y = R^{m,n} \quad \|y\| = (y_{ij} y_{ij})^{1/2} \quad i = 1:m, j = 1:n$$

Strang's first Lemma is defined

$$\|u - u_h\|_1 \le C \inf_{v_h \in V_h} [\|u - v_h\|_1 + \sup_{w_h \in V_h} \frac{a(v_h, w_h) - a_h(v_h, w_h)}{\|w_h\|_1}] \tag{13}$$

where u is the exact solution to the boundary value problem and u_h is the solution to the approximate weak form $a_h(w_h, u_h) = f(w_h)$ Because (8) is coercive (i.e. stable) (13) can be used to establish its rate of convergence. From hereon, let

$$v_h = \Pi_h u \tag{14}$$

be the interpolant of u thus from approximation theory [3] $\|u - v_h\|_1 \le Ch$. Hence, it will suffice to show that

$$\sup_{w_h \in V_h} \frac{a(v_h, w_h) - a_h(v_h, w_h)}{\|w_h\|_1} < Ch \tag{15}$$

Exploiting the symmetries of the elasticity tensor C the exact bilinear form (7) can be written in terms of the gradients

$$a(w_h, v_h) = \sum_{I=1}^{N_n} \sum_{e \in S_I} \frac{V^e}{4} \nabla w^e : C \nabla v^e \tag{16}$$

Furthermore, the modified bilinear form (8) can be rewritten

$$a_h(w_h, v_h) = a_h^n(w_h, v_h) + a_h^s(w_h, v_h) \tag{17}$$

where

$$a_h^n(w_h, v_h) = \sum_{I=1}^{N_n} \sum_{e \in S_I} \frac{V^e}{4} \nabla w^e : C \nabla v_h^I \tag{18}$$

$$a_h^s(w_h, v_h) = \sum_{I=1}^{N_n} \sum_{e \in S_I} \alpha^{e,I} \frac{V^e}{4} (\nabla w^I - \nabla w^e) : \tilde{C} (\nabla v^I - \nabla v^e) \tag{19}$$

and where (5) was used to eliminate $\delta\varepsilon^I$ in (18). Our first task is to show that

$$\sup_{w_h \in V_h} \frac{a(v_h, w_h) - a_h^n(v_h, w_h)}{\|w_h\|_1} < Ch \tag{20}$$

Equations (16) and (18) provide the following difference

$$a(v_h, w_h) - a_h^n(v_h, w_h) = \sum_{I=1}^{N_n} \sum_{e \in S_I} \frac{V^e}{4} \nabla w^e : C(\nabla v^e - \nabla v_h^I) \tag{21}$$

and by Cauchy Schwarz, the following inequality can be established

$$a(v_h, w_h) - a_h^n(v_h, w_h) \le M \sum_{I=1}^{N_n} \sum_{e \in S_I} \frac{V^e}{4} \|\nabla w^e\| \, \|\nabla v^e - \nabla v_h^I\| \tag{22}$$

where $\|\cdot\|$ is the norm $\|\nabla v\| = (v_{i,j} v_{i,j})^{1/2}$ as per (12) and M is the norm of the elasticity tensor $M = (C_{ijkl} C_{ijkl})^{1/2}$. In order to bound $\|\nabla v^e - \nabla v_h^I\|$ in (22), it is required that the exact solution u is sufficiently smooth ($u \in C^2$) such that following Taylor series expansion can be made about point x^e

$$\nabla u(x) = \nabla u(x^e) + \nabla^2 u(x^{e^*})(x - x^e) \tag{23}$$

where x^e is the coordinates for the centroid of the element e and $x^{e^*} \in \Omega^e$. For a 3D tetrahedral e, x^e corresponds to $\xi = (1/3, 1/3, 1/3)$ in the parent coordinates of the tetrahedral. Next, the following difference is computed using (3) and (4)

$$\nabla v^e - \nabla v^I = \nabla v^e - \nabla u(x^I) - \frac{1}{V^I} \sum_{\bar{e} \in S_I} \frac{V^{\bar{e}}}{4} (\nabla v^{\bar{e}} - \nabla u(x^I)) \tag{24}$$

where x^I is the coordinates of node I. Expanding $\nabla u(x^I)$ in terms of (23) and substituting into (24) yields

$$\begin{aligned} &\nabla v^e - \nabla v^I = \nabla v^e - \nabla u(x^e) - \nabla^2 u(x^{e^*})(x^I - x^e) \\ &- \frac{1}{V^I} \sum_{\bar{e} \in S_I} \frac{V^{\bar{e}}}{4} (\nabla v^{\bar{e}} - \nabla u(x^{\bar{e}})) + \sum_{\bar{e} \in S_I} \frac{V^{\bar{e}}}{4} (\nabla^2 u(x^{\bar{e}^*})(x^I - x^{\bar{e}}) \end{aligned} \tag{25}$$

