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THE ORTHOGONAL CONDUCTIVITY OF A TOROIDAL PLASMA

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MASTER

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The Orthogonal Conductivity of a Toroidal Plasma

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ABSTRACT

The orthogonal conductivity of a toroidal plasma is calculated in the fluid regime. If the damping time for toroidally directed angular momentum is τ_N, the orthogonal conductivity is shown to be $\sigma_\perp = (\rho c^2/B_p^2)/\tau_n$ for large τ_N. Here ρ is the mass density, c the speed of light and B_p the poloidal component of the magnetic field. For large τ_N, the flow induced by the orthogonal electric field is almost purely toroidal and of magnitude. $c\,(E_r - E_r^o)/B_p$ where E_r^o is the electric field required for ambipolar diffusion.

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I. INTRODUCTION

The orthogonal conductivity of a plasma has not been a popular topic for theoretical calculations. Indeed, the concept requires some explanation. The orthogonal conductivity is the constant of proportionality between the current perpendicular to the magnetic surfaces and the electric field perpendicular to the surfaces. It has been measured using probes in the B-3 stellarator[1] and the FM-1 spherator.[2] The primary importance of the concept occurs when a charged species is not well confined in a plasma device. Examples could be α particles from nuclear reactions or high energy ions from ion cyclotron heating or beam injection. To maintain charge neutrality the plasma must establish a back current equal to the electric current of the poorly confined species leaving the plasma.

The ideal MHD equations, which are customarily used in plasma physics, give zero orthogonal conductivity. This comes from $\vec{j}\cdot\vec{\nabla}p=0$ -- independent of the electric field -- and $\vec{\nabla}p$ being orthogonal to the magnetic surfaces.[3] This so-called intrinsic ambipolarity persists into the neo-classical regime for toroidally symmetric systems.[4]

The basic problem with obtaining a finite orthogonal conductivity is that the current crossing the magnetic surfaces is proportional to the damping time for toroidally directed angular momentum. In an idealized toroidally symmetric system, the symmetry gives an infinite damping time. However, there are a myriad of effects which in practice give a finite damping time for toroidal angular momentum. These include the drag of the neutrals, the orthogonal viscosity, magnetic pumping due to field ripple, and the convection of angular momentum by plasma diffusion.

The various effects which damp toroidal angular momentum can be represented by a phenomenological time constant. Mathematically this phenomonological theory closely resembles that with damping by neutrals alone. Consequently, of the various methods of damping toroidal angular momentum, we will retain only the neutral drag term. At the appropriate places, the changes that would occur if only toroidally directed momentum were damped and not all components of momentum as with neutral drag will be pointed out.

In addition to the non-ideal effects which can destroy toroidal angular momentum, there exists one very important non-ideal effect which conserves toroidal angular momentum, the parallel viscosity. In many plasma experiments, the parallel viscosity establishes flow equilibrium within a magnetic surface on a much shorter time scale than that for damping toroidal angular momentum. Consequently the effects of parallel viscosity will be included in the theory.

Previous theoretical work on the orthogonal conductivity has concentrated on the effect of the neutral drag, neglecting viscous effects. B. Lehnert in his 1963 review of rotating plasma confinement systems[5] gave the orthogonal conductivity of a cylinder, $\sigma_\perp = (\rho c^2/B^2)/\tau_N$. The ion-neutral collision time is τ_N, ρ is the plasma mass density, and B is magnetic field strength. S. Yoshikawa (1965) evaluated $\sigma_\perp$ in a torus and found the cylindrical result must be multiplied by a Pfirsch-Schlüter factor $(1 + 2q^2)$ with q the tokamak safety factor.[6]

Our results agree with Yoshikawa in the regime of negligible parallel viscosity. In the regime dominated by parallel viscosity we find $\sigma_\perp = (\rho c^2/B_p{}^2)/\tau_N$ with B_p being the poloidal magnetic field strength. For tokamaks, this is a substantial enhancement over both the cylindrical and the Yoshikawa result.

II. MODEL

We use the usual Knorr model[7] of a toroidal plasma for the magnetic field

$$\vec{B} = B_\theta \hat{\theta} + B_\phi \hat{\phi} , \; B_\theta = B_p(r)/R , \; B_\phi = B_T(r)/R \qquad (1)$$

where the major radius, $R = R_o - r \cos \theta$. We also define $\Theta(r) = B_\theta/B_\phi$, the inverse aspect ratio $\varepsilon = r/R_o$, and the safety factor $q = \varepsilon/\Theta$. The coordinate system is given in Fig. 1. The two fluid equations used are essentially linearized, time independent, Braginskii equations[8] with the addition of a neutral drag term for the ions.

