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MAXIMUM VOLUME-TO-STRESS RATIO FOR A TWO-RADII-CONTOUR DIAPHRAGM PUMP

R. D. Cheverton

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CONTENTS

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ABSTRACT

Recent experimental work with diaphragm pumps employing the two-radii type of contoured heads indicates that an optimum ratio of the two radii exists which provides a maximum ratio of displacement volume to stress. The purpose of this study was to determine by analytical methods whether an optimum design does exist and, if so, what it is. In order to do this, it was necessary to establish a reasonable criterion for failure. The proposed criterion considers the effect of biaxial stresses on fatigue failure through the use of the Mises-Hencky criterion for fatigue failure. By use of the proposed criterion, it was determined that an optimum ratio of the two radii does exist, its value being dependent on the ratio of diaphragm thickness to diaphragm deflection. Values for the optimum ratio of the two radii (where the ratio of radii is defined as the radius of the central portion of the diaphragm contour divided by the radius of the outer portion of the diaphragm) range from 1.94 to 7.33 as the ratio of diaphragm thickness to diaphragm deflection varies from 0.5 to 0.05, respectively.

INTRODUCTION

To optimize the design of diaphragm pumps from the standpoint of size, weight, displacement, and operating lifetime, it is necessary to maximize the ratio of displacement volume to diaphragm stress. Recent experimental work with diaphragm pumps employing the two-radii type of contoured heads indicates that an optimum ratio of the two radii exists which provides a maximum ratio of displacement volume to stress. Prior to this study there apparently has been no effort to establish by analytical means the existence of the optimum ratio of the two radii.

The treatment in this report is limited to the two-radii-contour type of diaphragm pump. In making the analysis an effective combined stress, based on the Mises-Hencky criterion of fatigue failure for combined stress, was used in calculating the volume-to-stress ratio.

METHOD OF ANALYSIS

Diaphragm Stresses

Figure 1 illustrates the geometrical features considered for the diaphragm-pump contoured head. For purposes of calculating the stresses, the diaphragm is divided into two regions. Region A is for $0 \leqq r \leqq za$, and region B is for $za \leqq r \leqq a$, where $z = R_1/(R_1 + R_2)$.

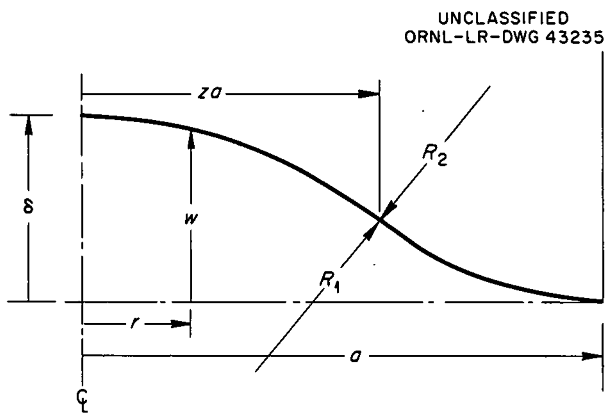

Fig. 1. Geometrical Features of Two-Radii-Type Diaphragm-Pump Contoured Head.

In the following analysis it is assumed that the diaphragm deflection curve matches the two-radii head contour perfectly. This assumption is valid since the maximum stress occurs when the diaphragm is fully deflected against the head contour. Therefore the deflection equations for both regions A and B are derived from the equation of a circle. They are presented here as Eqs. (1) and (2) (see "Nomenclature" at the end of this report for a definition of symbols):

$$w^A = \delta - R_1 + \left(R_1^2 - r^2\right)^{1/2} = \delta - r\left[\frac{1}{2}\left(\frac{r}{R_1}\right) + \frac{1}{8}\left(\frac{r}{R_1}\right)^3 + \ldots\right] \quad ; \qquad (1)$$

$$w^B = R_2 - \left[R_2^2 - (r-a)^2\right]^{1/2} = (a - r)\left[\frac{1}{2}\left(\frac{a-r}{R_2}\right) + \frac{1}{8}\left(\frac{a-r}{R_2}\right)^3 + \ldots\right] \quad . \qquad (2)$$

Provided that $r/R_1 << 1$ and $(a - r)/R_2 << 1$, Eqs. (1) and (2) are adequately approximated as follows:

$$w^A \cong \delta - \frac{r^2}{2R_1} \quad , \qquad (3)$$

$$w^B \cong \frac{(a-r)^2}{2R_2} \quad . \qquad (4)$$

Equations (3) and (4) were used in the derivation of the stress equations and in calculating the volumetric displacements of the pumps.