Taking the norm of the left and right sides of (25) and applying the triangle inequality yields the inequality

$$\begin{aligned} &\|\nabla v^e - \nabla v^I\| \le \|\nabla v^e - \nabla u(x^e)\| + \|\nabla^2 u(x^{e^*})\| h \\ &+ \frac{1}{V^I} \sum_{\bar{e} \in S_I} \frac{V^{\bar{e}}}{4} \|(\nabla v^{\bar{e}} - \nabla u(x^{\bar{e}}))\| + \frac{1}{V^I} \sum_{\bar{e} \in S_I} \frac{V^{\bar{e}}}{4} \|\nabla^2 u(x^{\bar{e}^*})\| h \end{aligned} \tag{26}$$

where h is the maximum element diameter and thus $\|x^I - x^e\| \leq h$ for all e. It turns out to be most convenient to bound (26) by the L_∞ norm; consequently, for some vector $u \in \mathbb{R}^3$

$$|u|_{2,\infty,\Omega^e} = \max_{ijk} \sup_{x \in \Omega^e} \left| \frac{\partial^2 u_i}{\partial x_j \partial x_k} \right| \tag{27}$$

Based on (14)and approximation theory [3,4], the following error bound is exploited

$$\left| \frac{\partial u_i}{\partial x_j}(x) - \frac{\partial v_i}{\partial x_j} \right|_{x \in \Omega^e} \leq Ch|u|_{2,\infty,\Omega^e} \quad \forall i = 1:3 \tag{28}$$

Using (3) and substituting

$$\|\nabla^2 u(x^e)\| = \left(\sum_{ijk} \left(\frac{\partial^2 u_i}{\partial x_j \partial x_k}(x^e) \right)^2 \right)^{1/2} \leq 3\sqrt{3}|u|_{2,\infty,\Omega^e} \tag{29}$$

and (28) into (26) yields

$$\begin{aligned} \|\nabla v^e - \nabla v^I\| &\leq (C^e + 3\sqrt{3})\, h\, |u|_{2,\infty,\Omega^e} \\ &\quad + \frac{1}{V^I} \sum_{\bar{e} \in S_I} \frac{V^{\bar{e}}}{4} (C^{\bar{e}} + 3\sqrt{3})\, h\, |u|_{2,\infty,\Omega^{\bar{e}}} \\ &\leq C\, h\, |u|_{2,\infty,\Omega} \end{aligned} \tag{30}$$

since $|u|_{2,\infty,\Omega}| \geq |u|_{2,\infty,\Omega^e}\ \forall e$. Substituting the result (30) into (22) and applying the Cauchy Schwartz inequality yields

$$\begin{aligned} \sum_{I=1}^{N_n} \sum_{e \in S_I} \frac{V^e}{4} \|\nabla w^e\| \, \|\nabla v^e - \nabla v^I\| &\leq \left(\sum_{I=1}^{N_n} \sum_{e \in S_I} \frac{V^e}{4} \|w^e\|^2 \right)^{1/2} \\ &\quad \times \left(\sum_{I=1}^{N_n} \sum_{e \in S_I} \frac{V^e}{4} C h^2 \, |u|^2_{2,\infty,\Omega} \right)^{1/2} \\ &\leq \left(\sum_{e=1}^{N_e} V^e \|w^e\|^2 \right)^{1/2} \left(\sum_{e=1}^{N_e} V^e C h^2 \, |u|^2_{2,\infty,\Omega} \right)^{1/2} \\ &\leq C\, |w_h|_{1,\Omega}\, |u|_{2,\infty,\Omega} V\, h \end{aligned} \tag{31}$$

where summations over nodes I and sets S_I were reorganized as summations over elements e as in (7). and the semi-norm is computed

$$|w_h|_{1,\Omega} = \left(\sum_{e=1}^{N_e} V^e \, \|\nabla w^e\|^2 \right)^{1/2} \tag{32}$$

since the ∇w^e is constant on Ω^e. The desired bound in (20) is a consequence of (31).