$\vec{\nabla} p_i = en(\vec{E} + \vec{v} \times \vec{B}/c) - en\eta\vec{j} + \vec{f}_i - \rho\vec{v}/\tau_N$ (2)

$\vec{\nabla} p_e = - en (\vec{E} + \vec{u} \times \vec{B}/c) + en\eta\vec{j} + \vec{f}_e$, $\vec{j} = en(\vec{v} - \vec{u})$ (3)

The force $f_{i,e}$ is the parallel viscosity given by Grimm and Johnson[9] -- one of the few expressions for this quantity which conserves toroidal angular momentum.

$$\vec{f} = \vec{\nabla} \cdot (\hat{b}\hat{b} - \frac{1}{3}\overleftrightarrow{\delta}) F \qquad (4)$$

with $\vec{b}$ a unit rector along the magnetic vector and $\overleftrightarrow{\delta}$ the identity tensor.

$$F = 2anT\tau \, [\hat{b} \cdot \vec{\nabla}(\hat{b}\cdot\vec{v}) - (\hat{b}\cdot\vec{\nabla}\hat{b})\cdot\vec{v} - \frac{\kappa}{3}\vec{\nabla}\cdot\vec{v}] \qquad (5)$$

with a = 3/4 for ions and 9/16 for electrons.[8] The bulk viscosity κ will not be required since our solutions will have $\vec{\nabla}\cdot\vec{v}=0$. The time constant τ refers to the self collision time of the species in question.

The solution will be derived by considering effects in two orders. In the lower order all dissipative effects are neglected. The lower order solution has arbitrary constants in the expressions for the current and velocity in a magnetic surface. The $\hat{b}$ components of the full fluid equations give two consistency relations which must be satisfied by the lower order solution and determine the arbitrary constants.

$$\int_0^{2\pi} \hat{b}\cdot(\vec{f}_i + \vec{f}_e)\, d\theta/2\pi = (\rho/\tau_N) \int_0^{2\pi} \hat{b}\cdot\vec{v}\, d\theta/2\pi \tag{6}$$

$$\int_0^{2\pi} \hat{b}\cdot\vec{f}_e\, d\theta/2\pi + en\eta \int_0^{2\pi} \hat{b}\cdot\vec{j}\, d\theta/2\pi = 0 \tag{7}$$

The $\hat{\phi}$ components of the full fluid equations give expressions for fluxes of electrons and ions across the magnetic surfaces in terms of the lower order equilibrium. The flux of electrons crossing a surface $n < u_r >$ is given by

$$\begin{aligned} < u_r > &= \int_0^{2\pi} (R/R_o)\, u_r\, d\theta/2\pi \\ &= (\eta c/B_p) \int_0^{2\pi} (R/R_o)^2\, j_\phi\, d\theta/2\pi \end{aligned} \tag{8}$$

The current crossing a magnetic surface $< j_r >$

$$\begin{aligned} < j_r > &= \int_0^{2\pi} (R/R_o)\, j_r\, d\theta/2\pi \\ &= \frac{\rho c}{B_p \tau_N} \int_0^{2\pi} (R/R_o)^2\, v_\phi\, d\theta/2\pi \end{aligned} \tag{9}$$

We have neglected terms in $< j_r >$ and $< u_r >$ which depend on the parallel viscosity. Toroidal symmetry implies ion and electrons cannot damp their own toroidal angular momentum; so

$$\int_0^{2\pi} (R/R_o)^2\, \hat{\phi}\cdot\vec{f}\, d\theta/2\pi = 0 \tag{10}$$

This will be explicitly demonstrated for the parallel viscosity of Grimm and Johnson. The expressions for $< j_r >$ and $< u_r >$ demonstrate their intimate relation with toroidal angular momentum conservation.

III. LOWER ORDER EQUILIBRIUM

The lower order fluid equations (without dissipation) can be written

$$\vec{\nabla} p = \vec{j} \times \vec{B}/c \quad , \qquad p = p_i + p_e \tag{11}$$

$$\frac{1}{en} \vec{\nabla} p_i - \vec{E} = \vec{v} \times \vec{B}/c \quad . \tag{12}$$

To have a steady-state solution, the pressure and the electric potential must be functions of r alone. That is, constant on a magnetic surface. Conservation of plasma and charge to the lower order requires

$$\vec{\nabla} \cdot \vec{j} = 0 \ , \quad \vec{\nabla} \cdot n\vec{v} = 0 \tag{13}$$

