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SUBJECT: Determination of the Six Turbulent Reynolds' Stresses by the Hot Wire Method for Arbitrary Turbulent Intensity and Geometry With Special Application to Axisymmetric Flow

TO: Distribution

FROM: R. P. Wichner and F. N. Peebles

ABSTRACT

A relationship is derived between the mean square fluctuating current of a hot wire anemometer and the six turbulent Reynolds' stresses in the stream-coordinate system without employing the usual low turbulent intensity approximation. The relatively simple result is a consequence of assuming proportionality between the wire current reading and the perpendicular velocity component instead of the non-linear dependence required by King's law. The assumption is valid for instruments equipped with the proper linearizing circuitry. The stream-coordinate Reynolds' stresses are then related to the cylindrical polar Reynolds' stresses.

An error analysis on the experimental determination of one of these stresses is indicated, but cannot be evaluated without further data.

DISCLAIMER

TABLE OF CONTENTS

NOMENCLATURE

A, B, . . . F,	Constants in equation (10)
a, b, . . . f,	Constants in equation (16)
I	HWA current, total
i	HWA current, fluctuating
V	Velocity, total
v	Velocity, fluctuating
$\hat{w}$	Unit vector in the direction of the wire
α, β, ϕ	Angles defined by figures 1 and 2

Subscripts

s, n, r	The three stream-coordinate directions
r, θ, z	The three cylindrical polar coordinates
p	Indicates perpendicular
$\underline{\quad}$	Indicates vector

Superscripts

$\hat{}$	Indicates unit vector
$\overline{\quad}$	Indicates time average

INTRODUCTION

For the past fifteen years, the hot wire anemometer (HWA) has been the principal tool for experimentally unraveling the structure of turbulent flows. Prior to this it was used to measure time-average velocities by application of King's rule, i.e., that the square of the current necessary to maintain the wire at a predetermined temperature varies linearly with the square root of the fluid velocity which cools the wire.

In order to use this principle to probe the rapidly time-varying velocities existing in turbulent flow, at least two basic changes were required. First the size of the sensing element had to be made as small as the smallest eddy size, and secondly, some means developed of sensing an interpretable signal. This latter was accomplished by applying a high speed electronic servo-mechanism to the bridge circuit containing the wire. In the "constant current method," the one which is electronically simplest and still most common, changes in the wire cooling rate are sensed as changes in resistance of the wire and fed to the servo-mechanism which properly adjusts the power supply to maintain a constant current in the wire. In the interpretation of the signal, which is the varying voltage across the bridge, account must be made for the fact that the ideal situation of truly constant current is only approached because of the finite response time required by the servo. The "constant current method" has the great disadvantage that in operation the wire temperature is fluctuating, and hence its thermal inertia damps the electrical signal somewhat.

The chief advantage of the "constant resistance method," which has recently come into widespread use, is that the effect of thermal inertia is greatly reduced and, consequently, the sensitivity is increased. The temperature variations of the wire are not entirely eliminated though, again because of the finite response time required by the servo. In this method, as the name implies, the servo-mechanism continually and rapidly readjusts the power supply to maintain the resistance (temperature) of the wire constant.

A second bonus of the constant resistance system is that the fragile heated element may now be more intimately mechanically supported (e.g., by plating the "wire" on an electrically non-conductory material) supposedly without altering the signal. Such intimate mechanical support is essential for use in high gas velocities and much lower liquid velocities, though it has not been determined if it is essential in the moderate liquid velocities encountered in fluid fueled reactors.

This memo discusses just one of the many statistical quantities used to describe turbulent flows which may be obtained by the HWA; these are the "one point double velocity correlations," or more commonly called the Reynolds' stresses. Though no complete survey of the extensive HWA literature was made, examination of some recent texts and papers shows that it is standard procedure to employ a "low turbulent intensity" approximation in the development of the equation relating the reading (i.e., the r.m.s. fluctuating current or voltage) to the Reynolds' stress. This restrictive approximation is necessitated by the extremely non-linear form of King's law. On the other hand, if it is assumed that the current reading is proportional to the perpendicular velocity, as is the case some commercially available instruments which contain linearizing circuits, the relationship between the r.m.s. fluctuating current reading and the Reynolds' stresses can be developed with complete rigor.

The treatment is one suggested by Traugott[2], who however goes on to employ a low turbulent intensity approximation.

