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NUMERICAL COMPUTATIONS FOR SELF-CONSISTENT ASTRON E LAYER WITH SLOWING DOWN OF ELECTRONS

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T. Seidman

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NUMERICAL COMPUTATIONS FOR SELF-CONSISTENT ASTRON E LAYER WITH SLOWING DOWN OF ELECTRONS

T. Seidman

Lawrence Radiation Laboratory, University of California

Livermore, California

In the calculation of the space and energy distributions (neglecting scattering effects) of the electron layer in Astron* occurs the pair of coupled, nonlinear, singular integro-differential equations,

$$\frac{d}{dg} P(g) = \frac{1}{g} \frac{\displaystyle\int \frac{\left[P(g) - Q(\rho)\right] \rho \, d\rho}{\sqrt{\rho^2 g^2 - \left[P(g)-Q(\rho)\right]^2}}}{\displaystyle\int \frac{\rho \, d\rho}{\sqrt{\rho^2 g^2 - \left[P(g)-Q(\rho)\right]^2}}} \qquad (1)$$

and

$$\frac{d}{d\rho}\left[\frac{1}{\rho}\frac{d}{d\rho} Q(\rho)\right] = \frac{S}{\rho} \int \frac{\left[Q(\rho)-P(g)\right] \left[g^2/\left(1 + g^2\right)^{3/2}\right] dg}{\sqrt{\rho^2 g^2 - \left[P(g)-Q(\rho)\right]^2} \displaystyle\int \frac{\rho \, d\rho}{\sqrt{\rho^2 g^2 - \left[P(g)-Q(\rho)\right]^2}}}, \qquad (2)$$

where the various integrals are taken between the singularities of the integrand subject to the condition $0 \leq \rho \leq 1$, $0 \leq g \leq \hat{g}$. To these are added the conditions

$$P(\hat{g}) = - \hat{g}$$
$$Q(1) = 0 \qquad (3)$$
$$Q\ (1) = G\hat{g}$$

and, in order to determine S, we add still another condition specifying the value of Q at some point, e.g.,

$$Q'\left(\frac{1 + s}{2}\right) = \frac{1}{4} G\hat{g}(1 + s), \qquad (4)$$

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* To be dealt with in a report by L. Tonks.

so that G, $\hat{g}$ and s are the parameters of the problem.

For convenience we let

$$\varphi(g) = g^2/(1 + g^2)^{3/2}$$

$$f(\rho,g) = \rho^2 g^2 - \left[P(g) - Q(\rho)\right]^2$$

$$R(g) = \int \frac{\rho\, d\rho}{\sqrt{f(\rho,g)}} \tag{5}$$

$$T(\rho) = \frac{1}{\rho}\int \frac{\left[Q(\rho) - P(g)\right] \varphi(g)\, dg}{R(g) \sqrt{f(\rho,g)}}$$

$$X(\rho) = G\hat{g} - S\int_{\rho}^{1} T(\rho')d\rho' = Q'(\rho)/\rho$$

Thus, Eqs. (1) and (2) become

$$P' = \frac{1}{gR} \int \frac{\left[P - Q\right] \rho\, d\rho}{\sqrt{f}} \tag{1'}$$

and

$$Q' = \rho X, \; X' = ST = \frac{S}{\rho}\int \frac{\left[Q - P\right] \varphi dg}{R \sqrt{f}} \tag{2'}$$