Since the deflection of the diaphragms considered is several times the thickness of the diaphragms (although small in comparison with other dimensions) the strain in the middle plane of the diaphragm could not be neglected. Thus the membrane stresses, as well as the bending stresses, were considered. The equations (see the Appendix for derivations) for the membrane stresses are as follows:

$$\frac{\sigma_{rM}^{A} a^2}{E\delta^2} = -\frac{1}{4z^2}\left(\frac{r}{a}\right)^2 - \frac{1}{(1-z)^2}\cdot\left[\ln z - z + \frac{z^2(1+\nu)}{4(1-\nu)} - \frac{z^3(1+\nu)}{6(1-\nu)} + \frac{11-13\nu}{12(1-\nu)}\right] , \quad (5)$$

$$\frac{\sigma_{tM}^{A} a^2}{E\delta^2} = \frac{\sigma_{rM}^{A} a^2}{E\delta^2} - \frac{1}{2z^2}\left(\frac{r}{a}\right)^2 , \quad (6)$$

$$\frac{\sigma_{rM}^{B} a^2}{E\delta^2} = \frac{1}{(1-z)^2}\left[\ln\frac{a}{r} + \frac{4}{3}\left(\frac{r}{a}\right) - \frac{1}{4}\left(\frac{r}{a}\right)^2 + \frac{z^2}{4}\left(\frac{2}{3}z - 1\right)\left(\frac{a}{r}\right)^2 + \right.$$

$$\left. + \frac{z^2(1+\nu)}{4(1-\nu)}\left(\frac{2}{3}z - 1\right) - \frac{11-13\nu}{12(1-\nu)}\right] , \quad (7)$$

$$\frac{\sigma_{tM}^{B} a^2}{E\delta^2} = \frac{1}{(1-z)^2}\left[\ln\frac{a}{r} + \frac{8}{3}\left(\frac{r}{a}\right) - \frac{3}{4}\left(\frac{r}{a}\right)^2 - \frac{z^2}{4}\left(\frac{2}{3}z - 1\right)\left(\frac{a}{r}\right)^2 + \right.$$

$$\left. + \frac{z^2(1+\nu)}{4(1-\nu)}\left(\frac{2}{3}z - 1\right) - \frac{23-25\nu}{12(1-\nu)}\right] . \quad (8)$$

The bending moments in a circular plate are represented approximately by the following equations:[1]

$$M_r = -D\left(\frac{d^2 w}{dr^2} + \frac{\nu}{r}\frac{dw}{dr}\right) , \quad (9)$$

$$M_t = -D\left(\frac{1}{r}\frac{dw}{dr} + \nu\frac{d^2 w}{dr^2}\right) . \quad (10)$$

Using Eqs. (3) and (4) and the relationship $\delta = a^2 z/2R_1$, the bending stresses are given by

$$\frac{\sigma_{rB}^{A} a^2}{E\delta^2} = \frac{\sigma_{tB}^{A} a^2}{E\delta^2} = \frac{h}{\delta z(1-\nu)} , \quad (11)$$

$$\frac{\sigma_{rB}^{B} a^2}{E\delta^2} = \frac{h}{\delta(1-z)(1-\nu^2)}\left[(1+\nu) - \frac{\nu a}{r}\right] , \quad (12)$$

$$\frac{\sigma_{tB}^{B} a^2}{E\delta^2} = \frac{h}{\delta(1-z)(1-\nu^2)}\left[(1+\nu) - \frac{a}{r}\right] . \quad (13)$$

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[1]S. Timoshenko, *Theory of Plates and Shells*, 1st ed., McGraw-Hill, New York, 1940.

Failure Criterion

During operation of the pump, the diaphragm is deflected from $-\delta$ to $+\delta$ in a continuous cycle. With the diaphragm on either side of the neutral position the sign of the membrane stresses is the same, but the bending stresses change sign as the diaphragm is deflected from one side to the other of the neutral position. Therefore the stress-vs-displacement curve is similar to that shown in Fig. 2. The problem now is one of selecting a suitable criterion for failure, where failure in this case may be defined as a fatigue crack. Since the fatigue strength of materials is greatly influenced by many variables such as surface finish and environment, and since there is not a great deal known about fatigue properties for combined stress conditions, the selection of a suitable failure criterion is difficult and is not likely to produce a criterion that is necessarily accurate for all cases. Therefore, in a somewhat arbitrary fashion, the Mises-Hencky[2] criterion for complete reversal of combined stresses was selected and is represented here by Eq. (14):

