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Final Report on subcontract B551021: Optimal AMG interpolation and Convergence theory

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Disclaimer

The goal of this project is to implement and study various techniques
for the construction of Algebraic Multigrid Methods (AMG) for the
solution of positive definite linear systems arising from the
discretizations   of   elliptic   partial   differential   equations
(PDEs). Both theoretical as well as practical implementation of the
methods that we have developed are based on compatible relaxation and
energy minimization.   The results that we have obtained are reported
in several publications resulting from the subcontract (references
[1]-[7]).

In [1] we have proposed an adaptive algebraic multigrid that uses
compatible relaxation technique to adaptively construct the set of
coarse variables. The nonzero supports for the coarse-space basis is
then determined by approximation of the so-called "ideal" two-level
interpolation operator   that   gives   maximal   error reduction per
iteration.   This approximation uses coarse grid basis, obtained by
minimizing a modified Frobenius norm of the coarse-level operator.
Our algorithm maintains multigrid-like optimality, without the need
for parameter tuning.   We have shown the efficacy and the robustness
of our approach on several test problems.

In a joint work with Vassilevski [2] we have designed an algorithm for
simultaneous   approximation of several   near   null space components,
which allows for more aggressive coarsening, and can be applied not
only to second order PDEs, but also to higher order (such as
discretizations of biharmonic equation) to produce highly accurate
interpolation operators and hence efficient multilevel methods.

We have also   studied optimal preconditioners   for discontinuous
Galerkin (DG) methods in [3]. The DG methods have many attractive
features, such as being locally conservative and in many cases more
accurate that the conventional conforming methods. A drawback is that
the number of unknowns in the resulting discrete system of equations
is much larger than the number of unknowns for a conforming method.
We have developed a unified approach for preconditioning DG methods
with the simplest possible discretizations, such as piece-wise
constant elements and conforming piece-wise linear elements.   The
family of preconditioners, that we have proposed and analyzed is based
on two level approach and leads to a significant reduction of the
problem size.   Moreover, our approach leads to coarse grid problems
that correspond to conforming methods, thus making possible the use of
already developed efficient multilevel methods for their solution.

We have also worked on the iterative methods for singular, nearly
singular and semidefinite linear systems ([4] and [6]).   The known
analysis and convergence results of such methods has been largely
based on the sufficient conditions known as P-regularity (or weak
regularity) of the corresponding matrix splittings. We have introduced
new, more refined conditions that are not only sufficient, but also
necessary for energy norm convergence.   These conditions basically
amount to assuming that the error is damped in a subspace (related to
the pseudo-inverse of the approximating matrix), and that the range of
the original matrix is contained in the range of the approximating
one. Such assumptions are naturally true for most iterative methods,
including multigrid and domain decomposition methods. In [6] we have
carried over these results to the case of nearly singular systems and
we have shown how the techniques developed can be utilized in the
analysis of iterative methods for linear systems corresponding to the
augmented Lagrangian methods applied to PDEs discretized by mixed and
hybrid finite element methods.

In [5] we proposed a new method for the efficient solution of linear
systems arising in discretizations of second order elliptic PDEs by a
generalized finite element method (GFEM). This has been an open
question for some time (last 5-6 years, since GFEM gained popularity),
and we were able to deliver solution to it and in particular relate

GFEM discretizations for scalar equations to the discretizations of systems of PDE. The efficient preconditioner that we propose, uses auxiliary (fictitious) space techniques and an additive BPX (Bramble, Pasciak and Xu~1988) preconditioner for the problems in auxiliary space.

The work on Helmholtz equation [7] can be viewed as a preliminary study towards targeting more complicated problems, such as indefinite Maxwell equations. The exact controllability framework (Due R. Glowinski and J. Lions) is based on recasting the Helmholtz equation in its original form (periodic solutions in time of a wave equation). Such solutions are found by minimizing a convex functional using a continuous version of the Conjugate Gradient (CG) algorithm. On every CG iteration it is required to solve a scalar second order PDE (like Laplace equation). The new technique, proposed by us is to use approximate solution instead, provided by few algebraic multigrid V-cycle iterations. Same procedure carries over to the solution of the indefinite Maxwell equations, and our goal is to further implement the inexact CG and the controllability methods to the indefinite Maxwell equation for wide range of wave numbers.

References:

[1] J. Brannick and L. Zikatanov. Algebraic multigrid methods based on compatible relaxation and energy minimization. Lecture Notes in Computational Science and Engineering, Springer Verlag (to appear).

[2] P. S. Vassilevski and L. Zikatanov. Multiple vector preserving interpolation mappings in algebraic multigrid. SIAM J. Matrix Anal. Appl., v. 27(4):1040--1055 (2006)

[3] V. Dobrev, R. Lazarov, P. Vassilevski, and L. Zikatanov. Multilevel preconditioning of 2nd order elliptic discontinuous Galerkin problems. Accepted for publication in Num. Lin. Alg. Appl.

[4] Y-J. Lee, J. Wu, J. Xu and L. Zikatanov. On the convergence of iterative methods for semidefinite linear systems. Accepted for publication in SIAM J. Matrix Anal. Appl.

[5] D. Cho and L. Zikatanov. Uniform preconditioning for generalized finite element equations. Submitted to Journal of Numerical Mathematics.

[6] Y-J. Lee, J. Wu, J. Xu and L. Zikatanov. Robust iterative methods for nearly singular systems. Completed but not submitted yet.

[7] D. Cho and L. Zikatanov. Conjugate Gradient method with inexact multiplication and controllability methods for the solution the indefinite Helmhotz equation. In preparation.