and Eq. (4) becomes

$$X\left(\frac{1+s}{2}\right) = \frac{G\hat{g}}{2} \;, \tag{4'}$$

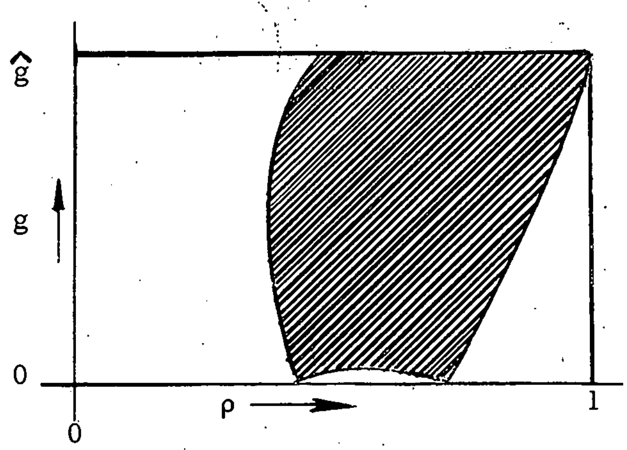
so

$$S = G\hat{g} \Big/ 2\int_{\frac{1+s}{2}}^{1} T(\rho)d\rho \;. \tag{6}$$

We are here concerned with a method for the numerical computation of an approximate solution for these equations. To this end we introduce quantities $\left\{P_j^n, \hat{P}_j^n, R_j^n, \varphi_j, Q_k^n, X_k^n, T_k^n, f_{j,k}^n, S^n\right\}$ for $n = 0,1,\ldots$, $j = 0,\ldots, J$ and $k = 0,\ldots, K$ which will approximate $\left\{P(g_j), P'(g_j), R(g_j), \varphi(g_j), Q(\rho_k), X(\rho_k), T(\rho_k), f(\rho_k, g_j), S\right\}$, respectively, where $g_j = j\Delta$, $\rho_k = k\delta$, $\Delta = \hat{g}/J$, and $\delta = 1/K$. These are defined recursively in n (i.e., by an iteration scheme); initially a $\mathcal{Q}^{-1} = (Q_0^{-1},\ldots, Q_k^{-1})$ is assumed, then the other quantities are calculated till $\mathcal{Q}^0$ can be computed and so on. The sequence is: from $\mathcal{Q}^{n-1}$

and $P_J^n = -\hat{g}$ one computes $f_{J,k}^n$ for $k = K, K-1, \ldots$, and then, by a numerical integration, computes R_J^n and $\hat{P}_J^n$ and so P_{J-1}^n, repeating this for $j = J, J-1, \ldots$ to obtain $\mathcal{f}^n, \mathcal{R}^n$, and $\mathcal{P}^n$; from these $\mathcal{T}^n$ can be computed by numerical integrations over k (i.e., ρ); S^n is then determined by a numerical analogue of Eq. (6), and, finally, $\mathcal{X}^n$ and $\mathcal{Q}^n$ can then be found by a final numerical (indefinite) integration.

The intervals over which the above-mentioned integrations are to be taken depend on the region on the j,k lattice for which $\mathcal{f}^n = \left\{ f_{j,k}^n \right\}$ is positive. It is convenient to discuss this (as well as certain other points) before describing the details of the calculation. From physical considerations and experimental numerical calculation (done in the case $G = 4$, $\hat{g} = 10$, $s = 0.7$), it appears that $f(\rho, g)$ is positive in a region of the form of the shaded area in Diagram A:



(A)

The integrals involved in the definition of R(g) and P′(g) are taken over the intersection of horizontal lines (g = constant) with this shaded area. Due to the factor $1/\sqrt{f}$ occurring in the integrands these integrals are singular, but if $f_\rho = \partial f/\partial\rho \neq 0$ is assumed at the point on this line at which $f = 0$, this is an integrable singularity. What is, perhaps, more serious is the fact that lines near the bottom (g = constant close to zero) intersect the region in two disjoint intervals. This bifurcation corresponds to the separation of the low-energy electrons into two classes. On physical grounds there is reason to believe, first, that the contribution to the final result of such low-energy electrons is negligible and, second, that the phenomenon will not occur when, eventually, the scattering effects are introduced. Since a correct treatment of this area would involve a difficult modification of the equations, and since the difference should in any case be negligible, it was decided to stop calculation at the beginning of bifurcation, i.e., to assume, at least mathematically, that these low-energy electrons simply cease to exist.

Certain asymptotic approximations may now be made. One notes, first, that if $P'(\hat{g})$ is greater than -1 (which we observe, experimentally, to be the case), then $g' = \frac{dg}{d\rho}\Big|_1 = -f_\rho/f_g = \hat{g}(G-1)/(P'+1)$ is positive and finite, where $g = g(\rho)$ gives the curve along which $f(\rho,g) = 0$; this to be taken in the neighborhood of the upper right-hand corner of Diagram A. It follows that for small ε

$$T(1-\varepsilon) = \frac{\hat{g}\,\varphi(\hat{g})}{R(\hat{g})}\int_{\hat{g}-g'\varepsilon}^{\hat{g}} \frac{dg}{\sqrt{f(1-\varepsilon,g)}} + \mathcal{O}(\varepsilon)$$
$$= \alpha\int_0^{\varepsilon} \frac{d\xi}{\sqrt{\varepsilon-\xi}} + \mathcal{O}(\varepsilon), \tag{7}$$

where α is a real, positive constant. Thus, as ε approaches 0,

$$T(1-\varepsilon) = \beta\sqrt{\varepsilon} + \mathcal{O}(\varepsilon), \tag{7'}$$