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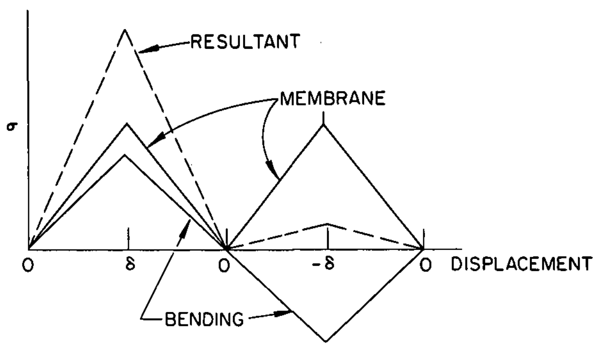

Fig. 2. Stress-Displacement Curves for a Diaphragm Deflected from $+\delta$ to $-\delta$.

$$\sigma_e \geqq \sqrt{\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2} \quad . \tag{14}$$

Here σ_1 and σ_2 are the principal stresses, and σ_e is the endurance stress for the material, assuming complete reversal of stresses. Thus, the effect of combined stresses on fatigue is considered. The equation implies that, if more than about 10^7 cycles of reversed stresses are desired without a fatigue failure, σ_1 and σ_2 must be such as to produce a value on the right-hand side of Eq. (14) not greater than the endurance limit of the material. Therefore, Eq. (14) provides an effective combined stress, σ_c, that might be useful in comparing diaphragm designs, where

$$\sigma_c \equiv \sqrt{\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2} \quad . \tag{15}$$

As mentioned above, Eq. (14) is strictly applicable only for complete reversal of stresses, and as indicated in Fig. 2, such reversal does not exist for the diaphragm

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[2]M. Hetényi, *Handbook of Experimental Stress Analysis*, p 450, Wiley, New York, 1950.

pumps. In order to treat the actual case, or a slight modification thereof, use is made of Gerber's parabola or, more precisely, the modified Goodman diagram. The diagram used is illustrated in Fig. 3.

The equation for the diagonal line in Fig. 3 is

$$\frac{1}{K} = \frac{\sigma_r}{\sigma_e} + \frac{\sigma_{ave}}{\sigma_{ult}} \ , \tag{16}$$

where

$$\sigma_r = \frac{\sigma_{max} - \sigma_{min}}{2} \ ,$$

$$\sigma_{ave} = \frac{\sigma_{max} + \sigma_{min}}{2} \ ,$$

σ_e = endurance limit of material,

σ_{ult} = ultimate strength of material,

K = safety factor or experimental correlation factor.

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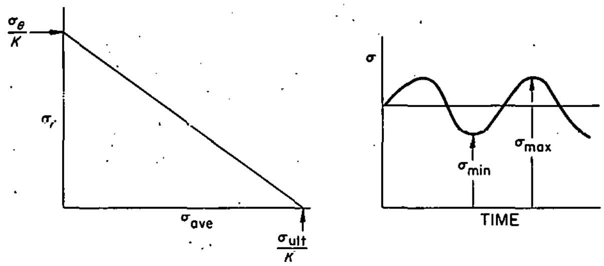

Fig. 3. Application of Modified Goodman Diagram to Diaphragm Analysis.

The hypothesis is that a point anywhere on or below the diagonal line indicates that an essentially infinite number ($>10^7$) of cycles is permitted without fatigue failure. Referring now to Fig. 2, it is observed that the diaphragm cycle consists of two stress peaks having different amplitudes. The smaller peak may be neglected, provided that the maximum peak gives a point on or below the diagonal line in Fig. 3. Under these conditions $\sigma_{min} = 0$, and Eq. (16) can be rearranged to yield

$$\sigma_{max} = \frac{2\sigma_e \sigma_{ult}}{K(\sigma_e + \sigma_{ult})} \ . \tag{17}$$

In Eq. (17) σ_{max} is considered to be an effective endurance limit for the diaphragm. Therefore, substituting σ_{max} for σ_e in Eq. (14), the proposed failure criterion is given by the relation

$$(\sigma_e)_{effective} = \frac{2\sigma_e \sigma_{ult}}{K(\sigma_e + \sigma_{ult})} \geqq \sqrt{\sigma_1^2 + \sigma_2^2 - \sigma_1 \sigma_2} \ . \tag{18}$$

If σ_e and σ_{ult} are known for a particular material that is subjected to a set of specified conditions, then values for σ_1 and σ_2, which satisfy Eq. (18), can be obtained by the appropriate selection of values for the parameters in Eqs. (5) through (13). The value of K should reflect the accuracy with which σ_e and σ_{ult} are known, as well as the validity of the failure criterion, and should be as close to unity as possible to obtain the maximum volumetric displacement for a given pump.