The next task is to show that

$$\sup_{w_h \in V_h} \frac{a_h^s(v_h, w_h)}{\|w_h\|_1} < Ch \tag{33}$$

Applying the results from (30),(31) and the Cauchy Schwarz and triangle inequalities to (19) provides the following bound

$$\begin{aligned} a_h^s(w_h, v_h) &\leq M \sum_{I=1}^{N_n} \sum_{e \in S_I} \alpha^{e,I} \frac{V^e}{4} \|\nabla w^I - \nabla w^e\| \, \|\nabla v^e - \nabla v^I\| \\ &\leq \hat{\alpha} M \sum_{I=1}^{N_n} \sum_{e \in S_I} \frac{V^e}{4} (\|\nabla w^I\| + \|\nabla w^e\|) \, \|\nabla v^e - \nabla v^I\| \\ &\leq \hat{\alpha} M \sum_{I=1}^{N_n} \sum_{e \in S_I} \frac{V^e}{4} \|\nabla w^I\| (C_1 \, |u|_{2,\infty,\Omega}\, h) + \hat{\alpha} M C_2 |w|_{1,\Omega}\, |u|_{2,\infty,\Omega} V\, h \end{aligned} \tag{34}$$

where $\hat{\alpha} = \max_{e,I} \alpha^{e,I}$. At this point it remains to bound the remaining summation from the last inequality in (34). Using the definition for nodal volume (4) and nodal strain (3) gives

$$\begin{aligned} \sum_{I=1}^{N_n} \sum_{e \in S_I} \frac{V^e}{4} \|\nabla w^I\| &\leq \sum_{I=1}^{N_n} \sum_{e \in S_I} \frac{V^e}{4} \Big\| \frac{1}{V^I} \sum_{\hat{e} \in S_I} \frac{V^{\hat{e}}}{4} \nabla w^{\hat{e}} \Big\| \\ &\leq \sum_{I=1}^{N_n} \sum_{e \in S_I} \frac{V^e}{4V^I} \sum_{\hat{e} \in S_I} \frac{V^{\hat{e}}}{4} \|\nabla w^{\hat{e}}\| \\ &\leq \sum_{I=1}^{N_n} \sum_{\hat{e} \in S_I} \frac{V^{\hat{e}}}{4} \|\nabla w^{\hat{e}}\| \\ &\leq \left(\sum_{I=1}^{N_n} \sum_{\hat{e} \in S_I} \frac{V^{\hat{e}}}{4} \|\nabla w^{\hat{e}}\|^2 \right)^{1/2} \left(\sum_{I=1}^{N_n} \sum_{\hat{e} \in S_I} \frac{V^{\hat{e}}}{4} \right)^{1/2} \\ &\leq |w_h|_{1,\Omega} V \end{aligned} \tag{35}$$

The desired bound in (33) is a arrived at by substituting the result of (35) into (34). Substitution of (20) and (33) into (13) provides the final result

$$\|u - u_h\|_1 \leq Ch \tag{36}$$

provided sufficient smoothness of the exact solution $u \in C^2 \cap (H^1)^3$.

Table 1. EIGENFREQUENCIES FROM TET AND HEX MESHES

Modes	$\alpha = 0.05$	$\alpha = 0$	hex
Mode 1	0.258	0.209	0.258
Mode 10	0.424	0.236	0.404
Mode 16	0.452	0.248	0.482