where β is another positive constant and, in particular,

$$X'(1) = ST(1) = 0. \tag{8}$$

On the other side of Diagram A let us consider the nature of $T(\rho)$. It is clear that the integral that defines T in (5) makes sense only as far as the shaded region extends. From physical considerations it appears appropriate to define T as identically zero for smaller ρ. If one lets ρ_{min} be the smallest ρ for which there exist g such that $f(\rho,g) \geq 0$ one has, then,

$$T(\rho) = 0 \qquad \rho < \rho_{min} \tag{9}$$

For $T(\rho)$ as $\rho \downarrow \rho_{min}$ there are two possibilities depending on whether (a) there is a bulge (as shown in Diagram A) so that $\left.\frac{dg}{d\rho}\right|_{\rho_{min}}$ becomes infinite, or (b) $\frac{dg}{d\rho}$ is less than 0 as ρ approaches ρ_{min} and $g = g(\rho)$ approaches $\hat{g}$.

We consider case (a) first. Let g_* be equal to $g(\rho_{min})$. It is more convenient, here, to think of ρ as a function of g, $\rho = \rho(g)$, in this region so $\rho_{min} = \rho(g_*)$ and $\rho' = \left.\frac{d\rho}{dg}\right|_{g_*} = 0$. Clearly $\rho'' = \left.\frac{d^2\rho}{dg^2}\right|_{g_*} > 0$. Then, letting $\rho - \rho_{min} = \varepsilon^2$ and $g - g_* = \gamma$, one has

$$f(\rho,g) = f(\rho_{min} + \varepsilon^2, g_* + \gamma) = f_\rho\Big|_{\rho_{min}} \left[\varepsilon^2 - \frac{\rho''}{2}\gamma^2\right] + \mathcal{O}(\varepsilon^2),$$

and, letting $\tilde{\gamma} = \gamma\sqrt{\rho''/2}$, we have

$$T(\rho_{min} + \varepsilon^2) = \frac{\sqrt{(2/\rho'')}\ \varphi(g_*)\ \left[Q(\rho_{min}) - P(g_*)\right]}{\sqrt{f_\rho}\ \ \rho_{min}\ R(g_*)} \int_{-\varepsilon}^{\varepsilon} \frac{d\tilde{\gamma}}{\sqrt{\varepsilon^2 - \tilde{\gamma}^2}} + \mathcal{O}(T)$$

$$= \beta/\varepsilon + \mathcal{O}(1/\varepsilon), \tag{10a}$$

where β is a positive constant. Thus

$$T(\rho_{min} + \varepsilon) \sim 1/\sqrt{\varepsilon}\ . \tag{10a$'$}$$

In case (b) one has, by an analysis essentially the same as that used to find $T(1 - \varepsilon)$, that

$$T(\rho_{min} + \varepsilon) \sim \sqrt{\varepsilon}. \tag{10b$'$}$$

The one remaining bit of asymptotic analysis required is that, in calculating the approximation to R, if $f^n_{j,k} \geqslant 0$, $f^n_{j,k-1} > 0$, $f^n_{j,k+1} < 0$, and ρ_* is such that $f(\rho_*, g_j) = 0$, then

$$\int_{(k-3/2)\delta}^{\rho_*} \frac{\rho\, d\rho}{\sqrt{f(\rho,g_j)}} = \rho_k \frac{\delta}{\sqrt{f^n_{j,k-1}}} \left(\frac{5}{2} + \frac{f^n_{j,k}}{f^n_{j,k} - f^n_{j,k+1}}\right) + \mathcal{O}(\delta), \tag{11}$$

and similarly for the other integrations over elements at the singularities of $1/\sqrt{f}$ which are involved in computing the approximations to R, P$'$, and T.

We are now ready to go through the description of the iteration procedure. This starts with the computation of the φ_j, g_j, $g_j^2 = (g_j)^2$, ρ_k, $\rho_k^2 = (\rho_k)^2$, and an initial set $\mathcal{Q}^{-1} = \left\{Q_k^{-1}\right\}$; after that all the iterations proceed identically. The initial $Q(\rho)$ is chosen as a polynomial satisfying $Q(1) = 0$, $X(1) = G\hat{g}$, $X\left((1+s)/2\right) = G\hat{g}/2$ with $X(\rho)$ linear $\left[X'' = 0\right]$. The general step of the iteration scheme goes as follows.