Optimization of Volume-to-Stress Ratio

To minimize the size of a pump, the displacement-volume-to-stress ratio should be as large as possible. The existence of a value of z that would produce a maximum volume-to-stress ratio was postulated by Hise[3] on the basis of considerable experimental work. Examination of Eq. (19),

$$\frac{V}{2\delta a^2} = 2\pi \left[\frac{z^2}{2} - \frac{z^3}{4} + \frac{1}{1-z}\left(\frac{1}{12} - \frac{z^4}{4} + \frac{2z^3}{3} - \frac{z^2}{2}\right)\right] \quad , \tag{19}$$

which represents the pump displacement volume from $-\delta$ to $+\delta$, and Eqs. (5) through (13), which represent the diaphragm stresses, indicates that an optimum value of z would depend only on the dimensionless ratio b/δ.

When calculating the volume-to-stress ratio for a particular pump having fixed values for z and b/δ, the maximum stress with respect to r/a must be used, yielding the minimum volume-to-stress ratio for the particular pump design. Using Eqs. (15) and (19), the latter volume-to-stress ratios were calculated and plotted against z in Fig. 4 for several values of b/δ. It is observed that optimum values of z do exist for the model being considered in this study.

If a pump is designed with an optimum z, there will be two points at which the maximum stress occurs: one at the center of the diaphragm and one somewhere in region B, the exact location depending on the value of z and b/δ. If the pump has a z less than the optimum, the maximum stress will be at the center of the diaphragm, and if z is greater than optimum, the maximum stress will be somewhere in region B.

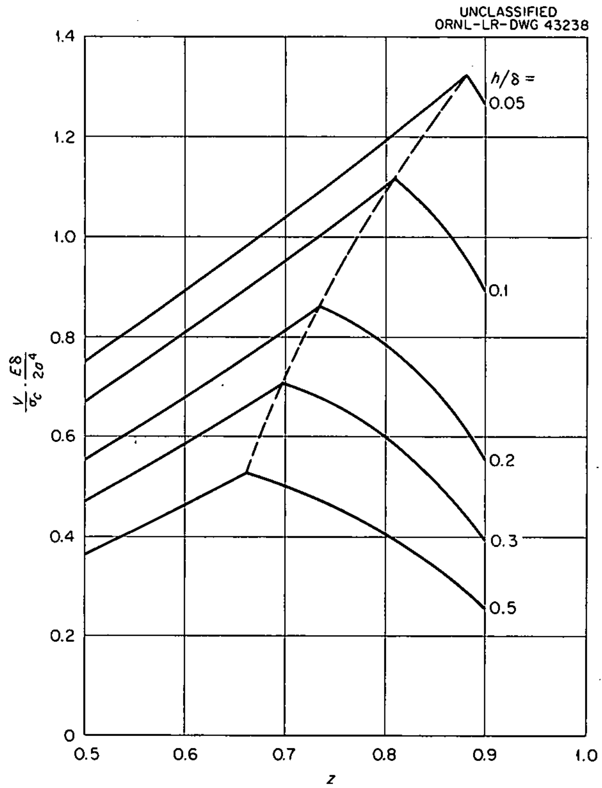

Fig. 4. Volume-Stress Ratio vs z for Various Values of b/δ, Using Combined Stress, σ_c.

[3]E. C. Hise, ORNL, private communication, November 1958.

Evaluation of Method for Determining Optimum z

How accurately does the above analysis determine the optimum value of z? Avoiding a comparison with fatigue-failure-type experimental data at this time, the question might best be answered in terms of the accuracy of the individual stresses and of the criterion used for predicting failure. A very limited amount of experimental data acquired by Zerby and Stevens[4] indicates that actual radial stresses in region A are about 25% less than calculated and those in region B about 10% greater. This results in about a 15% increase in the optimum value of z, assuming that Eq. (18) is an adequate criterion for predicting failure. Whether or not the same results apply for all values of b/δ is not known since an insufficient amount of data is available. The adequacy of Eq. (18) is questionable and will remain so until experimental data from diaphragm tests prove its validity. For this reason, Eq. (18) was compared with a simple maximum-stress criterion of failure, for which the maximum principal stress, σ_p, replaces the radical term in Eq. (18) to give Eq. (20):

$$\sigma_{max} = \frac{2\sigma_e \sigma_{ult}}{K(\sigma_e + \sigma_{ult})} = \sigma_p \quad . \qquad (20)$$

The results, illustrated in Fig. 5, show that the optimum value of z, using the maximum-combined-stress criterion defined by Eq. (15), gives optimum z values 1.5% ($b/\delta = 0.5$) to 6.6% ($b/\delta = 0.1$) greater than those obtained using the maximum-principal-stress criterion. The close agreement between the two methods does not necessarily indicate the accuracy of either. However, it does indicate that either method is probably equally good for computing optimum z values.