Determination of the Reynolds' Stresses in Stream-Coordinates

Before we can proceed with the development of the relation between the electrical signal and fluctuating velocity, some assumption must be made as to the manner in which the wire is cooled by the fluid. Almost universally it is assumed that only that component of the velocity that is perpendicular to the wire is effective in cooling it. Actually this cannot be strictly true for even when the perpendicular component is zero (i.e., flow is along the axis of the wire) some cooling takes place. Hinze[1] gives the following expression:

$$V^2_{effective} = V^2_{perpendicular} + C_1 V^2_{parallel} \qquad (1a)$$

The value of the constant, C_1, is given as ranging between 0.09 and 0.01, decreasing with increasing velocity. Hinze goes on to state that it appears to be possible to ignore the parallel velocity component, though, when the axis of the wire is more than 20° from the direction of the velocity. Equation (1a) then reduces to:

$$V_{effective} = V_{perpendicular} \qquad (1b)$$

In extending this assumption to turbulent flow, we will place the wire outside of a 20° cone about the mean velocity vector and state that only the component of the instantaneous velocity that is perpendicular to the wire is effective in cooling it. Since often the instantaneous velocity will be closer than 20° to the wire axis, and hence out of the range of applicability of equation (1b), this assumption is not entirely satisfactory, and the results so obtained will undoubtedly be somewhat warped. However, there seems to be no practical alternative.

Now, in order to continue, a relationship between the instantaneous current and instantaneous perpendicular velocity is required. The most common starting point is King's law, which strictly applies to time-averaged turbulent flow (and perhaps steady laminar flow)

$$\overline{I}^2 = C_2 + C_3 \sqrt{\overline{V}_p} \quad , \tag{2a}$$

and extend its applicability to each turbulent instant, i.e.,

$$I^2(t) = C_2 + C_3 \sqrt{V_p(t)} \tag{2b}$$

This, however, presents a difficult road to follow to the end without making some limiting assumption to get rid of the square root term.

On the contrary, if we assume that our HWA contains the very convenient linearizing circuit now commerically available, the mathematical continuation becomes simple and can be carried out with rigor.

The linearizing circuit alters King's law to read:

$$k\overline{I} = \overline{V}_p \tag{3a}$$

Equation (3a) is an experimentally observed fact in a HWA system which contains this circuit. As before, we must extend the range of applicability of equation (3a) to each <u>instant</u>, i.e.,

$$kI(t) = V_p(t) \tag{3b}$$

and, equation (3b) must be recognized as one of the necessary assumptions on which the following development is based.

Dividing $I(t)$ and $V_p(t)$ into mean and fluctuating components and squaring yields:

$$k^2 \left[\bar{I} + i(t) \right]^2 = \left[\bar{V}_p + v_p(t) \right]^2$$

or $$k^2 \left[\bar{I}^2 + 2\bar{I}i(t) + i^2(t) \right] = \bar{V}^2_p + 2\bar{V}_p\, v_p(t) + v^2_p(t)$$

Time-averaging eliminates the two terms linear in the fluctuating component.

$$k^2\bar{I}^2 + k^2\,\overline{i^2} = \bar{V}^2_p + \overline{v^2_p} \tag{3c}$$

Combining equations (3a) and (3c) yields the simple and useful result:

$$k^2\,\overline{i^2} = \overline{v^2_p} \tag{4}$$

The problem then becomes one of expressing the quantity $\overline{v^2_p}$ in terms of velocity fluctuations in convenient directions, e.g., the coordinate axes directions.

Referring to figure 1, $\hat{n}$, $\hat{s}$, $\hat{r}$ are the unit vectors along the three stream-coordinate axes; $\hat{s}$ is also the direction of the mean velocity, $\underline{\bar{V}}_s$. (The top bar signifies a time-average quantity; the bottom bar, a vector. See p 3 for nomenclature.) The direction of the unit vector $\hat{w}$ along the axis of the wire is specified by angles α and β.

The instantaneous velocity is given by:

$$\underline{V}(t) = \underline{\bar{V}}_s + \underline{v}_s(t) + \underline{v}_n(t) + \underline{v}_r(t) \tag{5}$$

where the small v's are the three fluctuating components of the velocity which when added to the mean velocity yield the actual velocity, $\underline{V}(t)$.

The velocity vector perpendicular to the wire is then:

$$\underline{V}_p(t) = \left[\underline{\bar{V}}_s + \underline{v}_s(t) + \underline{v}_n(t) + \underline{v}_r(t)\right] - \hat{w}\left[\left(\underline{\bar{V}}_s + \underline{v}_s(t) + \underline{v}_n(t) + \underline{v}_r(t)\right) \cdot \hat{w}\right] \tag{6}$$