Since there is no radial component of velocity in the lower order solution

$$\vec{\nabla} \cdot n\vec{v} = n\vec{\nabla} \cdot \vec{v} = 0 \tag{14}$$

The divergence conditions on velocity and current affect only the θ component due to the symmetry in ϕ. They require

$$j_\theta = J_p(r)\ R_o/R \quad , \quad v_\theta = V_p(r)\ R_o/R \tag{15}$$

with J_p and V_p arbitrary functions of r. The radial components of fluid equations imply

$$j_\phi = \frac{1}{\Theta} [J_p \frac{R_o}{R} - J_D \frac{R}{R_o}] \qquad (16)$$

$$v_\phi = \frac{1}{\Theta} [V_p \frac{R_o}{R} - V_E \frac{R}{R_o}] \qquad (17)$$

with

$$J_D(r) = \frac{C}{B_T} \frac{dp}{dr} \quad , \quad V_E(r) = - \frac{C}{B_T}(E_r - \frac{1}{en} \frac{dp_i}{dr}) \quad . \qquad (18)$$

The arbitrary functions $J_p(r)$ and $V_p(r)$ are constants on a magnetic surface and must be evaluated in terms of J_D and V_E with the consistency relations, Eqs. 6 and 7. Once this is done Eqs. 15, 16, and 17 give an exact solution to the lower order equations which is consistent with the full fluid equations.

IV. CONSISTENCY RELATIONS

The equilibrium derived in Sec. III is symmetric in the ion and electron velocities. Therefore, the integrals required by the consistency relations, Eqs. 6 and 7, have identical structure. In Appendix A, using this lower order equilibrium, we will show that

$$\int_0^{2\pi} \hat{b} \cdot \vec{f} \, d\theta/2\pi = - a \frac{\Theta}{(1+\Theta^2)^{1/2}} \frac{nT\tau}{R_o^2} V_p \qquad (19)$$

In Appendix B , is a derivation of

$$\int_0^{2\pi} \hat{b} \cdot \vec{v} \, d\theta/2\pi = \frac{1}{(1+\Theta^2)^{1/2}} \frac{1}{\Theta} [\frac{1+\Theta^2}{(1-\varepsilon^2)^{1/2}} V_p - V_E] \qquad (20)$$

These expressions are exact except a above differs from the values given in Sec. II by terms of order ε^2.

To evaluate the consistency relation given by Eq. 6 we define several dimensionless parameters.

$$\alpha_i = a_i \frac{T_i}{m_i} \frac{\tau_i \tau_N}{(qR_o)^2} \tag{21}$$

This parameter is formed by dividing the coefficient in the ion viscosity integral by the coefficient ρ/τ_N from the neutral drag integral. A similar coefficient is defined for the electrons α_e, and $\alpha = \alpha_i + \alpha_e$. The ratio, δ_o, of the ion and electron viscosities is of some importance

$$\delta_o = \alpha_e/\alpha_i$$

$$= \frac{a_e}{a_i} Z \frac{T_e \tau_e}{T_i \tau_i} \tag{22}$$

The parameter δ_o is quite small [$\sim (m_e/m_i)^{1/2}$] if $T_i \gtrsim T_e/5$. With these definitions, Eq. 6 gives

$$-q^2 (\alpha_i V_p + \alpha_e U_p) = \frac{1}{\Theta^2} [\frac{1+\Theta^2}{(1-\varepsilon^2)^{1/2}} V_p - V_E] \tag{23}$$

Assuming $\varepsilon \ll 1$, but Θ arbitrary yields

$$[1 + \Theta^2 (1 + q^2 [\tfrac{1}{2} + \alpha_i])] V_p = V_E - \Theta^2 q^2 \alpha_e U_p \tag{24}$$

If only toroidally directed momentum were damped, the $1 + \Theta^2$ in Eq.(23) would be Θ^2 and $1 + q^2 (\frac{1}{2} + \alpha_i)$ in Eq. (24) would be $q^2 (\frac{1}{2} + \alpha_i)$.

The consistency relation, Eq. (7), requires a new dimensionless parameter to compare the electron viscosity and the resistivity. This is

$$\alpha_\eta = a_e \frac{nT_e\tau_e}{(qR_o)^2} \frac{1}{\eta} = a_e \frac{T_e}{m_e} \left(\frac{\tau_e}{qR_o} \right)^2 \qquad (25)$$

Assuming $\epsilon << 1$, but Θ arbitrary gives

$$[1 + \Theta^2(1 + \frac{1}{2} q^2)] V_p = V_D + \{1 + \Theta^2[1 + q^2(\frac{1}{2} + \alpha_\eta)]\}U_p \qquad (26)$$

with $V_D = J_D/en$. To validly use the two fluid equations α_η must be must less than one. Consequently it will be dropped. However, the results for $\alpha_\eta \gtrsim 1$ are of some interest despite their lack of validity and will be discussed later.