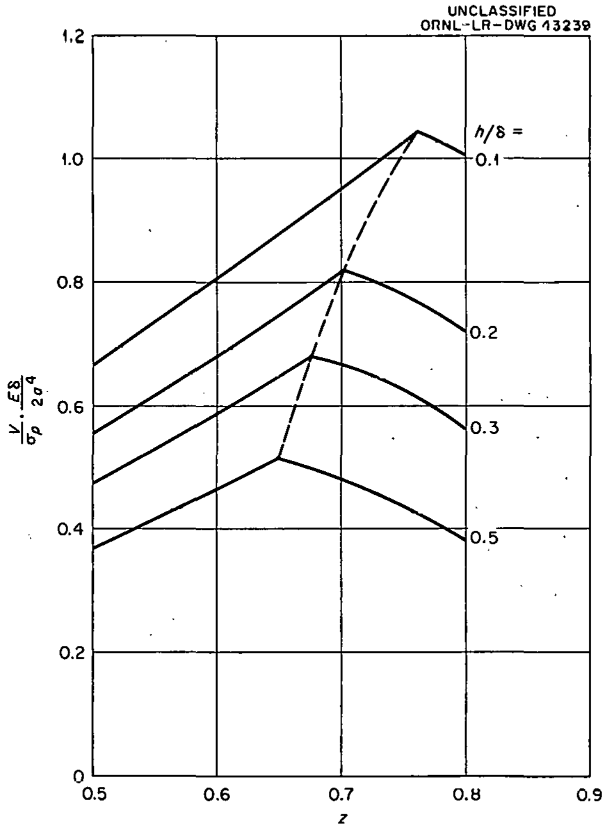

Fig. 5. Volume-Stress Ratio vs z for Various Values of b/δ, Using Maximum Principal Stress, σ_p.

[4]P. N. Stevens, "Pulsafeeder Diaphragm Studies," inter-company correspondence, ORNL, September 1953.

Appendix

DERIVATION OF EQUATIONS FOR MEMBRANE STRESSES IN A DIAPHRAGM

The following analysis[5] is applicable to a diaphragm that is deflected a distance equal to several times the diaphragm thickness, in which case the strain in the mid-plane must not be neglected; the deflection, however, is considered small in comparison with other deminsions. A system such as this is typical of many types of diaphragm pumps.

The equilibrium and continuity equations for a diaphragm of the type described above (considering membrane stresses only) are derived from a force balance on an element of the diaphragm and from Hooke's law, respectively.[6] Consider the element in Fig. 6, subjected to the membrane forces N_r and N_t. A summation of the forces in the radial direction gives

$$N_r r \, d\theta + 2N_t \, dr \frac{d\theta}{2} = \left(N_r + \frac{dN_r}{dr} dr\right) (r + dr) \, d\theta \ ,$$

or

$$N_r - N_t + r \frac{dN_r}{dr} = 0 \ . \tag{1A}$$

From Hook's law,

$$N_r = \frac{Eh}{1 - \nu^2} (e_r + \nu e_t) \ , \tag{2A}$$

$$N_t = \frac{Eh}{1 - \nu^2} (e_t + \nu e_r) \ . \tag{3A}$$

Referring to Fig. 7, the radial unit elongation of the element due to the radial displacement u is du/dr. The unit elongation due to the normal displacement w is

[5]This method of analysis is similar to that used by C. D. Zerby (unpublished analysis, January 1953) and P. N. Stevens ("Pulsafeeder Diaphragm Studies," inter-company correspondence, ORNL, September 1953).

[6]S. Timoshenko, *Theory of Plates and Shells*, 1st ed., McGraw-Hill, New York, 1940.

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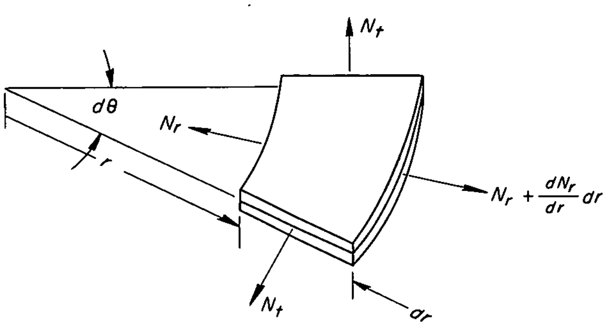

Fig. 6. Forces on Element of Diaphragm.