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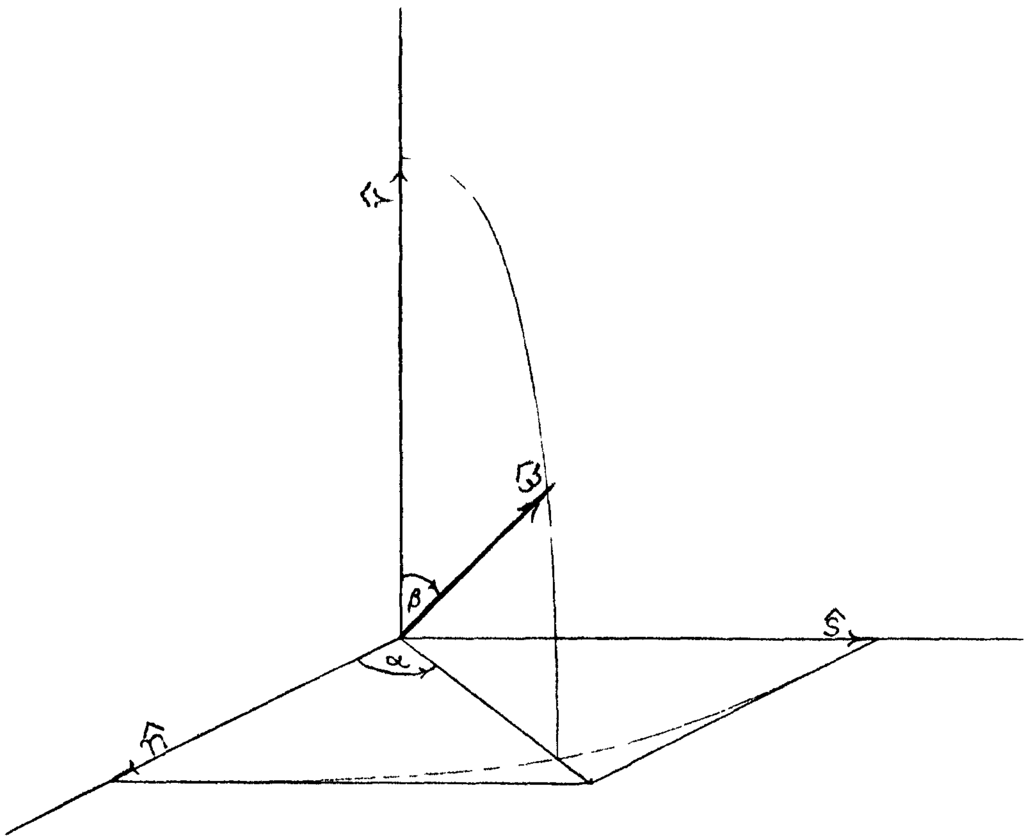

THE WIRE AND STREAM – COORDINATE AXES

FIG. 1

The second term on the RHS of equation (6) is the time-varying velocity vector parallel to the wire, which when subtracted from the total velocity vector yields the perpendicular component, $\underline{V}_p(t)$. Now separate $\underline{V}_p(t)$ into a time average and fluctuating part.

$$\underline{V}_p(t) = \overline{\underline{V}}_p + \underline{v}_p(t) \tag{7}$$

Equation (7) when combined with (6) yields:

$$\underline{v}_p(t) = \left[\underline{v}_s(t) + \underline{v}_n(t) + \underline{v}_r(t)\right] - \hat{w}\left[\left(\underline{v}_s(t) + \underline{v}_n(t) + \underline{v}_r(t)\right) \cdot \hat{w}\right] \tag{8}$$

The square of the magnitude of this vector is simply the dot product, $\underline{v}_p(t) \cdot \underline{v}_p(t)$. Utilizing the relation

$$\hat{w} = (\sin\alpha \sin\beta)\,\hat{s} + (\cos\alpha \sin\beta)\,\hat{n} + (\cos\beta)\,\hat{r} \tag{9}$$

we obtain by carrying through the vector algebra (see Appendix A)

$$\underline{v}_p(t) \cdot \underline{v}_p(t) = Av^2{}_s(t) + Bv^2{}_n(t) + Cv^2{}_r(t) + Dv_s(t)\,v_n(t)$$

$$+ Ev_s(t)v_r(t) + Fv_n(t)\,v_r(t) \tag{10a}$$

where:

$$\begin{aligned} A &\equiv 1 - \sin^2\alpha \sin^2\beta \\ B &\equiv 1 - \cos^2\alpha \sin^2\beta \\ C &\equiv \sin^2\beta \\ D &\equiv -\sin 2\alpha \sin^2\beta \\ E &\equiv -\sin\alpha \sin 2\beta \\ F &\equiv -\cos\alpha \sin 2\beta \end{aligned} \tag{10b}$$

Apply the time-average operator $\frac{1}{T}\int_0^T dt$ to equation (10) to yield:

$$\overline{v^2_p} = A\,\overline{v^2_s} + B\,\overline{v^2_n} + C\,\overline{v^2_r} + D\,\overline{v_s v_n} + E\,\overline{v_s v_r} + F\,\overline{v_n v_r} \qquad (11)$$

Equation (11) is the sought after relation between the magnitude of the mean square fluctuating perpendicular component and the stream-coordinate Reynolds' stresses.

Combining equations (4) and (11) yields

$$k\,\overline{i^2} = A\,\overline{v^2_s} + B\,\overline{v^2_n} + C\,\overline{v^2_r} + D\,\overline{v_s v_n} + E\,\overline{v_s v_r} + F\,\overline{v_n v_r} \qquad (12)$$

which describes the relation between the HWA signal and the six stream-coordinate Reynolds' stresses.

