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Bent Marshak Waves

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Abstract

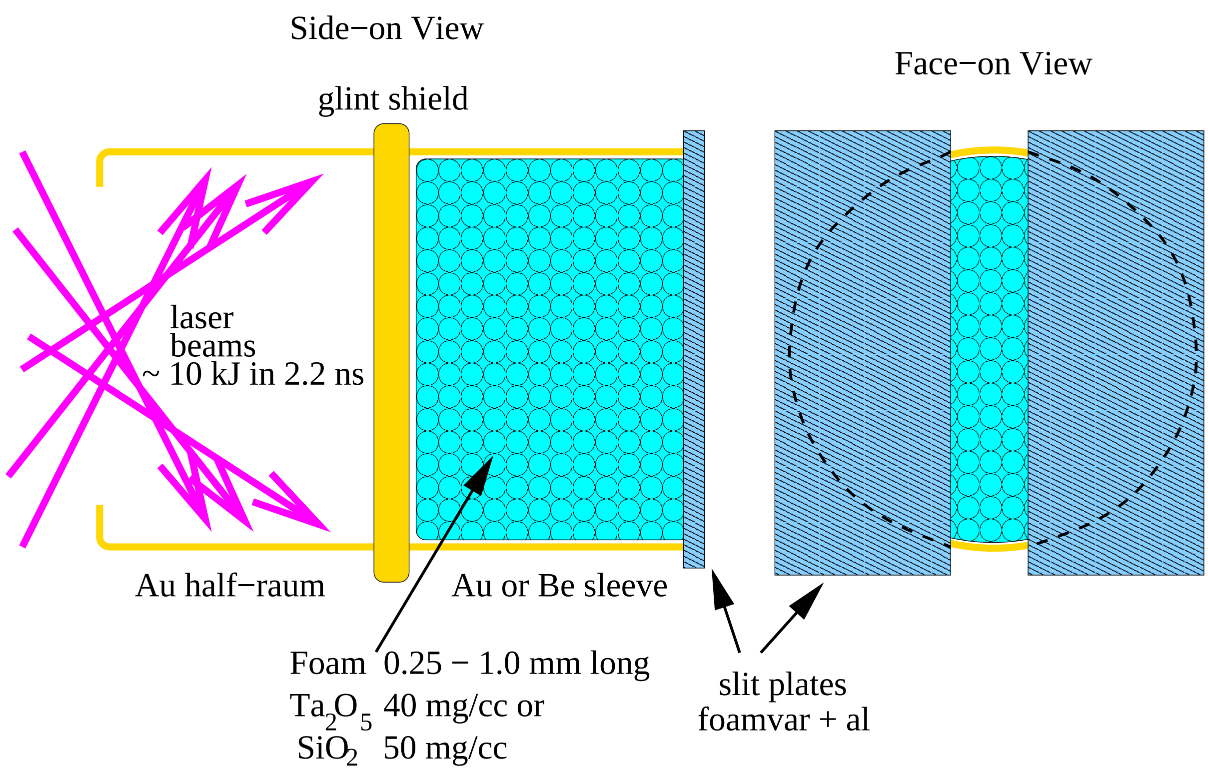
Radiation driven heat waves (Marshak Waves[1]) are ubiquitous in astrophysics and terrestrial laser driven high energy density plasma physics (HEDP) experiments. Generally, the equations describing Marshak waves are so nonlinear, that solutions involving more than one spatial dimension require simulation. However, in this paper we show how one may analytically solve the problem of the two-dimensional nonlinear evolution of a Marshak wave, bounded by lossy walls, using an asymptotic expansion in a parameter related to the wall albedo and a simplification of the heat front equation of motion.[2] Three parameters determine the nonlinear evolution, a modified Markshak diffusion constant, a smallness parameter related to the wall albedo, and the spacing of the walls. The final nonlinear solution shows that the Marshak wave will be both slowed and bent by the non-ideal boundary. In the limit of a perfect boundary, the solution recovers the original diffusion-like solution of Marshak. The analytic solution will be compared to a limited set of simulation results and experimental data.