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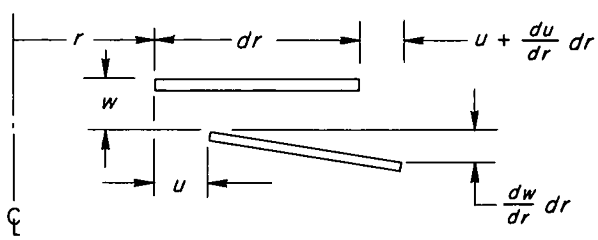

Fig. 7. Displacements of Element of Diaphragm.

$1/2(dw/dr)^2$. In the circumferential direction the unit elongation is just u/r. Therefore, the tangential and radial strain components are given by

$$e_t = \frac{u}{r} , \tag{4A}$$

$$e_r = \frac{du}{dr} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2 . \tag{5A}$$

Adding Eqs. (2A) and (3A) and making use of Eqs. (4A) and (5A) gives

$$\frac{1}{Eb}(N_r - \nu N_t) = e_r = \frac{du}{dr} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2 , \tag{6A}$$

$$\frac{1}{Eb}(N_t - \nu N_r) = e_t = \frac{u}{r} . \tag{7A}$$

Differentiating Eq. (7A) gives

$$Eb\frac{du}{dr} = N_t + r\frac{dN_t}{dr} - \nu\left(N_r + r\frac{dN_r}{dr}\right) . \tag{8A}$$

From Eq. (1A),

$$N_t = N_r + r\frac{dN_r}{dr} . \tag{1Aa}$$

Differentiating Eq. (1Aa) and multiplying by r,

$$r\frac{dN_t}{dr} = 2r\frac{dN_r}{dr} + r^2\frac{d^2N}{dr^2} . \tag{9A}$$

Substituting Eqs. (1Aa) and (9A) into (8A),

$$Eb\frac{du}{dr} = N_r(1-\nu) + r\frac{dN_r}{dr}(3-\nu) + r^2\frac{d^2N_r}{dr^2} . \tag{10A}$$

Substituting Eqs. (1Aa) and (10A) into (6A),

$$\frac{d^2N_r}{dr^2} + \frac{3}{r}\frac{dN_r}{dr} + \frac{Eb}{2r^2}\left(\frac{dw}{dr}\right)^2 = 0 . \tag{11A}$$

The deflection curve $w(r)$ for the fully deflected diaphragm is represented by the arcs of two different diameter circles, thus dividing the diaphragm into the two regions A and B. The solution of Eq. (11A) in the two regions follows.

Region A

For region A,

$$w = \delta - \frac{r^2}{2R_1} , \tag{12A}$$

$$\left(\frac{dw}{dr}\right)^2 = \left(\frac{r}{R_1}\right)^2 . \tag{13A}$$

Substituting Eq. (13*A*) into (11*A*),

$$\frac{d^2 N_r}{dr^2} + \frac{3}{r}\frac{dN_r}{dr} + \frac{Eh}{2R_1^2} = 0 . \tag{14A}$$

For convenience, let

$$N_r = y , \qquad r = x , \qquad \frac{Eh}{2R_1^2} = K_1 .$$

Then

$$\frac{d^2 y}{dx^2} + \frac{3}{x}\frac{dy}{dx} + K_1 = 0 . \tag{14Aa}$$

The complementary solution of Eq. (14*Aa*) is obtained from the homogeneous equation

$$\frac{d^2 y}{dx^2} + \frac{3}{x}\frac{dy}{dx} = 0 . \tag{15A}$$

In Eq. (15*A*) let

$$v = \frac{dy}{dx} , \qquad \frac{dv}{dx} = \frac{d^2 y}{dx^2} .$$

Then

$$\frac{dv}{dx} + \frac{3v}{x} = 0 ,$$

which gives

$$v = Cx^{-3} ,$$

or

$$\frac{dy}{dx} = Cx^{-3} ,$$

which gives

$$y_c = C_1 x^{-2} + C_2' . \tag{16A}$$

The particular solution of Eq. (14*Aa*) is obtained by letting

$$\frac{dy}{dx} = Cx \quad , \qquad \frac{d^2 y}{dx^2} = C \ .$$

The result is

$$y_p = -\frac{K_1 x^2}{8} + C_3 \ . \tag{17A}$$

The general solution for Eq. (14*Aa*) is

$$y = C_1 x^{-2} - \frac{K_1 x^2}{8} + C_2 \ . \tag{18A}$$

Region B

For region B,

$$w = \frac{(a - r)^2}{2R_2} \ , \tag{19A}$$

$$\left(\frac{dw}{dr}\right)^2 = \frac{(a - r)^2}{R_2^2} \ . \tag{20A}$$

Substituting Eq. (20*A*) into (11*A*),

$$\frac{d^2 N_r}{dr^2} + \frac{3}{r}\frac{dN_r}{dr} + \frac{Eb(a - r)^2}{2R_2^2 r^2} = 0 \ . \tag{21A}$$