To determine these stresses experimentally, it is necessary to take a set of six readings $(\overline{i^2}_j,\ j = 1, 2 \ldots 6)$ each one taken at a different wire orientation $\left[(\alpha, \beta)_j,\ j = 1 \ldots 6\right]$, and solve the resulting six simultaneous linear equations.

$$k^2\,\overline{i^2_j} = A_j\,\overline{v^2_s} + B_j\,\overline{v^2_n} + C_j\,\overline{v^2_r} + D_j\,\overline{v_s v_n} + E_j\,\overline{v_s v_r} + F_j\,\overline{v_n v_r}$$

$$\text{where } j = 1, 2, \ldots 6 \qquad (13)$$

In the next section, equation (13) will be specialized to the cylindrical polar coordinate system and solved for the cylindrical polar Reynolds' stresses.

Consider now the case of a wire oriented perpendicular to the mean flow and parallel to $\hat{n}$; i.e., $\alpha = 0$, $\beta = 90°$. Equation (11) reduces to:

$$k\,\overline{i^2} = \overline{v^2_s} + \overline{v^2_r} \qquad (11a)$$

Thus, when the linear response relation (equation 3b) is the starting point, the HWA signal, $\overline{i^2}$, is equally sensitive to the cooling effects of the velocity fluctuations in both perpendicular directions. When King's law is is the starting point, (as it would be in all HWA's not equipped with the above-mentioned linearizing circuit) the equation equivalent to equation (13), which is of necessity derived by assuming that the turbulent intensity is small, predicts that the HWA signal depends solely on the fluctuations in the direction of the mean velocity. (See references 1, 2, 3)

Specialization to Cylindrical Polar Coordinates

For convenience and without loss of generality, a cylindrical polar system is selected whose origin and radial axis ($\hat{r}$) coincide with those of the stream-coordinate system used above. Therefore, only one number, the angle ϕ, (see figure 2) is required to fix the position of the polar axes relative to the stream axes.

The following are the relationships between the velocities along the polar axes to those along the stream-coordinate axes.

$$\left.\begin{aligned} v_s &= v_z \cos\phi + v_\theta \sin\phi \\ v_n &= -v_z \sin\phi + v_\theta \cos\phi \\ v_r &= v_r \end{aligned}\right\} \quad (14)$$

By squaring, cross-multiplying and time averaging equations (14), the following relationships are obtained.

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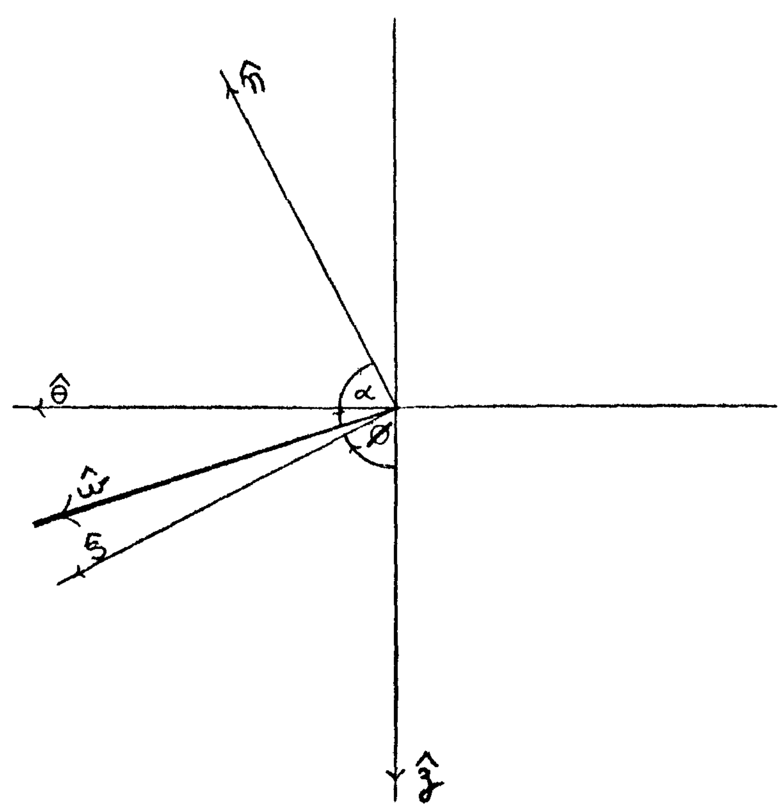

RELATION OF ANGLES IN THE $\hat{R}$– PLANE

FIG. 2

$$\left.\begin{aligned}
\overline{v^2_s} &= \overline{v^2_z} \cos^2 \phi + \overline{v^2_\theta} \sin^2 \phi + 2\, \overline{v_z v_\theta} \sin \phi \cos \phi \\
\overline{v^2_n} &= \overline{v^2_z} \sin^2 \phi + \overline{v^2_\theta} \cos^2 \phi - 2\, \overline{v_z v_\theta} \sin \phi \cos \phi \\
\overline{v^2_r} &= \overline{v^2_r} \\
\overline{v_s v_n} &= -\frac{1}{2}\, \overline{v^2_z} \sin 2\phi + \overline{v_z v_\theta} \cos 2\phi + \frac{1}{2}\, \overline{v^2_\theta} \sin 2\phi \\
\overline{v_s v_r} &= \overline{v_z v_r} \cos \phi + \overline{v_\theta v_r} \sin \phi \\
\overline{v_n v_r} &= -\, \overline{v_z v_r} \sin \phi + \overline{v_\theta v_r} \cos \phi
\end{aligned}\right\} \quad (15)$$