[1] Marshak, R.E., Phys. Fluids, **1**, 24, (1958)

[2] J.H. Hammer and M.D. Rosen, Phys. Plasmas, **10**, 1829 (2003)

Supersonic radiation transport experiments have revealed an unexpected boundary dependence

Ω experiments by Back, C. A., *et al.* [Phys. Plasmas, **7**, p. 2126, 2000] demonstrated significant curvature of the radiation front profile in test samples.



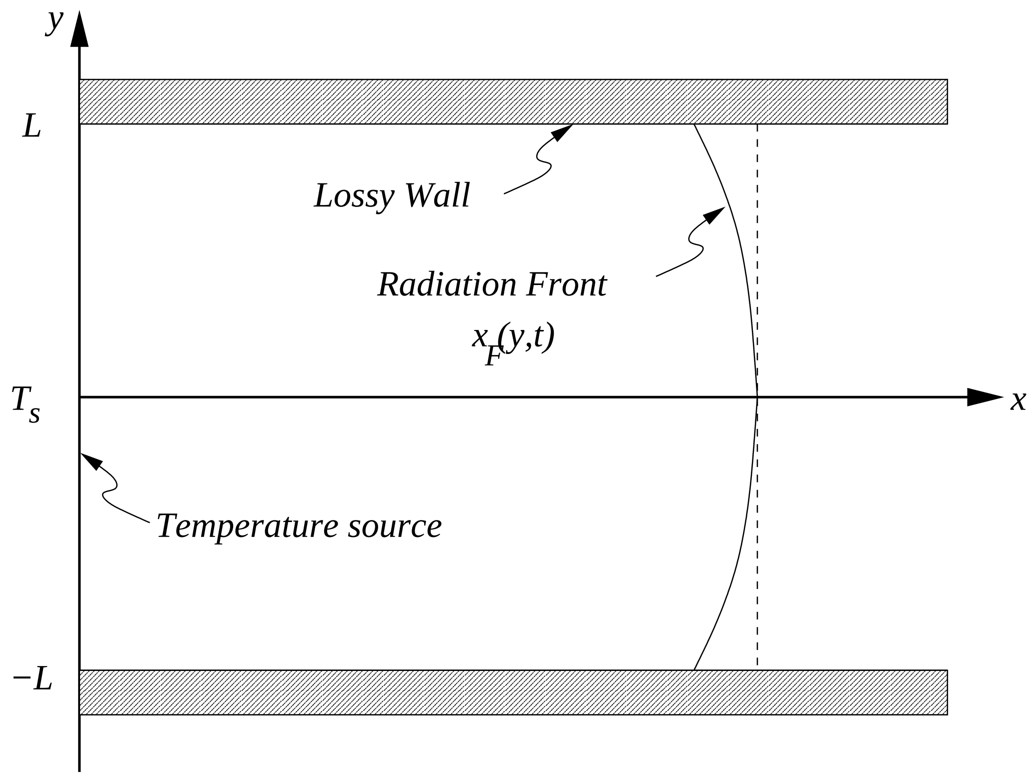
mfp ∼ 0.25 mm in these targets

A 1D streaked spectrometer records photon energies of 550 eV

Further work by Back, C. A., *et al.* [paper FO1.00009 of this conference] shows that the curvature of the radiation front depends upon the composition of the sleeves surrounding the aerogel foam.

We treat this problem in 2D planar geometry

Assuming a supersonic radiation front with temperature (T) independent opacity, $\nabla^2 T^4 = 0$



T must be of the form

$$T^4 = \sum_{n=0}^{\infty} \cos(k_n y) \left[A_n(t) e^{k_n x} + B_n(t) e^{-k_n x} \right]$$

It is natural to choose form of the front to be

$$x_F(y,t) = \sum_{n=0}^{\infty} c_n \cos(k_n y)$$

The non-ideal boundary generates an eigenvalue problem

It is assumed that the boundary at $y = \pm L$ has a constant albedo, a. The energy flux into the boundary is the difference between the absorption and re-emission, so

$$\left.\frac{\partial T^4}{\partial y}\right|_{y=L} = \frac{3}{4}\rho\kappa(a-1)T^4\Big|_{y=L}$$