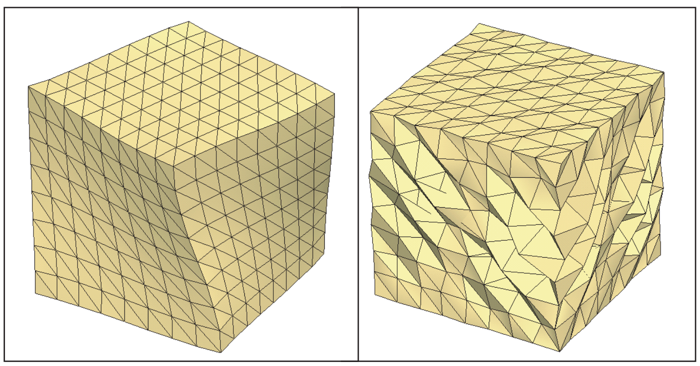

Figure 1. FIRST EIGENMODE: $\alpha = 0.05$ (LEFT) AND $\alpha = 0$ (RIGHT)

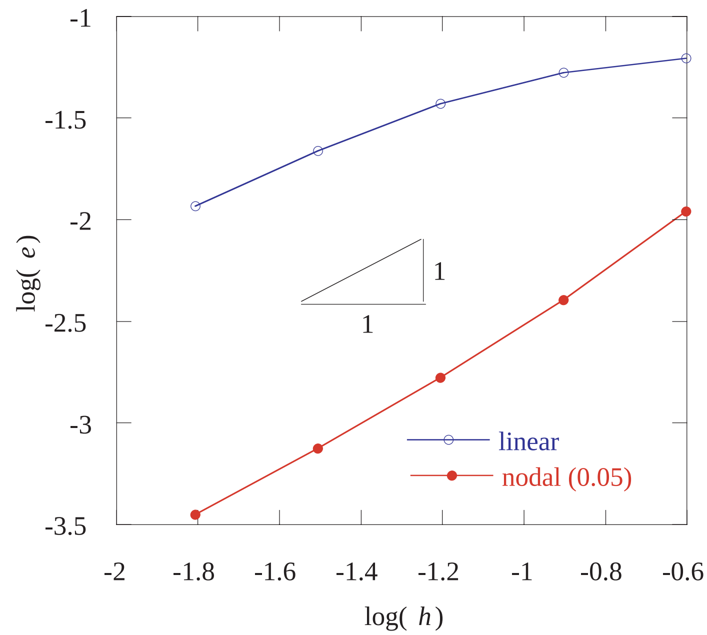

Figure 2. DISCRETIZATION ERROR FOR CANTILEVER BEAM

RESULTS

The following example problems demonstrate the necessity of the stabilization added to the nodal integration scheme and the good convergence characteristics of the proposed approach. A uniform stabilization parameter was employed throughout such that $\alpha^{e,I} = \alpha$ with $\alpha = 0.05$

Eigenanalysis

An eigenanalysis reveals the spurious modes of the nodal integration approach (i.e. $\alpha = 0$). The eigenvalues and first eigenmode of a 1x1x1 block with $\rho = 1$, $E = 1$ and $\nu = 0.499$ are shown in Table 1 and Fig. 1 respectively. The eigenfreqencies are compared to a mesh composed of 512 incompatible modes hexahedral elements. With $\alpha = 0$, the spurious modes are not zero energy modes since the method is still stable in the L^2 norm but not the H^1 norm. On the other hand, it is seen that further mesh refinement yields convergence to the wrong eigenfrequencies with $\alpha = 0$. Whereas, the stabilized formulation converges to the correct eigenfrequencies.

Asymptotic Error

A standard benchmark is a cantilever beam $\nu = 0.499$ loaded in shear as described in [5]. The energy of the discretization error is plotted in Fig. 2 for the standard linear triangle element and the nodally integrated triangle with $\alpha = 0.05$. The nodally integrated element actually converges at a rate of 1.2 and is clearly superior to the linear triangle.

Plasticity

The Cook membrane Fig. 3 (see [6] for dimensions) with Von Mises plasticity ($E = 70$, $\nu = 0.333$, $\sigma_y = 0.243$ and a linear hardening modulus $E_t = 0.15$) is considered. Meshes with $n = 8, 16, 32, 64$ elements along the edges are considered and Fig. 3 shows the plastic strain for the $n = 16$ case. The stabilization stiffness $\tilde{C}$ (8) was chosen as follows: $\tilde{\mu} = \alpha E_t/(2(1+\nu))$ and $\tilde{\lambda} = \alpha\lambda$ where $\lambda = E/((1+\nu)(1-2\nu))$ with $\alpha = 0.05$. The tip displacement versus n is shown plotted in Fig. 4 for the nodally integrated triangle, linear triangle and the QM6 incompatible modes quadrilateral. The linear tetrahedral is very stiff whereas the nodally integrated triangle performs as well or better than the incompatible modes quadrilateral.