Equation (15) when substituted into (13) yields the second set of desired equations, those relating the six HWA readings $(\overline{i^2_j})$ to the cylindrical polar Reynolds' stresses. The algebra is carried out in Appendix B.

$$k^2\, \overline{i^2_j} = a_j\, \overline{v^2_z} + b_j\, \overline{v^2_\theta} + c_j\, \overline{v^2_r} + d_j\, \overline{v_z v_\theta} + e_j\, \overline{v_z v_r} + f_j\, \overline{v_\theta v_r}$$

$$\text{where} \quad j = 1 \ldots 6 \qquad (16)$$

and

$$\left.\begin{aligned}
a &= 1 - \sin^2\beta\,(\cos^2 \phi \sin^2\alpha + \sin^2 \phi \cos^2\alpha - 1/2 \sin 2\phi \sin 2\alpha) \\
b &= 1 - \sin^2\beta\,(\sin^2 \phi \sin^2\alpha + \cos^2 \phi \cos^2\alpha + 1/2 \sin 2\phi \sin 2\alpha) \\
c &= \sin^2\beta \\
d &= \sin^2\beta\,(\sin 2\phi \cos 2\alpha - \sin 2\alpha \cos 2\phi) \\
e &= \sin 2\beta\,(\sin \phi \cos\alpha - \sin\alpha \cos \phi) \\
f &= -\sin 2\beta\,(\sin \phi \sin\alpha + \cos \phi \cos\alpha)
\end{aligned}\right\} \quad (17)$$

Let D be the determinant of the coefficients and ${}_iD_j$ be D with the j'th column replaced by the six current readings, i.e.,

$$D \equiv \begin{vmatrix} a_1 & b_1 & \cdots & f_1 \\ \vdots & & & \\ a_6 & \cdots & \cdots & f_6 \end{vmatrix} \quad \text{and}$$

$$_iD_1 \equiv \begin{vmatrix} \overline{i^2}_1 & b_1 & \cdots & f_1 \\ \vdots & & & \\ \overline{i^2}_6 & b_6 & \cdots & f_6 \end{vmatrix} \quad \text{etc.}$$

Therefore the solutions to equations (16) may be written:

$$\left.\begin{aligned}
\overline{v^2_z} &= \frac{k^2 \; {}_iD_1}{D} && \text{(a)} \\
\overline{v^2_\theta} &= \frac{k^2 \; {}_iD_2}{D} && \text{(b)} \\
\overline{v^2_r} &= \frac{k^2 \; {}_iD_3}{D} && \text{(c)} \\
\overline{v_z v_\theta} &= \frac{k^2 \; {}_iD_4}{D} && \text{(d)} \\
\overline{v_z v_r} &= \frac{k^2 \; {}_iD_5}{D} && \text{(e)} \\
\overline{v_\theta v_r} &= \frac{k^2 \; {}_iD_6}{D} && \text{(f)}
\end{aligned}\right\} (18)$$

Error Analysis

The above development is useful in itself since it relates, in a very general geometric situation, the HWA signal to the six Reynolds' stresses without employing the low turbulent intensity approximation. It remains to be shown, however, that it is practical to obtain the Reynolds' stresses in this manner since each answer is dependent on:

1. Six readings of the mean square current meter.
2. Six readings of the angular vernier for α.
3. A prior determination of the angle ϕ by a directional impact probe.
4. Accurate knowledge of the angle β. (In the planned experimental routine, the angle β will remain fixed as α is varied.)
5. Assurance that the radial position, r, does not vary during the six readings.
6. Validity of assumption (1b).
7. Validity of assumption (3b).

There are, of course, other sources of error to which all HWA determinations may be subject, such as instantaneous non-uniformity of velocity along wire, conduction losses to wire supports, etc. These are discussed in reference (4). We will investigate the effects of (1) (2) (3). Errors introduced by (6) and (7) are unknown but may be significant.

The analytical expression for the error involved in the experimental determination of $\overline{v_r v_\theta}$ is given below. Unfortunately, as will be shown, it is sensitive to the actual values of the six mean current readings, $\overline{i^2}_1$, . . . $\overline{i^2}_6$, so numerical evaluation must wait until some data are taken.