For convenience, let

$$N_r = y \quad , \qquad r = x \quad , \qquad \frac{Eb}{2R_2^2} = K_2 \ .$$

Then

$$x^2 \frac{d^2 y}{dx^2} + 3x \frac{dy}{dx} = -K_2 (a - x)^2 \ . \tag{21Aa}$$

The complementary solution of Eq. (21*Aa*) is obtained from

$$\frac{d^2 y}{dx^2} + \frac{3}{x}\frac{dy}{dx} = 0 \ ,$$

which is the same as Eq. (15*A*). Therefore,

$$y_c = C_3 x^{-2} + C_4 \ . \tag{22A}$$

To obtain the particular solution of Eq. (21*Aa*), let

$$x = e^t \quad , \qquad \frac{dx}{dt} = e^t = x \quad ,$$

and let

$$\frac{dy}{dx} = \frac{dy}{dt}\,\frac{dt}{dx} = \frac{dy}{dt}\,\frac{1}{x} \ , \tag{23A}$$

$$\frac{d^2y}{dx^2} = \frac{1}{x^2}\,\frac{d^2y}{dt^2} - \frac{1}{x^2}\,\frac{dy}{dt} \ . \tag{24A}$$

Substituting Eqs. (23A) and (24A) into (21Aa),

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} = -K_2(a - e^t)^2 \ . \tag{25A}$$

The solution has the form

$$y = At + Be^t + Ce^{2t} \ ,$$

or

$$y = -K_2\left(\frac{a^2t}{2} - \frac{2ae^t}{3} + \frac{e^{2t}}{8}\right) \ ,$$

or

$$y_p = -K_2\left(\frac{a^2}{2}\ln x - \frac{2ax}{3} + \frac{x^2}{8}\right) \ . \tag{26A}$$

The general solution for Eq. (21Aa) is

$$y = C_3x^{-2} + C_4 - K_2\left(\frac{a^2}{2}\ln x - \frac{2ax}{3} + \frac{x^2}{8}\right) \ . \tag{27A}$$

Substituting $y = N_r$ and $x = r$ into Eqs. (18A) and (27A),

$$N_r^A = \frac{C_1}{r^2} - \frac{K_1r^2}{8} + C_2 \ , \tag{18Aa}$$

$$N_r^B = \frac{C_3}{r^2} + C_4 - K_2\left(\frac{a^2}{2}\ln r - \frac{2ar}{3} + \frac{r^2}{8}\right) \ . \tag{27Aa}$$

Differentiating Eqs. (18Aa) and (27Aa) and substituting into (1Aa) gives

$$N_t^A = -\frac{C_1}{r^2} - \frac{3}{8}K_1r^2 + C_2 \ , \tag{28A}$$

$$N_t^B = -\frac{C_3}{r^2} + C_4 - K_2\left(\frac{a^2}{2}\ln r + \frac{a^2}{2} - \frac{4ar}{3} + \frac{3}{8}r^2\right) \ . \tag{29A}$$

Boundary Conditions

The boundary conditions are:

1. Forces must be finite for all values of r.
2. At $r = za$, $N_r^A = N_r^B$ and $N_t^A = N_t^B$.
3. At $r = a$, $e_t = 0$; thus from Eq. (7A), $N_t^B = \nu N_r^B$.

From boundary condition 1, $C_1 = 0$. From boundary condition 2,

$$C_3 = -\frac{z^2 a^4}{2} K_2 \left[\frac{1}{2} - \frac{2z}{3} + \frac{z^2}{4}\left(1 - \frac{K_1}{K_2}\right)\right] \quad , \tag{30A}$$

$$C_2 = C_4 - \frac{K_2 a^2}{2}\left[\ln za + \frac{1}{2} - 2z + \frac{z^2}{2}\left(1 - \frac{K_1}{K_2}\right)\right] \quad . \tag{31A}$$

From boundary condition 3,

$$C_4 = \frac{C_3(1+\nu)}{a^2(1-\nu)} + \frac{K_2 a^2}{2}\left[\ln a - \frac{11 - 13\nu}{12(1-\nu)}\right] \quad . \tag{32A}$$