The surface constants V_p and J_p can be expressed in terms of V_E and J_D using Eq. (24) and (26).

$$V_p = \frac{1}{1+\Theta^2[1+q^2(\frac{1}{2}+\alpha)]} [V_E + \frac{\Theta^2 q^2 \alpha_e}{1 + \Theta^2(1 + \frac{1}{2} q^2)} V_D] \qquad (27)$$

with $V_D = J_D/en$ and

$$J_p = \frac{1}{1 + \Theta^2(1 + \frac{1}{2} q^2)} J_D \qquad (28)$$

Neglecting neutral momentum damping in all but the toroidal direction would change $1 + q^2(\frac{1}{2} + \alpha)$ in Eq. (27) into $q^2(\frac{1}{2} + \alpha)$.

V. PERPENDICULAR CURRENT

The consistency relations evaluated in Sec. IV eliminated all arbitrariness in the lower order equilibrium solution and the perpendicular current can be evaluated using Eq. (9). The required integral is performed in Appendix B giving

$$\int_0^2 (R/R_o)^2 \, v_\phi \, d\theta/2\pi = \frac{1}{\Theta} [V_p - (1 + \frac{3}{2} \varepsilon^2) V_E] \tag{29}$$

The substitution of V_p from Eq. (27) gives $< j_r >$ in terms of V_E and V_D or equivalently in terms of E_r and $dp_{i,e}/dr$. The expression $< j_r >$ is of the form

$$< j_r > = \sigma_\perp (E_r - E_r^{\,o}) \tag{30}$$

with $E_r^{\,o}$ being the electric field required for ambipolar diffusion. If in the expression for $E_r^{\,o}$ we assume that $\Theta << 1$ as well as $\varepsilon << 1$, we find

$$E_r^{\,o} = \frac{1}{en} [\frac{dp_i}{dr} - \delta \frac{dp_e}{dr}]/(1 + \delta) \tag{31}$$

$$\delta = \frac{q^2 \alpha_e}{1 + q^2 (2 + \alpha_i)}$$

The parameter δ measures the fraction of the diamagnetic current carried by each species. If the solution is dominated by parallel viscosity ($\alpha \gg 1$ or $\tau_N \to \infty$), then $\delta = \delta_o$ defined in Eq. (22). The orthogonal conductivity is given by

$$\sigma_\perp = \frac{\rho c^2}{B^2} \frac{1}{\tau_N} (1 + \Theta^2) \frac{1 + (2+\alpha) q^2}{1 + \Theta^2 + \varepsilon^2 \alpha} \tag{32}$$

for arbitrary Θ but $\varepsilon \ll 1$. This expression has several interesting limits

$$\sigma_\perp = \frac{\rho c^2}{B^2} \frac{1}{\tau_N} (1 + 2q^2) \quad , \quad \alpha \ll 1 \tag{33}$$

$$\sigma_\perp = \frac{3}{4} \frac{\rho c^2}{B^2} (1 + \delta_o) \frac{T_i}{m_i} \frac{\tau_i}{R_o^2} \quad , \quad 1 \ll \alpha \ll 1/\varepsilon^2 \tag{34}$$

$$\sigma_\perp = \frac{\rho c^2}{B_p^2} \frac{1}{\tau_N} \quad , \quad \alpha \gg 1/\varepsilon^2 \tag{35}$$

The intermediate regime $\sigma_\perp$, $1 \ll \alpha \ll 1/\varepsilon^2$, is independent of τ_N. Viewed as a function of $1/\tau_N$, $\sigma_\perp$ has two regions of constant slope separated by a plateau.

If only toroidal momentum damping is included in the theory, $\delta = \alpha_e/(2 + \alpha_i)$, and the factor $[1 + (2 + \alpha) q^2]/[1 + \Theta^2 + \varepsilon^2 \alpha]$ in Eq. (32) becomes $[(2 + \alpha) q^2]/[1 + \varepsilon^2 \alpha]$.

VI. PERPENDICULAR ELECTRON FLUX

The flux of electrons across the magnetic surfaces is given by Eq. (8). The integral in this expression can be evaluated with Eq. (29), for the current and velocity behave analogously in the lower order equilibrium (V_E goes to J_D).

The expression for J_p in terms of J_D is given in Eq. (28). With a small amount of algebra, one finds

$$< u_r > = - \frac{\eta c^2}{B^2} (1 + 2q^2) \frac{dp}{dr} , \qquad (36)$$

the well-known Pfirsch-Schlüter result.[10]

The electrons diffuse in accordance with the Pfirsch-Schlüter formula independent of the electric field while the ion diffusion is reduced or enhanced in response to electric fields. This result, of course, comes from the model of the neutral drag acting only on the ions.