The total error in $\overline{v_r v_\theta}$, $\Delta \overline{v_r v_\theta}$, is related to the errors in the determination of α_j, $\overline{i^2}_j$, and ϕ by:

$$\Delta\overline{v_r v_\theta} = \sum_{j=1}^{6} \left[\frac{\partial \overline{v_r v_\theta}}{\partial \alpha_j} \Delta\alpha_j + \frac{\partial \overline{v_r v_\theta}}{\partial \overline{i^2_j}} \Delta\overline{i^2_j} \right] + \frac{\partial \overline{v_r v_\theta}}{\partial \phi} \Delta\phi \qquad (19)$$

In equation (19), the absolute value of each term is taken to make the sum. It is assumed now that $\Delta\alpha_1 = \ldots \Delta\alpha_6$ and $\Delta\overline{i^2_1} = \ldots = \Delta\overline{i^2_6}$, it can be shown that (see Appendix C) the relative error $\Delta\overline{v_r v_\theta}/\overline{v_r v_\theta}$ is given by:

$$\frac{\Delta\overline{v_r v_\theta}}{\overline{v_r v_\theta}} = \sum_{j=1}^{6} \left\{ \left[\frac{\partial}{\partial\alpha_j} ({}_iD_6) - \frac{{}_iD_6}{D} \frac{\partial D}{\partial\alpha_j} \right] \Delta\alpha + \frac{{}^5D^j}{{}_iD_6} \Delta\overline{i^2} \right\} + \frac{1}{\overline{v_r v_\theta}} \frac{\partial\overline{v_r v_\theta}}{\partial\phi} \Delta\phi \qquad (20)$$

where ${}^5D^j$ is the fifth order determinant formed from ${}_iD_6$ by eliminating the j^{th} row and sixth column. Note that $\frac{\partial}{\partial\alpha_j}({}_iD_6)$ is simply ${}_iD_6$ with the j^{th} row differentiated with respect to α_j.

In the apparatus that will be used in future experiments, α will be determined by a machinist's angular vernier accurate to $\pm 0.1^\circ$, or 0.038% of the full scale reading. The angle ϕ will be determined by a two-dimensional, directional impact probe whose sensitivity varies with fluid velocity. At reasonable velocites (approximately 8 ft/sec or greater) experience indicates that $\Delta\phi$ is in the order of $\pm 0.2^\circ$ or $\pm 0.056\%$ of the full scale.

The error introduced by the reading of the RMS current meter, probably greatly exceeds this. The instrument that will be used has 0 to 5 milliampere scale with a likely error of ± 0.025 ma associated with each reading. The relative error is therefore $\pm 1/2\%$; a factor of 18 greater than the relative error in the determination of α, and a factor of 9 greater than the relative error of ϕ. Equation (20) may therefore be approximated by:

$$\frac{\Delta \overline{v_r v_e}}{\overline{v_r v_e}} \approx \sum_{j=1}^{6} \frac{{}^{5}D^{j}}{{}_{i}D_{6}} \, \Delta \overline{i^2} \tag{20a}$$

As stated above, the value of the relative error, $\Delta \overline{v_r v_e} / \overline{v_r v_e}$, is sensitive to the value of the readings, $\overline{i^2}_1 \ldots \overline{i^2}_6$. For example, if $\overline{i^2}_1 = \overline{i^2}_2 = \ldots = \overline{i^2}_6$, the determinant ${}_iD_6$ is zero (since $c_j = \sin^2\beta$ = constant also) and the relative error is infinite.

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1. Hinze, J. O., Turbulence, p 103, McGraw-Hill (1959).

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3. Hubbard, P. G., Doctoral Dissertation, University of Iowa, p 27 (1954).

4. Hinze, J. O., Turbulence, Chapter II.

Appendix A

The Derivation of the Expression for $\overline{v^2_p}$. (Equation 10)

$$\underline{v}_p = \left[\underline{v}_s + \underline{v}_n + \underline{v}_r\right] - \hat{w}\left[(\underline{v}_s + \underline{v}_n + \underline{v}_r)\cdot\hat{w}\right] \tag{8}$$

The time-dependence indication is omitted for simplicity.

$$\hat{w} = (\sin\alpha\,\sin\beta\,)\hat{s} + (\cos\alpha\,\sin\beta\,)\hat{n} + (\cos\beta\,)\hat{r} \tag{9}$$

$$\therefore\ \underline{v}_p\cdot\underline{v}_p = \Big[\hat{s}(v_s - X\sin\alpha\sin\beta\,) + \hat{n}(v_n - X\cos\alpha\sin\beta\,)$$

$$+ \hat{r}(v_r - X\cos\beta\,)\Big]\cdot\Big[\text{ itself }\Big] \tag{A1}$$

where $X \equiv (v_s + V_n + v_r)\cdot\hat{w} = v_s\sin\alpha\,\sin\beta + v_n\cos\alpha\,\sin\beta + v_r\cos\beta$

$$\therefore\ \underline{v}_p\cdot\underline{v}_p = (v_s - X\sin\alpha\,\sin\beta\,)^2 + (v_n - X\cos\alpha\,\sin\beta\,)^2 + (v_r - X\cos\beta\,)^2$$