Using the general solution for T^4, one finds the eigenvalue condition

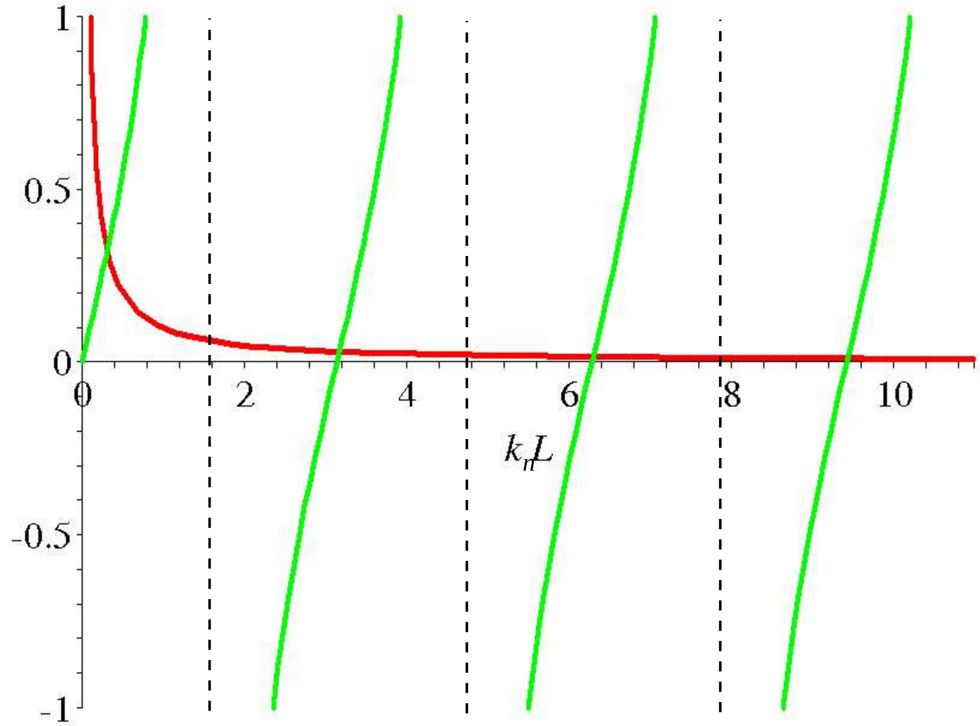
$$\tan(k_n L) = \frac{\varepsilon}{k_n L}$$

where $\varepsilon = \frac{3}{4}\rho\kappa L(1-a) \ll 1$ is a useful asymptotic expansion parameter.

The eigenvalues are nearly $n\pi$

Expanding the tangent, we look for the correction

$$k_n L \;=\; n\pi + \phi_n$$
$$\tan(n\pi + \phi_n) \;\approx\; \phi_n + \frac{1}{3}\phi_n^3 + \ldots$$



Solving the eigenvalue condition obtains

$$\phi_n \approx -\frac{n\pi}{2} + \sqrt{\left(\frac{n\pi}{2}\right)^2 + \varepsilon}$$

Since $\phi_0 > 0$ we must take the + root.

Note that ϕ_n rapidly becomes small

A good expansion parameter, $\sqrt{\varepsilon}$, is apparent

$$\phi_0 = \sqrt{\varepsilon}$$
$$\phi_1 \approx \frac{\varepsilon}{\pi} - \frac{\varepsilon^2}{\pi^3} + \ldots$$

Generally for $n > 0$,

$$\phi_n \approx \frac{\varepsilon}{n\pi} - \frac{\varepsilon^2}{n^3\pi^3} + \ldots$$

Thus, we will formally expand in $\sqrt{\varepsilon}$ and solve order by order

$$A_n = A_n^{(0)} + \sqrt{\varepsilon} A_n^{(\frac{1}{2})} + \varepsilon A_n^{(1)} + \ldots$$
$$B_n = B_n^{(0)} + \sqrt{\varepsilon} B_n^{(\frac{1}{2})} + \varepsilon B_n^{(1)} + \ldots$$

