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ABSTRACT

The Vlasov-ion, fluid-electron model of Freidberg for studying the linear stability of hot-ion pinch configurations is here extended to include electron pressure. Within the framework of an adiabatic electron-gas picture, it is shown that this model is still amenable to the numerical methods described by Lewis and Freidberg.

I. INTRODUCTION

Freidberg[1] has formulated a Vlasov-ion, fluid-electron model for examining the linear stability of hot-ion pinch configurations to low-frequency perturbations. His closed, linear, homogeneous system of equations reads (MKS units),

$$\mu_o^{-1}\left[(\nabla\times B_o)\times\vec{B}_1 + (\nabla\times\vec{B}_1)\times\vec{B}_o\right] + \nabla(\vec{\xi}_\perp\cdot\nabla p_o)$$

$$= i\omega e\left[n_o\vec{\xi}_\perp\times\vec{B}_o + \int(\vec{E}_o+\vec{v}\times\vec{B}_o)\frac{\partial f_o}{\partial\varepsilon}S_1\, d^3v\right] \qquad (1)$$

$$\vec{B}_1 = \nabla\times(\vec{\xi}_\perp\times\vec{B}_o) \qquad (2)$$

$$S_1 = e\int_{-\infty}^{t}\vec{\xi}'_\perp\cdot(\vec{E}_o+\vec{v}\times\vec{B}_o)'dt' \quad , \qquad (3)$$

where $f_o\left[\frac{1}{2}m_i v^2 + e\phi_o(r)\right] \equiv f_o(\varepsilon)$ is the equilibrium ion distribution function, corresponding to density $n_o(\vec{r})$, pressure $p_o(\vec{r})$, embedded in equilibrium electric and magnetic fields $\vec{E}_o(\vec{r})$ and $\vec{B}_o(\vec{r})$. Here, a particular ion at time t has mass m_i, charge e, position $\vec{r}$, and velocity $\vec{v}$. The past-time orbit of such an ion in the equilibrium fields defines the trajectory $\vec{r}\,(t')$, $v\,(t')$, traversed by the orbit integral of Eq. (3).

The perturbation has been assumed to depend on time through $\exp(-i\omega t)$. The complex frequency ω constitutes the eigenvalue of the homogeneous linear system in Eqs. (1) through (3) when suitable boundary conditions are required on the perturbation $\vec{\xi}_1$. The assumptions of low frequency, long wavelength ($\ll \lambda_D$), and small phase velocity ($\ll c$) have been invoked to justify quasi-neutrality and neglect of displacement current.

In the derivation of Eqs. (1) through (3), the electrons were regarded as a massless, pressureless fluid. Therefore, the electron macroscopic velocity $\vec{u}$ obeyed the equation

$$\vec{E} + \vec{u}\times\vec{B} = 0 \, . \qquad (4)$$

In spite of the negligible electron temperature, Coulomb collisions of the electrons were also neglected.

It is the purpose of this report to generalize Eqs. (1) through (3) so as to include nonvanishing electron pressure in an equation for electron momentum of the type given in Eq. (4). This must then be supplemented by the energy equation, or equivalently, the adiabatic law. This local, fluid-electron model must be founded upon "frequent" electron-electron collisions.

We are concerned with gross perturbations of the pinch, such that $L_\perp \sim$ pinch radius, where $L_\perp$ is a characteristic length across the (basically theta-pinch) magnetic field. Macroscopic drifts, as well as guiding center drifts, are characterized by the ion diamagnetic velocity, $u_\perp \sim r_i \left|\frac{\nabla n}{n}\right| v_{thi} \sim (r_i/L_\perp) v_{thi}$, where r_i is the thermal-ion gyroradius, and v_{thi} is the ion thermal velocity. Within the spirit of the FLR ordering, we expect that the pinch inhomogeneity plays a fundamental role in the low-frequency, large-scale perturbations of interest. Therefore, we expect the perturbation frequency to scale as $\omega \sim u_\perp/L_\perp \sim (r_i/L_\perp)^2 \omega_{ci}$, where ω_{ci} is the ion gyrofrequency. These FLR scalings have been freely used to estimate the importance of various terms in our electron model, as described later. However, no small ion gyroradius expansion is ever made in deriving our final set of equations.

For future reference, we now write Eqs. (1) and (3) in a slightly different form. One defines a new orbit integral, S, by

$$S(\vec{r},\vec{v},t) \equiv \int_{-\infty}^{t} m_i \vec{v}' \cdot \frac{d\vec{\xi}_\perp'}{dt'}\, dt' \qquad (5)$$

and notes that S_1 of Eq. (3) can be written

$$S_1 = m_i \vec{v}\cdot\vec{\xi}_\perp - S \quad . \qquad (6)$$

Substitution of Eq. (6) into Eq. (1) then yields

$$F_{MHD}(\vec{\xi}_\perp) = - i\omega e \int (\vec{E}_o + \vec{v}\times\vec{B}_o)\, \frac{\partial f_o}{\partial \epsilon}\, S\, d^3v \ , \qquad (7)$$

where $\vec{F}_{MHD}(\vec{\xi}_\perp)$ represents the left side of Eq. (1) and is formally identical to the MHD force operator for incompressible displacements. Equations (2), (5), (7), and the definition of $F_{MHD}(\xi_\perp)$ constitute the linear, homogeneous system of equations that govern the perturbations of the hot-ion pinch configuration.