$$\left.\begin{aligned}\therefore\ \underline{v}_p\cdot\underline{v}_p = {} & v^2_s - 2v_s X\sin\alpha\,\sin\beta + X^2\sin^2\alpha\,\sin^2\beta + v^2_n \\ & - 2v_n X\cos\alpha\,\sin\beta + X^2\cos^2\alpha\,\sin^2\beta + v^2_r \\ & - 2v_r X\cos\beta + X^2\cos^2\beta\end{aligned}\right\} \tag{A2}$$

from (A1)

$$\left.\begin{aligned}X^2 = {} & v^2_s\sin^2\alpha\,\sin^2\beta + v^2_n\cos^2\alpha\,\sin^2\beta + v^2_r\cos^2\beta \\ & + 2v_s v_n\sin\alpha\,\sin^2\beta\,\cos\alpha + 2v_s v_r\sin\alpha\,\sin\beta\,\cos\beta \\ & + 2v_n v_r\cos\alpha\,\sin\beta\,\cos\beta\end{aligned}\right\} \tag{A3}$$

Collecting the factors of X^2 in (A3) simplifies:

$$X^2 \left(\sin^2\alpha \sin^2\beta + \cos^2\alpha \sin^2\beta + \cos^2\beta\right) = X^2$$

So, equation (A2) may be written:

$$\underline{v}_p \cdot \underline{v}_p = v^2_s - 2v_s X \sin\alpha \sin\beta + v^2_n - 2v_n X \cos\alpha \sin\beta$$

$$+ v^2_r = 2v_r X \cos\beta + X^2 \qquad \text{(A4)}$$

Using (A1), collecting terms in (A4), and time averaging yields:

$$\overline{\underline{v}_p \cdot \underline{v}_p} = \overline{v^2_s} \left(1 - \sin^2\alpha \sin^2\beta\right) + \overline{v^2_n} \left(1 - \cos^2\alpha \sin^2\beta\right)$$

$$+ \overline{v^2_r} \left(\sin^2\beta\right) + \overline{v_s v_n} \left(- 2 \sin\alpha \cos\alpha \sin^2\beta\right) \qquad \text{(A5)}$$

$$+ \overline{v_s v_r} \left(- 2 \sin\alpha \sin\beta \cos\beta\right) + \overline{v_n v_r} \left(- 2 \cos\alpha \sin\beta \cos\beta\right)$$

or

$$\overline{v^2_p} = A\, \overline{v^2_s} + B\, \overline{v^2_n} + C\, \overline{v^2_r} + D\, \overline{v_s v_n} + E\, \overline{v_s v_r} + F\, \overline{v_n v_r} \qquad (10)$$

where

$$A = 1 - \sin^2\alpha \sin^2\beta$$

$$B = 1 - \cos^2\alpha \sin^2\beta$$

$$C = \sin^2\beta$$

$$D = - 2 \sin\alpha \cos\alpha \sin^2\beta = - \sin 2\alpha \sin^2\beta$$

$$E = - 2 \sin\alpha \sin\beta \cos\beta = - \sin\alpha \sin 2\beta$$

$$F = - 2 \cos\alpha \sin\beta \cos\beta = - \cos\alpha \sin 2\beta$$

Appendix B

Derivation of the Cylindrical Polar Reynolds' Stresses from the Stream-Coordinate Reynolds' Stresses. (Equation 16)

Combining equations (15) and (13)

$$\overline{\underline{v}_p \cdot \underline{v}_p} = A \left[\overline{v^2_z} \cos^2 \phi + \overline{v^2_\theta} \sin^2 \phi + 2 \, \overline{v_z v_\theta} \sin \phi \cos \phi \right] +$$

$$B \left[\overline{v^2_z} \sin^2 \phi + \overline{v^2_\theta} \cos^2 \phi - 2 \, \overline{v_z v_\theta} \sin \phi \cos \phi \right] +$$

$$C \left[\overline{v^2_r} \right] +$$

$$D \left[- \frac{1}{2} \overline{v^2_z} \sin 2 \phi + \overline{v_z v_\theta} \cos 2 \phi + \frac{1}{2} \overline{v^2_\theta} \sin 2 \phi \right] +$$

$$E \left[\overline{v_z v_r} \cos \phi + \overline{v_\theta v_r} \sin \phi \right] +$$

$$F \left[- \overline{v_z v_r} \sin \phi + \overline{v_\theta v_r} \cos \phi \right]$$

then collecting terms:

$$\overline{\underline{v}_p \cdot \underline{v}_p} = a \, \overline{v^2_z} + b \, \overline{v^2_\theta} + c \, \overline{v^2_r} + d \, \overline{v_r v_\theta} + e \, \overline{v_z v_r} + f \, \overline{v_\theta v_r} \qquad \text{(B1)}$$

where

$$a \equiv \cos^2 \phi + B \sin^2 \phi - \frac{1}{2} D \sin 2 \phi$$

$$b \equiv A \sin^2 \phi + B \cos^2 \phi + \frac{1}{2} D \sin 2 \phi$$

$$c \equiv C$$

$$d \equiv 2 A \sin \phi \cos \phi - 2 B \sin \phi \cos \phi + D \cos 2 \phi$$

$$e \equiv E \cos \phi - F \sin \phi$$

$$f \equiv E \sin \phi + F \cos \phi$$

And using (7a), yields for the value of the constants:

$$a = \cos^2\phi - \cos^2\phi \sin^2\alpha \sin^2\beta + \sin^2\phi - \sin^2\phi \cos^2\alpha \sin^2\beta$$
$$+ \sin^2\phi \sin\alpha \cos\alpha \sin^2\beta$$

$$= 1 + \sin^2\beta \left[- \cos^2\phi \sin^2\alpha - \sin^2\phi \cos^2\alpha + \frac{1}{2} \sin 2\phi \sin 2\alpha \right]$$

$$b = \sin^2\phi - \sin^2\phi \sin^2\alpha \sin^2\beta + \cos^2\phi - \cos^2\phi \cos^2\alpha \sin^2\beta$$
$$- \sin 2\phi \sin\alpha \cos\alpha \sin^2\beta$$

$$= 1 + \sin^2\beta \left[- \sin^2\phi \sin^2\alpha - \cos^2\phi \cos^2\alpha - \frac{1}{2} \sin 2\phi \sin 2\alpha \right]$$

$$c = C = \sin^2\beta$$

$$d = \sin 2\phi \left[1 - \sin^2\alpha \sin^2\beta - 1 + \cos^2\alpha \sin^2\beta \right]$$
$$- 2 \sin\alpha \cos\alpha \sin^2\beta \cos 2\phi$$

$$= \sin^2\beta \left[\sin 2\phi \cos 2\alpha - \sin 2\alpha \cos 2\phi \right]$$

$$e = - \sin\alpha \sin 2\beta \cos\phi + \sin\phi \cos\alpha \sin 2\beta$$

$$= \sin 2\beta \, (\sin\phi \cos\alpha - \sin\alpha \cos\phi)$$

$$f = - \sin\phi \sin\alpha \sin 2\beta - \cos\phi \cos\alpha \sin 2\beta$$

$$= \sin 2\beta \, (- \sin\phi \sin\alpha - \cos\phi \cos\alpha)$$

Equation (Bl) may be written:

$$\overline{v^2_p} = a\,\overline{v^2_z} + b\,\overline{v^2_\theta} + c\,\overline{v^2_r} + d\,\overline{v_z v_\theta} + e\,\overline{v_z v_r} + f\,\overline{v_\theta v_r}$$

which yields equation (16) when combined with equation (4).

Appendix C

Derivation of the Relative Error in the Experimental Determination of $\overline{v_r v_\theta}$. (Equation 20)

From equation (18f)

$$\left.\begin{array}{l} \dfrac{\partial \overline{v_r v_\theta}}{\partial \alpha_1} = \dfrac{k^2}{D^2}\left[D \dfrac{\partial}{\partial \alpha_1}\left[{}_iD_6 \right] - {}_iD_6 \dfrac{\partial D}{\partial \alpha_1} \right] \\ \quad\vdots \qquad\qquad\qquad \vdots \\ \dfrac{\partial \overline{v_r v_\theta}}{\partial \alpha_j} = \dfrac{k^2}{D^2}\left[D \dfrac{\partial}{\partial \alpha_j}\left[{}_iD_6 \right] - {}_iD_6 \dfrac{\partial D}{\partial \alpha_j} \right] \end{array}\right\} \qquad (C1)$$

also:

$$\frac{\partial \overline{v_r v_\theta}}{\partial \overline{i^2_1}} = \frac{k^2}{D}\frac{\partial}{\partial \overline{i^2}}\left({}_iD_6\right) = \frac{k^2}{D}\begin{vmatrix} a_1 & \cdots & e_1 & 1 \\ a_2 & & & 0 \\ \vdots & & & \vdots \\ a_6 & \cdots & e_6 & 0 \end{vmatrix}$$

$$= \frac{k^2}{D}\begin{vmatrix} a_2 & \cdots & e_2 \\ \vdots & & \\ a_6 & \cdots & e_6 \end{vmatrix} \equiv \frac{k^2}{D}\,{}^5D^1 \qquad (C2)$$

substituting (C1) and (C2) into (19) yields for $\Delta\alpha_j$ = constant and $\Delta\overline{i^2_j}$ = constant:

$$\Delta\overline{v_r v_\theta} = \sum_{j=1}^{6}\left\{\left[\frac{k^2}{D^2}\left(D \frac{\partial}{\partial\alpha_j}\ {}_iD_6\right) - {}_iD_6\ \frac{\partial D}{\partial\alpha_j}\right]\Delta\alpha + \frac{k^2}{D}\ {}^5D^j\,\Delta\overline{i^2}\right\}$$

$$+ \frac{\partial \overline{v_r v_\theta}}{\partial \phi}\,\Delta\phi \qquad (C3)$$

Dividing equation (C3) through by $\overline{v_r v_\theta}$ given by equation (18f) yields equation (20).

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