The boundary conditions at $x = 0$ and $x = x_F$ will determine A_n and B_n

On the source side ($x = 0$) $T = T_s$

$$T_s^4 = \sum_{n=0}^{\infty} (A_n + B_n) \left[\cos\left(\frac{n\pi y}{L}\right) \left(1 - \frac{\phi_n^2 y^2}{2L^2} + \frac{\phi_n^4 y^4}{24L^4} - \dots\right) - \sin\left(\frac{n\pi y}{L}\right) \left(\frac{\phi_n y}{L} - \frac{\phi_n^3 y^3}{6L^3} + \dots\right)\right]$$

Breaking down the above equation order-by-order and using orthogonality, one finds

$$A_n + B_n = \begin{cases} T_s^4 \left(1 + \frac{\varepsilon}{3}\right) + O(\varepsilon^{\frac{3}{2}}),\ n = 0 \\ \frac{2(-1)^n \varepsilon T_s^4}{n^2\pi^2} + O(\varepsilon^{\frac{3}{2}}),\ \ n > 0 \end{cases}$$

The boundary condition at the curved front is more complex

Taylor expansion around the front allows one to approximately satisfy $T(x_F, y) = 0$

$$\underbrace{T^4(x_F, y)}_{0} \approx T^4(c_0, y) + (x_F - c_0) \left.\frac{\partial T^4}{\partial x}\right|_{c_0}$$

With this weak curvature expansion, the front B.C. is recast as

$$T^4(c_0, y) \approx \left[c_0 - \sum_{n=0}^{\infty} c_n \cos(k_n y)\right] \left.\frac{\partial T^4}{\partial x}\right|_{c_0}$$

Using the expression for T^4 one eventually finds through order-by-order solution that

$$A_n e^{k_n c_0} + B_n e^{-k_n c_0} = 0 + O(\varepsilon^{\frac{3}{2}})$$

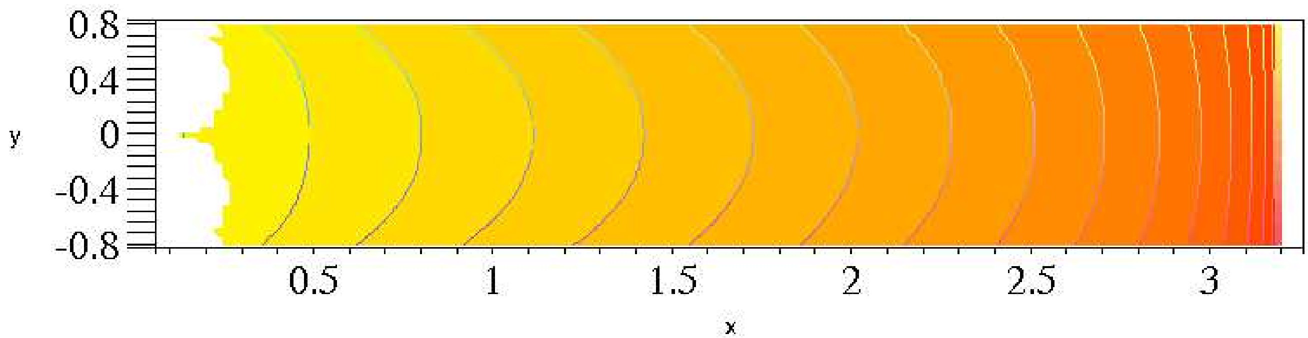
The lowest order solution shows an edge cooling

Solving for A_n and B_n one finds the temperature structure is given by

$$\frac{T^4}{T_s^4} = -\left(1+\frac{\varepsilon}{3}\right)\cos\left(\sqrt{\varepsilon}\frac{y}{L}\right)\frac{\sinh\left[k_0(x-c_0)\right]}{\sinh(k_0 c_0)}$$

$$+\frac{4\varepsilon}{\pi^2}\sum_{n=1}^{\infty}\cos(k_n y)\frac{(-1)^{n+1}}{n^2}\frac{\sinh\left[k_0(x-c_0)\right]}{\sinh(k_0 c_0)}+O(\varepsilon^{\frac{3}{2}})$$