The effect of large α_η, Eq. (25), will now be considered. While considering this effect, we will assume $V_E = 0$ and $\delta_o << 1$ to simplify the calculations. One finds if $\Theta<<1$ that

$$< u_r > = - \frac{\eta c^2}{B^2} \frac{1+q^2(2+\alpha_\eta)}{1+\varepsilon^2\alpha_\eta} \frac{dp}{dr} \qquad (37)$$

As noted earlier, this equation has no validity in a classical theory for $\alpha_\eta \gtrsim 1$. Suppose, however, we ignore the limits of classical theory and consider $\alpha_\eta >> 1/\varepsilon^2$. Then we find diffusion by a pseudoclassical law

$$< u_r > = - \frac{\eta c^2}{B_p^2} \frac{dp}{dr} \qquad (38)$$

This is $1/\varepsilon^{1/2}$ larger than neo-classical diffusion.[11] If the classical theory were valid in the long mean free path limit, it would predict an even larger diffusion coefficient than the neo-classical theory. For $\alpha_\eta >> 1/\varepsilon^2$, the classical theory also predicts a bootstrap current , $j_\phi = - c(dp/dr)/B_p$. This result also is $1/\varepsilon^{1/2}$ larger than the neo-classical value.[12]

The classical theory taken into the long mean free path regime is even more at variance with the Pfirsch-Schlüter theory than is the neo-classical theory. This comes from the classical formula for the parallel viscosity increasing without limit as the mean free path increases. The neo-classical theory appears to differ from the classical primarily in the parallel viscosity with α_η bounded from above by $1/\varepsilon^{3/2}$.

The pseudoclassical diffusion coefficient given in Eq. (38) would be valid if the mean free path for electron-electron collisions were shorter than the connection length qR_o while the mean free path for electrons losing momentum to ions were much longer. Enhancement of collision times by instabilities could have this effect.

VII. FLOW PATTERN

The flow pattern induced by the electric field across the magnetic surfaces can be determined using Eqs. (15), (17), and (27). The diamagnetic velocity, which appears in Eq. (27), can be replaced by the electric field required for ambipolar diffusion E_r^o. One finds under the assumption $\Theta<<1$, $\varepsilon<<1$, that

$$V_p = \frac{V_E - V_E^{\circ}}{1 + \Theta^2\,[1 + q^2(1/2 + \alpha)]} + V_E^{\circ}\,(1 + 3/2\,\varepsilon^2) \quad (39)$$

with v_E° calculated with the field E_r° (See Eqs. (18) and (31)). The second term in Eq. (39) is the poloidal rotation which would occur under ambipolar conditions. In the classical regime, viscous effects are negligible on the diamagnetic current distribution so this term is not affected by parallel viscosity.

The toroidal component of velocity can be accurately given for $\varepsilon \ll 1$ by

$$v_\phi = v_\phi^{\circ} + v_\phi^{1}\, q \cos\theta \quad (40)$$

with $v_\phi^{\circ} = (V_p - V_E)/\Theta$ and $v_\phi^{1} = V_p + V_E$. Eq. (39) can be used to evaluate v_ϕ° and v_ϕ^{1} giving

$$v_\phi^{\circ} = -\frac{\varepsilon^2\alpha}{1+\varepsilon^2\alpha}\;\frac{V_E - V_E^{\circ}}{\Theta} \quad (41)$$

$$v_\phi^{1} = \frac{2 + \varepsilon^2\alpha}{1 + \varepsilon^2\alpha}\,V_E + \frac{\varepsilon^2\alpha}{1 + \varepsilon^2\alpha}\,V_E^{\circ} \quad (42)$$

For purposes of evaluating v_θ one can simplify Eq. (39) to

$$v_\theta = \frac{v_E - v_E^{\circ}}{1 + \varepsilon^2\alpha} + v_E^{\circ} \quad (43)$$

Equations (41) through (43) are valid even if the momentum damping is only in the toroidal direction.

The toroidal flow, v_ϕ , is dominated by the cos Θ term for $\epsilon\alpha<2$. For $\alpha >> 1/\epsilon^2$, however, the toroidal flow is unidirectional and approximately equal to $v_\phi = (c/B_p)(E_r - E_r^O)$. The poloidal flow, V_θ, is just the $\vec{E} \times \vec{B}$ flow for $\alpha << 1/\epsilon^2$. However, for $\alpha >> 1/\epsilon^2$ it drops to the diamagnetic drift velocity v_E^O and is independent of the applied electric field $E_r - E_r^O$.

VIII. DISCUSSION

The toroidal orthogonal conductivity $\sigma_\perp$ relates the average electric current crossing a magnetic surface $< j_r >$ and the electric field across the surface E_r. The current $< j_r >$ interacts with the poloidal magnetic field to give a torque in the toroidal direction. Consequently, to have a finite orthogonal conductivity, there must be a finite damping time τ_N for toroidal angular momentum to balance this torque.