DISCUSSION

A stabilized nodally integrated tetrahedral element formulation was developed. It was shown analytically and numerically that the proposed formulation was stable and optimally convergent. Although the element is not shown to be LBB stable, it performs well in some cases where nearly incompressible or plastic material were used. More studies will be done for more general material cases such as nonlinear hardening and large deformations.

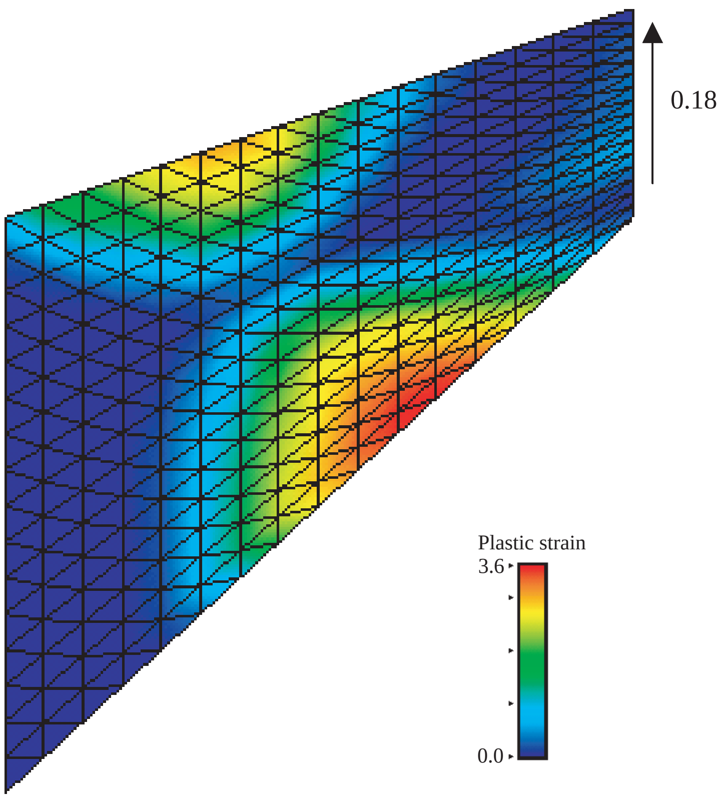

Figure 3. COOK MEMBRANE: EFFECTIVE PLASTIC STRAIN $n = 16$

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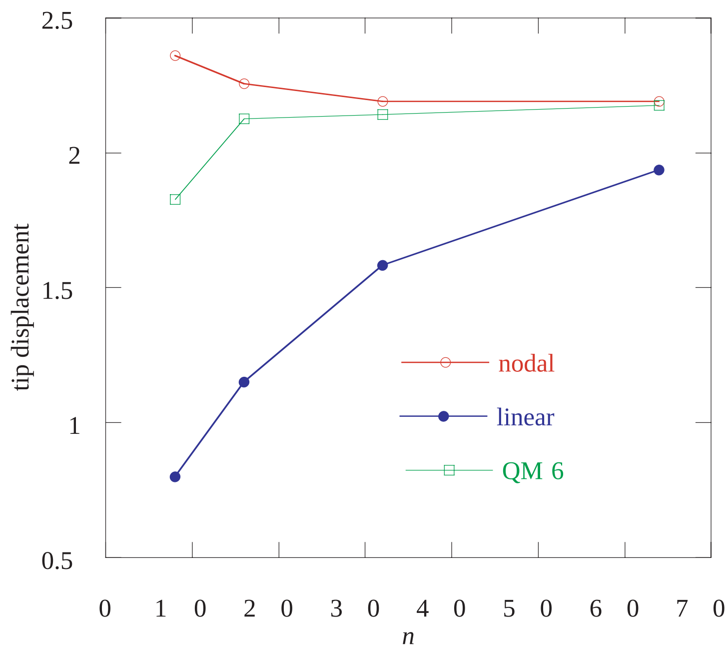

Figure 4. TIP DISPLACEMENT VERSUS ELEMENTS PER SIDE