First we consider the computation of $\mathcal{f}^n, \mathcal{R}^n, \hat{\mathcal{P}}^n$, and $\mathcal{P}^n$ from $\mathcal{Q}^{n-1}$. We know from Eq. (3) that $P_J^n = -\hat{g}$. It is then possible to compute $f_{J,K}^n = 0$, $f_{J,K-1}^n = \rho_{K-1}^2 g_J^2 - \left[P_J^n - Q_{K-1}^{n-1}\right]^2 > 0$, and so on for $k = K, K-1, \ldots$. As the integration is between the zeroes of f, this need only continue until $f_{J,k} \leqslant 0$. Let the first such k reached be $\kappa = \kappa^n$; then one computes only $f_{J,k}^n$ for $k = K, K-1, \ldots, \kappa$. One now does a numerical integration to obtain simultaneously approximations to $UI_J^n = \int (P-Q)\rho \sqrt{f}^{-1} d\rho$ and $R_J^n = \int \rho \sqrt{f}^{-1} d\rho$. These are done by first finding the contributions from the elements at the end-points of the interval by using Eq. (11) and such similar formulas, and then obtaining the contributions from the interior elements of the interval by the trapezoidal rule. Thus, e.g., if $K = 90$ and $\kappa^n = 50$, one would have

$$R_J^n = \delta \left[\frac{1}{\sqrt{f_{J,89}^n}} \left(\frac{5}{2} + \frac{0}{?}\right) + \frac{\rho_{51}}{\sqrt{f_{J,52}^n}} \left(\frac{5}{2} + \frac{2\, f_{J,51}^n}{f_{J,51}^n - f_{J,50}^n}\right) + \sum_{k=53}^{88} \frac{\rho_k}{\sqrt{f_{J,k}^n}} \right], \quad (12)$$

and similarly for the upper integral. Then

$$\hat{P}_J^n = \frac{UI_J^n}{g_J R_J^n} \quad (13)$$

and

$$P_{J-1}^n = P_J^n - \hat{P}_J^n \Delta . \quad (14)$$

It is now possible to compute $f_{J-1,k}^n$ for $k = K, \ldots, E_{J-1}^n$, where $E_J^n = \kappa^n$ and

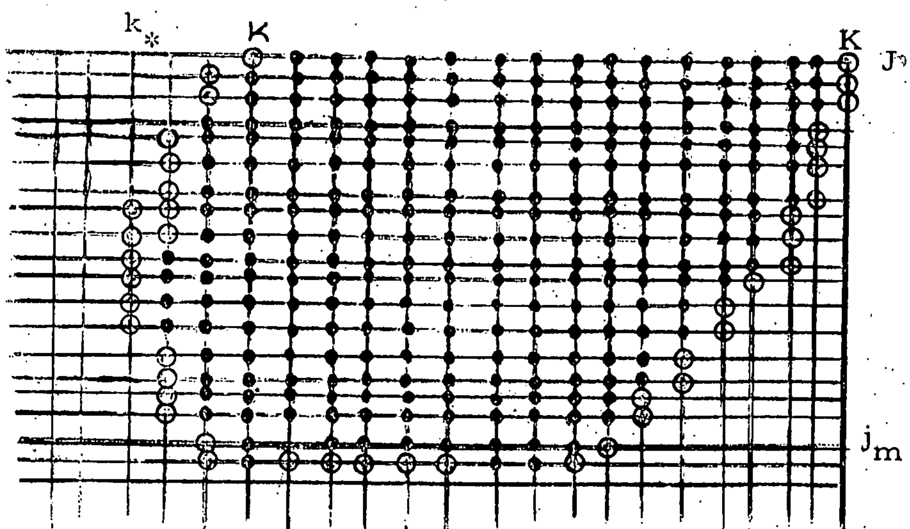
E^n_J is correspondingly defined for the line $j=J-1$. We also let F^n_{J-1} be the last k (working down from K) for which $f_{J-1,k} \leq 0$. Clearly $F^n_J = K$. Then, assuming, e.g., that $F^n_{J-1} = 90$ and $E^n_{J-1} = 49$, one would have

$$R^n_{J-1} = \delta\left[\frac{\rho_{89}}{\sqrt{f^n_{J-1,88}}}\left(\frac{5}{2} + \frac{2\,f^n_{J-1,89}}{f^n_{J-1,89} - f^n_{J-1,90}}\right) + \frac{\rho_{50}}{\sqrt{f_{J-1,51}}}\left(\frac{5}{2} + \frac{2\,f^n_{J-1,50}}{f^n_{J-1,50} - f^n_{J-1,49}}\right) + \sum_{k=52}^{87}\frac{\rho_k}{\sqrt{f^n_{J-1,k}}}\right], \qquad (12')$$