Although our calculations are valid for arbitrary τ_N in fluid regime, the situation of most practical interest is the limit of weak damping of toroidal angular momentum. Under this condition, the parallel viscosity dominates the flow pattern induced by the $\vec{E} \times \vec{B}$ drift. One finds $\sigma_\perp = (\rho c^2/B_p^2)/\tau_N$ with ρ the plasma mass density and B_p the poloidal magnetic field. The induced flow is toroidal with $v_\phi = (c/B_p)(E_r - E_r^O)$ and E_r^O equal to the electric field required for ambipolar diffusion.

In the fluid theory the important parameter for determining the importance of the parallel viscosity is $\alpha_i = \frac{3}{4}(T_i/m_i)\ \tau_i \tau_N/(qR_o)^2$. Here T_i and m_i are the temperature mass of the ions, τ_i is the ion-ion collision time and qR_o is the connection length for the magnetic field. Parallel viscosity is dominate if $\alpha_i >> 1/\epsilon^2$ and negligible if $\alpha_i << 1$. The regime of negligible parallel viscosity has been studied by S. Yoshikawa assuming damping of momentum by neutrals.[6] The orthogonal conductivity in this regime is $\sigma_\perp = (\rho c^2/B^2)(1+2q^2)/\tau_N$ with B the magnitude of the magnetic field and q the well-known tokamak safety factor, $q = (rB_T)/(R_O B_p)$. In the Yoshikawa regime the induced poloidal velocity is the value expected from the $\vec{E} \times \vec{B}$ drift. There is, however, a much larger toroidal flow than the $\vec{E} \times \vec{B}$ flow. This comes from the requirement of zero flow divergence in the surface. One finds $v_\phi = 2qV_E \cos\theta$ where V_E is the magnitude of the $\vec{E} \times \vec{B}$ flow and θ is the poloidal angular distance. In the intermediate regime $1 << \alpha_i << 1/\epsilon^2$, the orthogonal conductivity is independent of τ_N.

The orthogonal conductivity has been measured on the B3 stellarator[1] and the FM-1 spherator.[2] The B3 operated in the regime $\alpha_i << 1$. The neutral density was not well enough known to test the theory with great accuracy, however, the data were consistent with the Yoshikawa orthogonal conductivity. The neutral density was thought low enough in the B3 to require Yoshikawa's Pfirsch-Schlüter factor $(1+2q^2)$. The FM-1 experiments were done with $\alpha_i >> 1/\epsilon^2$. The results for $\sigma_\perp$ were quite close to those predicted by the theory. However, the poloidal field dominates the toroidal field in FM-1; so the $1/B_p^2$ scaling was not properly tested.

If τ_N is large enough compared to the ion collision time, one would expect the results for $\alpha_i >> 1/\epsilon^2$ to hold in the neo-classical regime. Neo-classically, the toroidal velocity tends to relaxes to $v_\phi = (c/B_p)(E_r - E_r^o)$ on an ion collision time scale and the induced poloidal velocity goes to zero.[13] The relation between $< j_r >$ and the damping time for toroidal angular momentum is just toroidal torque balance and should hold in all regimes.

An interesting analogy was found between the classical theory extended to mean free paths long compared to the connection length and the neo-classical theory. Indeed, the primary calculational difference between the classical and neo-classical theory appears to be in the parallel viscosity. The long mean free path classical theory predicts diffusion according to the pseudo-classical law --$(B/B_p)^2$ faster than diffusion in cylinder-- which is even faster than the neo-classical results.

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APPENDIX A

In this appendix the required integrals of the parallel viscous force will be evaluated. The parallel viscous force was given in Eqs. (4) and (5). First the expression for F , Eq. (5), will be evaluated with the velocity distribution of Eqs. (15) and (17). To do this we must evaluate $(\vec{b}\cdot\vec{\nabla})(\vec{b}\cdot\vec{v})$ and $(\hat{b}\cdot\vec{\nabla}\hat{b})\cdot\vec{v}$. Using $\hat{b} = (\hat{\phi} + \Theta\hat{\theta})/(1 + \Theta^2)^{1/2}$

$$\hat{b}\cdot\vec{\nabla}(\vec{b}\cdot\vec{v}) = -\frac{\sin\theta}{R_o}\left[V_p\left(\frac{R_o}{R}\right)^2 + \frac{V_E}{1+\Theta^2}\right] \tag{A1}$$