Contours of $T(x,y)$



We can now generate the front equation of motion

For supersonic radiation fronts, Hammer and Rosen found the simplified equation of motion

$$\rho e \dot{\mathbf{x}}_F = -\frac{4\sigma}{3\kappa\rho} \nabla T^4 \Big|_{x_F}$$

- Internal energy of material at T_s — e
- Opacity of cold material — κ
- Density of cold material — ρ
- Stefan-Boltzman constant — σ

Using our expressions for $x_F(y,t)$ and $T(x,y)$ we can find the unknowns c_n

A set of nonlinear equations for the front shape are generated

$$\dot{c}_0 = \frac{4\sigma T_s^4}{3\kappa\rho^2 e}\left(1+\frac{\varepsilon}{3}\right)\frac{\sqrt{\varepsilon}}{L\sinh\left(\frac{\sqrt{\varepsilon}}{L}c_0\right)} + O(\varepsilon^2)$$

$$\dot{c}_1 = -\frac{8\sigma T_s^4}{3\kappa\rho^2 e}\frac{\varepsilon}{\pi^2}\frac{k_1}{\sinh(k_1 c_0)} + O(\varepsilon^3)$$

$$\vdots$$

$$\dot{c}_n = \frac{8\sigma T_s^4}{3\kappa\rho^2 e}\frac{\varepsilon(-1)^n}{n^2\pi^2}\frac{k_n}{\sinh(k_n c_0)} + O(\varepsilon^3)$$

Note that the diffusion constant of Marshak's wave is

$$D_M = \frac{8\sigma T_s^4}{3\kappa\rho^2 e}$$

Remarkably, the equation for c_0 is easily solvable

$$c_0(t) = \frac{L}{\sqrt{\varepsilon}} arccosh \left[\frac{D\varepsilon t}{2L} + 1 \right]$$

where $D = D_M(1 + \varepsilon/3)$ is a modified radiation diffusion constant.

The above solution is more easily understandable through expansion

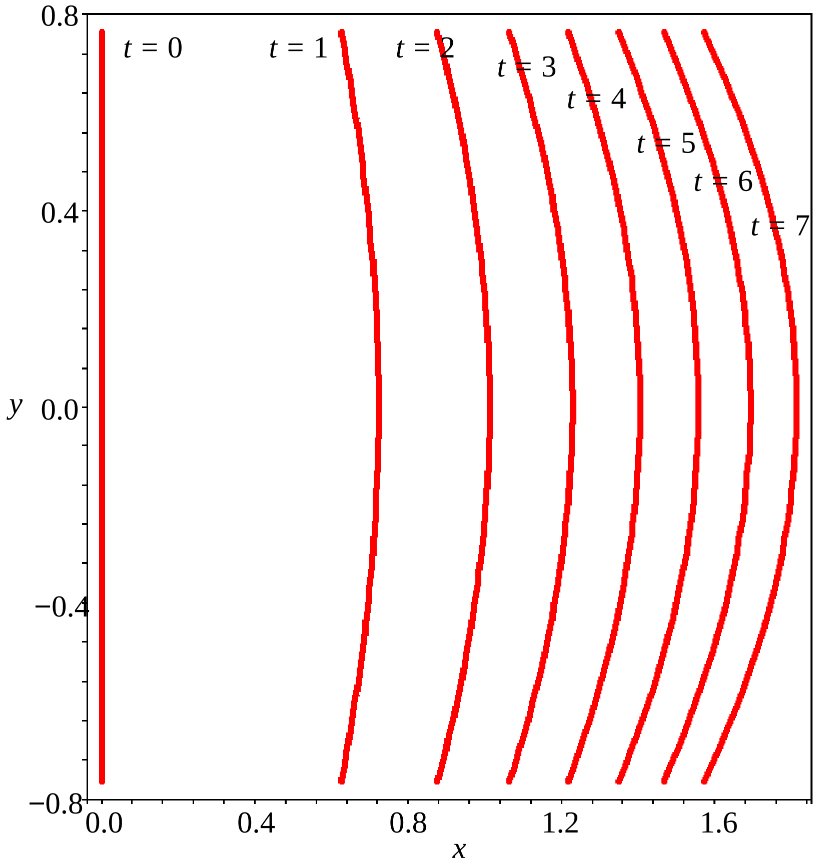
$$c_0(t) \approx \underbrace{\sqrt{Dt}}_{Marshak's\ soln} \underbrace{- \frac{L}{12}\sqrt{\frac{2}{\varepsilon}} \left(\frac{D\varepsilon t}{2L} \right)^{\frac{3}{2}} + \ldots}_{Drag-like\ slowing}$$