and similarly for UI^n_{J-1}. Then

$$P^n_{J-2} = P^n_{J-1} - \frac{UI^n_{J-1}\Delta}{g_{J-1}\,R^n_{J-1}}, \qquad (14')$$

and the process continues. This goes on for $j=J, J-1, \ldots, j_m$, where either j_m is equal to 0 or j_m is the line above that at which the bifurcation phenomenon is first observed. We arrive, then, at Diagram B:



where "o" denotes an $f^n_{J,k} \leq 0$, "•" denotes an $f^n_{j,k} > 0$, and unmarked points are those for which $f^n_{j,k}$ is not computed. Clearly $k_* = k^n_*$ is the next index below ρ_{min}. An approximation to ρ_{min} that is better than k_* is

$$\rho^n_* = \min\left\{E^n_j - f^n_{j,E^n_j}\Big/\left(f^n_{j,E^n_j+1} - f^n_{j,E^n_j}\right) : j=J,\ldots j_m\right\}, \qquad (15)$$

and one can also let g_*^n be the g_j for the corresponding j. One can now distinguish between case (a) and case (b) as described above according as $g_*^n \neq \hat{g}$ or $g_*^n = \hat{g}$, respectively; the possibility of $g_* = g(\rho_{min}) = \hat{g}$ but $\left.\frac{dg}{d\rho}\right|_{\rho_{min}} = \infty$ is negligible.

At this point it is possible to compute T_k^n for $k=K-1,\ldots,k_*^n + 1$ by a numerical integration exactly similar to that described above. Although the situation is unlikely to arise during the ρ integration (but is to be provided for anyway), in the g integration it is quite possible to come across intervals of integration involving too few mesh points for the direct application of formulas analogous to Eq. (12), and precaution must be taken to identify such cases and make the obvious modifications in the formulas. It must also be observed that in the g integration one can have end-points of intervals of integration at which the integrand is not singular; i.e., at $g = \hat{g}$ or g_{j_m}. At such points the trapezoidal rule is to be used right to the end. It should be noted that when this occurs at $\hat{g}$ one uses only a half interval whereas at g_{j_m} one uses a full interval. Thus an integral involving both would be actually, from $(j_m - 1/2)\Delta$ to $J\Delta = \hat{g}$.

It will be noted that one cannot obtain T_K^n by this method. On looking back to Eq. (8) one notes that $T(1) = 0$. For present purposes, however, it is more relevant to look back to Eq. (7′) and note that if one takes

$$T_K^n = \frac{1}{3} T_{K-1}^n \tag{16}$$

rather than $T_K^n = 0$, one obtains a more accurate approximation to X_{K-1}^n, etc., on integrating T.

It is now possible to obtain S^n from

$$S^n = - G\hat{g}/\delta \left[T_H^n + 2\, T_{H+1}^n + \ldots + 2\, T_{K-1}^n + T_K^n\right], \tag{6′}$$

where H is the integral part of $(1 + s)/2\delta$, and then to determine X_k^n and Q_k^n for $k=K,K-1,\ldots,k_*+1$ from $X_K^n = G\hat{g}$, $Q_K^n = 0$, and

$$\left.\begin{aligned} X_k^n &= X_{k+1}^n - \frac{1}{2}\,\delta\, S^n \left(T_k^n + T_{k+1}^n\right) \\ Q_k^n &= Q_{k+1}^n - \frac{1}{2}\,\delta \left(\rho_k\, X_k^n + \rho_{k+1} X_{k+1}^n\right) \end{aligned}\right\} . \tag{2''}$$

To obtain X_k^n and Q_k^n for $k \leqslant k_*^n$, one looks back to Eq. (10′) [(a) or (b), as is appropriate] and Eq. (9); so for $k \leqslant k_*$ in case (a), we have

$$X_k^n = X_{k_*}^n = X_{k_*+1}^n - 2\, S^n\, a_*\, T_{k_*+1}^n\,, \tag{17}$$

where $a_* = \left[(k_*+1)\delta - \rho_*^n\right]$. Case (b) is handled exactly as is case (a). While this is not strictly accurate, it was felt that the difference is unimportant. Similarly, one obtains

$$Q_{k_*}^n = Q_{k_*+1}^n - X_{k_*}\left(k_* + \frac{1}{2}\right)\delta^2 - 4S^n T_{k_*+1}^n a_*^2 \left(\frac{a_*}{5} + \frac{\rho_*^n}{3}\right) \tag{18}$$

from which Q_k^n can be determined for $k < k_*$ by the lower line of (2″).

This completes the cycle of the iteration and it is then possible to return to the computation of ρ^{n+1}, and so on.

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