Using

$$d\hat{\theta} = -\hat{r}\,d\theta + \hat{\phi}\sin\theta\,d\phi \tag{A2}$$

$$d\phi = (\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta})\,d\phi \tag{A3}$$

$$(1+\Theta^2)\hat{b}\cdot\vec{\nabla}\hat{b} = -\frac{\sin\theta}{R}(\hat{\theta} - \Theta\hat{\phi}) + \left(\frac{\cos\theta}{R} - \frac{\Theta^2}{r}\right)\hat{r} \tag{A4}$$

So that

$$(\hat{b}\cdot\vec{\nabla}\hat{b})\cdot\vec{v} = -\frac{\sin\theta}{R_o}\frac{V_E}{1+\Theta^2} \tag{A5}$$

Combining terms and remembering $\vec{\nabla}\cdot\vec{v} = 0$, Eq. (14), one finds

$$F = -2anT\tau\frac{\sin\theta}{R_o}V_p\left(\frac{R_o}{R}\right)^2 \tag{A6}$$

The viscous force can be written as

$$\vec{f} = F\vec{\nabla}\cdot(\hat{b}\hat{b} - \frac{1}{3}\overleftrightarrow{\delta}) + (\hat{b}\hat{b} - \frac{1}{3}\overleftrightarrow{\delta})\cdot\vec{\nabla}F \tag{A7}$$

Using the expressions for $d\hat{\theta}$ and $d\hat{\phi}$ given above as well as

$$d\hat{r} = \hat{\theta} d\theta - \hat{\phi} \cos\theta \, d\phi \quad \text{(A8)}$$

it is straight forward to show

$$(1+\Theta^2)\vec{\nabla}\cdot(\hat{b}\hat{b} - \frac{1}{3}\overleftrightarrow{\delta}) = \frac{\sin\theta}{R}[2\Theta\hat{\phi} - (1-\Theta^2)\hat{\theta}] + (\frac{\cos\theta}{R} - \frac{\Theta^2}{r})\hat{r} \quad \text{(A9)}$$

This implies

$$\hat{b}\cdot\vec{f} = \frac{\Theta}{(1+\Theta^2)^{1/2}}[\frac{2}{3}\frac{1}{r}\frac{\partial F}{\partial\theta} + F\frac{\sin\theta}{R}] \quad \text{(A10)}$$

$$\hat{\phi}\cdot\vec{f} = \frac{\Theta}{1+\Theta^2}[\frac{1}{r}\frac{\partial F}{\partial\theta} + 2F\frac{\sin\theta}{R}] \quad \text{(A11)}$$

The expression for $\hat{\phi}\cdot\vec{f}$ can be made simpler using the expression for F.

$$\hat{\phi}\cdot\vec{f} = -\frac{\Theta}{1+\Theta^2}\frac{2anT\tau}{R_o r}(\frac{R_o}{R})^2 V_p \cos\theta \quad \text{(A12)}$$

This last expression clearly obeys the condition that the parallel viscosity obey toroidal angular momentum conservation

$$\int_o^{2\pi}(R/R_o)^2 \, \hat{\phi}\cdot\vec{f} \, d\theta/2\pi = 0 \quad \text{(A13)}$$

To evaluate the integral of $\hat{b}\cdot\vec{f}$ over θ one needs to evaluate the integral $\int_o^{2\pi}(R_o/R)^3 \sin^2\theta \, d\theta/2\pi$. This integral is 1/2 $[1 + o(\varepsilon^2)]$ so

$$\int_o^{2\pi}\hat{b}\cdot\vec{f} \, d\theta/2\pi = -a\frac{\Theta}{(1+\Theta^2)^{1/2}}\frac{nT\tau}{R_o^2}V_p \quad \text{(A14)}$$

APPENDIX B

In this appendix the required integrals of the velocity distribution will be evaluated. The velocity distribution is given in Eqs. (15) and (17). The integral $\int_0^{2\pi} \hat{b}\cdot\vec{v}\, d\theta/2\pi$ will be evaluated first

$$\hat{b}\cdot\vec{v} = \frac{1}{(1+\Theta^2)^{1/2}\Theta} \left[\frac{R_o}{R} V_p \; (1+\Theta^2) - V_E \frac{R}{R_o} \right] \quad . \tag{B1}$$

To carry out the required integral we need

$$\int_0^{2\pi} \frac{R_o}{R}\, d\theta/2\pi = \frac{1}{(1-\varepsilon^2)^{1/2}} \tag{B2}$$

$$\int_0^{2\pi} \frac{R}{R_o}\, d\theta/2\pi = 1 \tag{B3}$$

This gives

$$\int_0^{2} \hat{b}\cdot\vec{v} \;\; d\theta/2\pi = \frac{1}{(1+\Theta^2)^{1/2}} \; \frac{1}{\Theta} \left[\frac{1+\Theta^2}{(1-\varepsilon^2)^{1/2}} V_p - V_E \right] \tag{B4}$$