Note that as $\varepsilon \to 0$ the solution reduces to the usual Marshak wave solution.

The non-ideal boundary produces a "drag" on the radiation, bending and slowing the front

The front is given by

$$x_F(y,t) = \frac{L}{\sqrt{\varepsilon}} arccosh\left[\frac{D\varepsilon t}{2L} + 1\right] \cos\left(\frac{\sqrt{\varepsilon}}{L} y\right) + H.O.T.$$

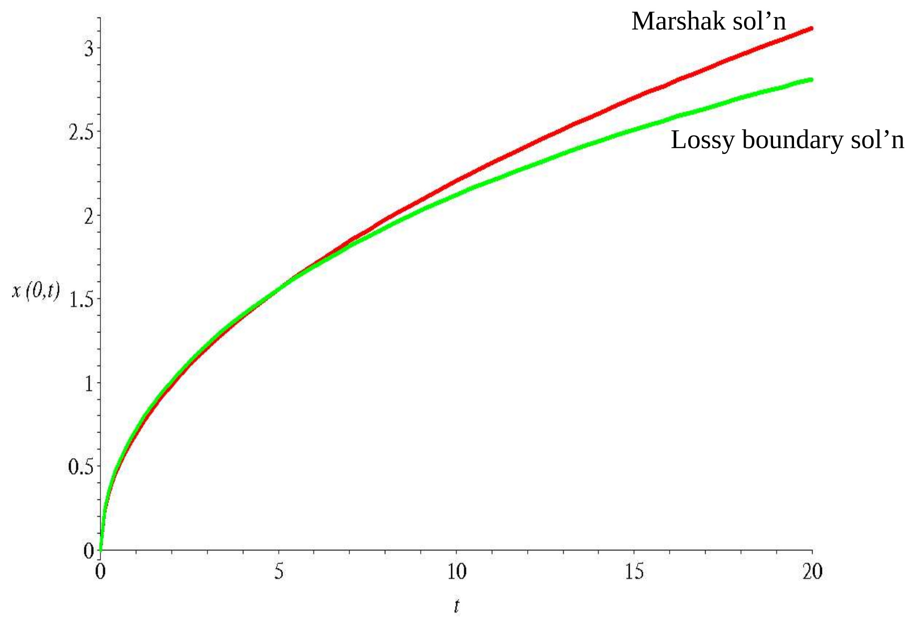


The front radius of curvature at $y = 0$ is

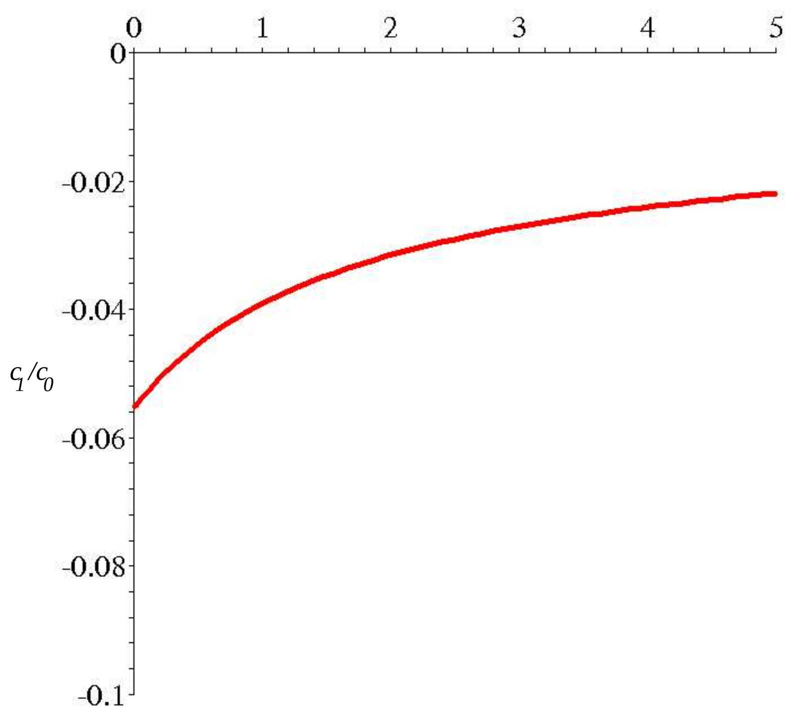
$$R_C \approx \frac{L}{\sqrt{\varepsilon}} \frac{1}{arccosh\left[\frac{D\varepsilon t}{2L} + 1\right]}$$