Substituting Eq. (30A) into (32A),

$$C_4 = \frac{K_2 a^2}{2}\left[-\frac{z^2(1+\nu)}{2(1-\nu)} + \frac{2z^3(1+\nu)}{3(1-\nu)} - \frac{z^4(1+\nu)}{4(1-\nu)}\left(1 - \frac{K_1}{K_2}\right) + \ln a - \frac{11 - 13\nu}{12(1-\nu)}\right] \quad . \tag{33A}$$

Substituting Eq. (33A) into (31A),

$$C_2 = \frac{K_2 a^2}{2}\left\{-\ln z + 2z - \frac{z^2}{2}\left[\left(1 - \frac{K_1}{K_2}\right) + \frac{1+\nu}{1-\nu}\right] + \frac{2z^3(1+\nu)}{3(1-\nu)} - \frac{z^4(1+\nu)}{4(1-\nu)}\left(1 - \frac{K_1}{K_2}\right) - \frac{17 - 19\nu}{12(1-\nu)}\right\} \quad . \tag{34A}$$

The following substitutions are made:

$$K_1 = \frac{Eb}{2R_1^2} \, , \qquad K_2 = \frac{Eb}{2R_2^2} \, , \qquad z = \frac{R_1}{R_1 + R_2} \, ,$$

$$\delta = \frac{a^2 z}{2R_1} \, , \qquad \therefore \; R_1^2 = \frac{a^4 z^2}{4\delta^2} \, , \qquad R_2^2 = \frac{(1-z)^2 a^4}{4\delta^2} \, ,$$

$$\sigma = \frac{N}{b} \, .$$

The stress equations are now given by

$$\frac{\sigma_r^A a^2}{\delta^2 E} = -\frac{1}{4z^2}\left(\frac{r}{a}\right)^2 - \frac{1}{(1-z)^2}\left[\ln z - z + \frac{z^2(1+\nu)}{4(1-\nu)} - \frac{z^3(1+\nu)}{6(1-\nu)} + \frac{11-13\nu}{12(1-\nu)}\right] , \qquad (35A)$$

$$\frac{\sigma_t^A a^2}{\delta^2 E} = \frac{\sigma_r^A a^2}{\delta^2 E} - \frac{1}{2z^2}\left(\frac{r}{a}\right)^2 , \qquad (36A)$$

$$\frac{\sigma_r^B a^2}{\delta^2 E} = \frac{1}{(1-z)^2}\left[\ln\frac{a}{r} + \frac{4}{3}\left(\frac{r}{a}\right) - \frac{1}{4}\left(\frac{r}{a}\right)^2 + \frac{z^2}{4}\left(\frac{2}{3}z - 1\right)\left(\frac{a}{r}\right)^2 + \right.$$

$$\left. + \frac{z^2(1+\nu)}{4(1-\nu)}\left(\frac{2}{3}z - 1\right) - \frac{11-13\nu}{12(1-\nu)}\right] , \qquad (37A)$$

$$\frac{\sigma_t^B a^2}{\delta^2 E} = \frac{1}{(1-z)^2}\left[\ln\frac{a}{r} + \frac{8}{3}\left(\frac{r}{a}\right) - \frac{3}{4}\left(\frac{r}{a}\right)^2 - \frac{z^2}{4}\left(\frac{2}{3}z - 1\right)\left(\frac{a}{r}\right)^2 + \right.$$

$$\left. + \frac{z^2(1+\nu)}{4(1-\nu)}\left(\frac{2}{3}z - 1\right) - \frac{23-25\nu}{12(1-\nu)}\right] . \qquad (38A)$$

NOMENCLATURE

a	Outside radius of diaphragm
A	Arbitrary constant; if superscript, denotes region A
B	Arbitrary constant; if superscript, denotes region B
C	Arbitrary constant, with or without subscripts or superscripts
D	Flexural rigidity of plate $= Eh^3/[12(1 - \nu^2)]$
e_r	Radial strain of middle plane
e_t	Tangential strain of middle plane
E	Young's modulus
h	Thickness of diaphragm
M_r	Radial bending moment per unit length of circumference
M_t	Tangential bending moment per unit length of circumference
N_r	Radial membrane force per unit length
N_t	Tangential membrane force per unit length
r	Radial distance
R_1	Contour radius, region A
R_2	Contour radius, region B
u	Radial component of displacement at a point in the middle plane
w	Normal component of displacement at a point in the middle plane
V	Volumetric displacement of pump
z	Dimensionless parameter $= R_1/(R_1 + R_2)$
δ	Maximum deflection at center of diaphragm
ν	Poisson's ratio
σ	Stress
σ_{rM}, σ_{rB}	Radial membrane and bending stresses, respectively
σ_{tM}, σ_{tB}	Tangential membrane and bending stresses, respectively

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