The other integral we must evaluate is $\int_0^{2\pi} (R/R_o)^2 v_\phi d\theta/2\pi$. Evaluation of this integral involves the integral

$$\int_0^{2\pi} (R/R_o)^3 \; d\theta/2\pi = 1 + \frac{3}{2}\varepsilon^2 \tag{B5}$$

Consequently

$$\int_0^{2\pi} \left(\frac{R}{R_o} \right)^2 v_\phi \; d\theta/2\pi = \frac{1}{\Theta}[V_p - (1+\frac{3}{2}\,\varepsilon^2) V_E] \tag{B6}$$

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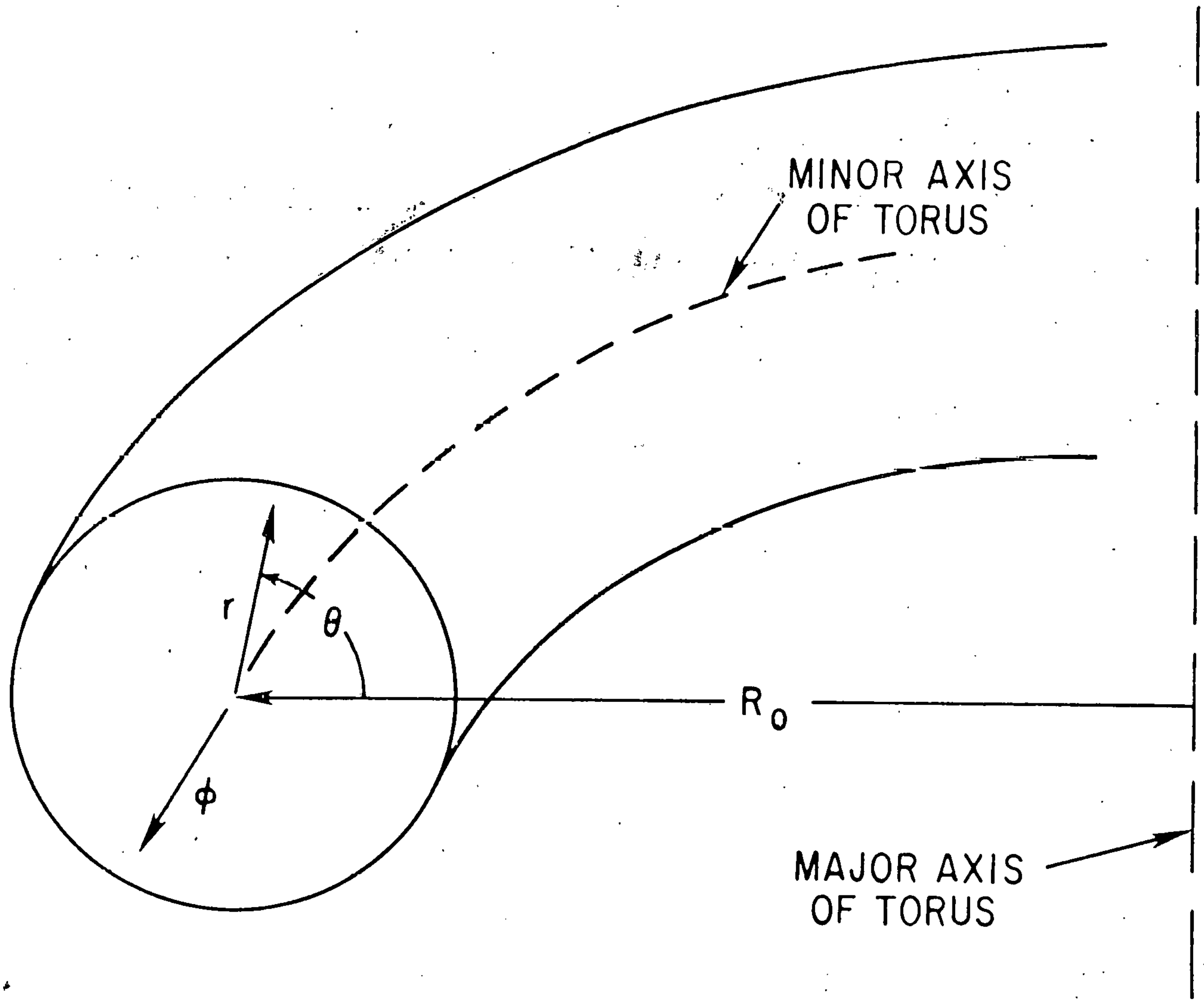

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Fig. 1. The toroidal coordinate system.

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