Higher order contributions to x_F become less-and-less important in time

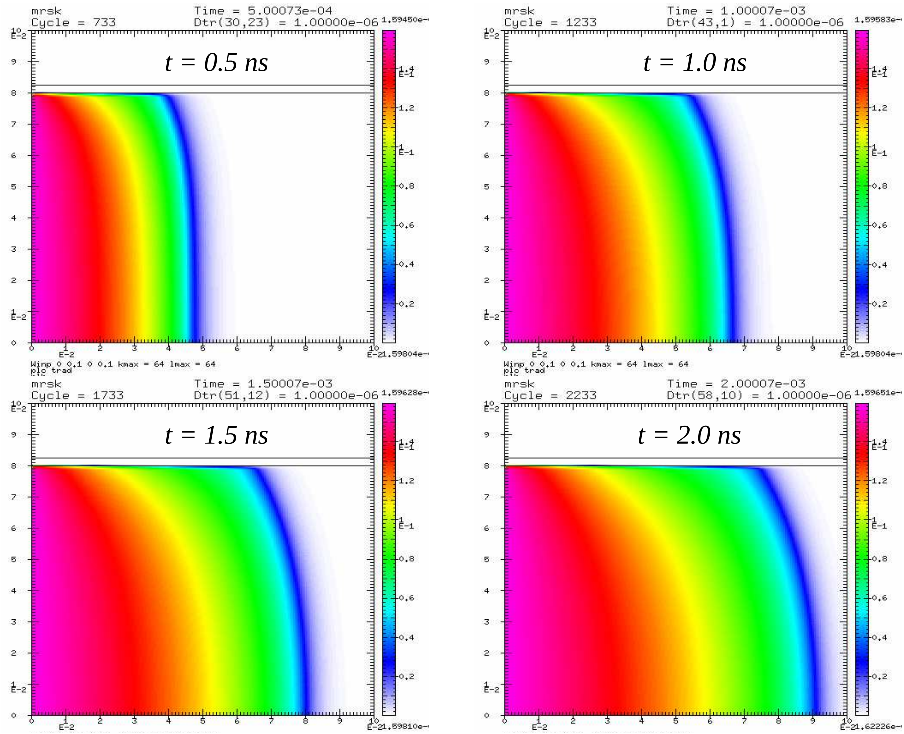
Slowing of $x_F(0,t)$ compared to Marshak

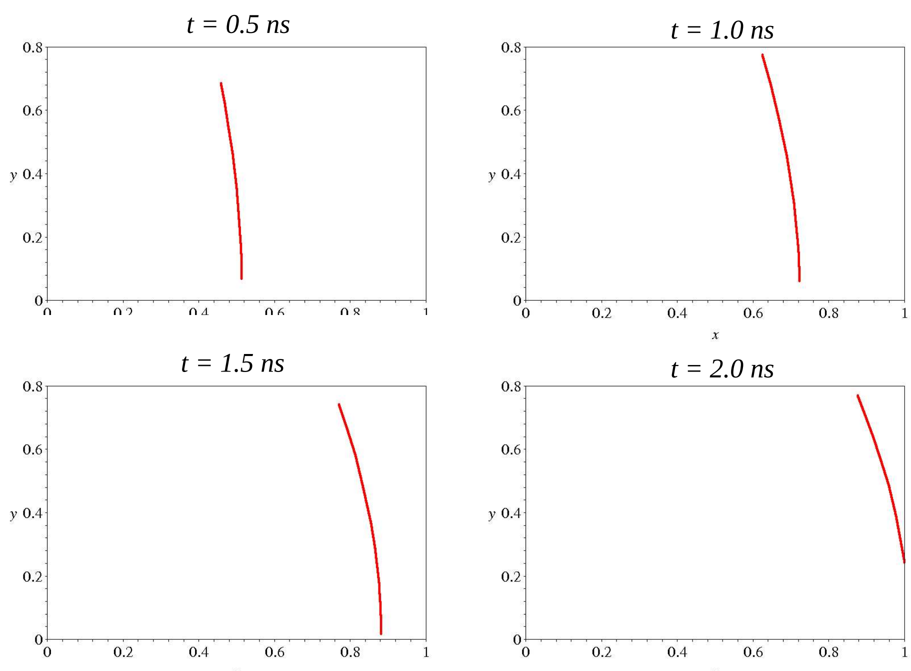


Ratio of $c_1(t)$ to $c_0(t)$



Analytic model compares well with simulation

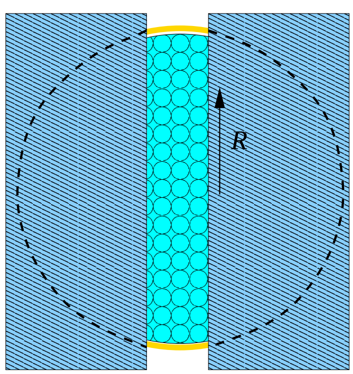


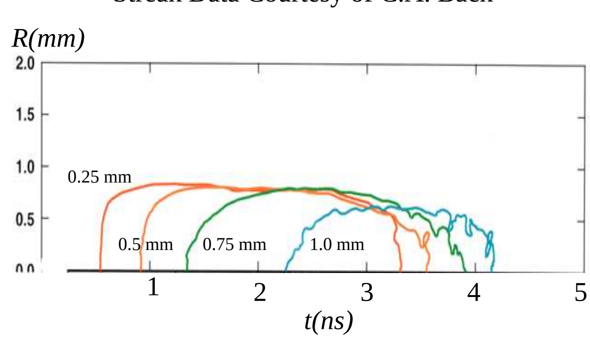


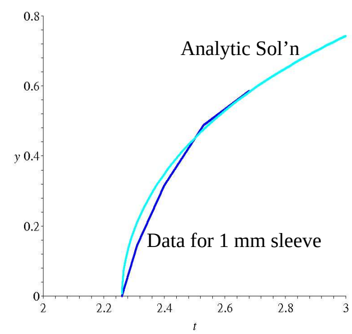
The analytic model also appears to fit experiment using reasonable parameters

Solving for $y(t)$ at a fixed x, we can compare to streak data

$$y(t) = \frac{L}{\sqrt{\varepsilon}} \cos^{-1} \left[\frac{x}{c_0(t)} \right] \Bigg|_{x=slit}$$







Data from the 0.8 mm radius and 1 mm long Au sleeve (optically thick case) experiment is well matched with $D = 0.46$ mm^2/ns and $\varepsilon = 0.3$–providing a measure of wall albedo.

A lossy boundary creates a "drag" on the radiation front

- A non-ideal boundary bends and slows Marshak waves.
- Three key parameters describe the full 2D behavior

- The tube radius: L

- A smallness parameter: $\varepsilon = \frac{3}{4}\rho\kappa L(1-a)$

- A diffusion constant: $D = \frac{8\sigma T_s^4}{3\rho^2\kappa e}\left(1+\frac{\varepsilon}{3}\right)$

- The cylindrical coordinate version can be obtained via the same derivation except with the trigonometric functions replaced by Bessel functions.
- To include $\kappa = \kappa(\rho, T)$ and $e = e(\rho, T)$, one must address the eigenvalue problem differently.

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Pre-prints?