We now remark on a certain property of the orbit integral S. Because the past-time equations of motion in the equilibrium fields have the form $d\vec{v}'/dt' = F(\vec{r}',\vec{v}')$, $d\vec{r}'/dt' = \vec{v}'$, i.e., because the time t' does not appear explicitly, it follows that the past-time, unperturbed orbit solutions will depend only on the time difference $(t-t')$. It is then a trivial matter to show that the dependence $\exp(-i\omega t)$, assumed for the perturbation quantity $\vec{\xi}_\perp(\vec{r},t)$, implies a like time dependence of the orbit integral $S(\vec{r},\vec{v},t)$.

Finally, we remark on the relevance of the electron and ion models that are to be used. The Scyllac full torus[2] has had the following properties: $T_i \sim 1$ keV, $T_e \sim (1/2)$keV, for the ion and electron temperatures; and densities like $n \sim 2 \times 10^{16}$ cm^{-3}. The observed instability growth time for an m = 1 perturbation is $t_{pert.} \approx (1/2)\ \mu s$.

From the above densities and temperatures, one calculates an ion-ion collision time, $t_{ii} \sim 1\ \mu s$ and an electron-electron collision time $t_{ee} \sim 0.1\ \mu s$. Therefore, a collisionless (i.e., Vlasov) model for the ion gas is (marginally) relevant. Furthermore, $t_{ee} \ll t_{pert.}$ justifies a local fluid model for the electrons. This is further justified by the small electron gyroradius compared to the pinch radius (perturbation scale length) and by the small mean free path λ_{ee} along the magnetic field (100 cm) compared to the wavelengths of interest along the field. (The most dangerous modes, according to both MHD theory and the Scyllac experiment, have very long wavelengths along the field. In the 4-m major radius torus, toroidal mode numbers n = 0,1,2 have been observed. These correspond to wavelengths $\gtrsim$ 1000 cm).

Although there exists in Scyllac an equilibrium-field characteristic length of ~ 40 cm in the toroidal direction, the fact that this is somewhat smaller than λ_{ee} may not be too important for our local fluid electron model, since it involves magnetic fields only 0.05 times as large as the main theta-pinch field.

The present model differs from earlier FLR treatments (Kennel and Greene;[3] Bowers and Haines[4]) in two respects. First, no small ion gyroradius expansion is made here. Second, the electrons are not assumed to be collisionless and hence obey a Vlasov equation. Rather, they are here described by a local fluid model which assumes $\omega < \nu_{ee}$ and $\lambda_{ee} < \ell_\parallel$, where ω and $\ell_\parallel$ characterize the time and space scales of the perturbation. This fluid model seems more consistent with conditions in the Scyllac full torus as described in Ref. #2. It is also more consistent with conditions in the ZT-1 experiment.

II. BASIC EQUATIONS

We have Vlasov ions,

$$\frac{\partial f}{\partial t} + \vec{v}\cdot\nabla f + \frac{e}{m_i}(\vec{E}+\vec{v}\times\vec{B})\cdot\nabla_{\vec{v}} f = 0 \quad , \qquad (8)$$

and we take the simplest possible fluid model for the electrons, namely,

$$\nabla P_e + ne\,(\vec{E}+\vec{u}\times\vec{B}) = 0 \quad . \qquad (9)$$

Electron inertia has been ignored because the frequencies of interest are small against the electron gyro- and plasma frequencies. In Eq. (9), $\vec{u}$ is the electron macroscopic velocity, and P_e is the electron pressure, assumed scalar.

Equation (9), the electron momentum equation, is to be supplemented by the electron continuity equation, and the electron energy equation with heat flux and ohmic dissipation ignored. Electron-ion energy exchange is also ignored. This constitutes a simple and convenient closure of the electron hierarchy of moment equations, and yields the standard adiabatic law for the electron fluid. As a special case (γ=1), one can also obtain a constant, uniform-temperature model.

The neglect of electron-ion friction in the momentum equation, and the neglect of the corresponding ohmic dissipation in the energy equation have been examined in detail. The required inequalities prove to be

$$\nu_{ei} \ll \omega_{ce} \qquad (i)$$

and

$$\frac{\nu_{ei}}{\omega} \ll \frac{\omega_{pe}^2 L_\perp^2}{c^2} \quad , \qquad (ii)$$

where ν_{ei} is the 90° Coulomb collision frequency with ions, ω_{ce} is the electron gyrofrequency, ω_{pe} is the electron plasma frequency, ω^{-1} is a perturbation time and $L_\perp$ is a characteristic length across the field. Inequality (ii) means that the perturbation time is much shorter than the resistive diffusion time. Both Eqs. (i) and (ii) are well satisfied in Scyllac.

The neglect in the energy equation of electron heat flux driven by electron temperature gradients has been examined in detail. The required inequality proves to be

$$\frac{\lambda_{ee}^2}{L_\parallel^2} \ll \frac{\omega}{\nu_{ei}} \quad . \qquad (iii)$$

Here $\lambda_{ee} \sim \lambda_{ei}$ is the electron mean free path for 90° Coulomb deflections and $L_\parallel$ is a characteristic length along the field. It is not clear that Eq. (iii) will always be satisfied for the perturbations of interest to us.

The neglect of electron-ion energy relaxation in the electron energy equation proves to require

$$(\nu_{ei}/\omega_{ce}) \ll (r_i/L_\perp)^2 \quad , \qquad (iv)$$

where r_i is a thermal-ion gyroradius. (Here, we always use the FLR ordering, $\omega \sim (r_i/L_\perp)^2\omega_{ci}$, to estimate the perturbation frequency.) Inequality (iv) is well satisfied for typical Scyllac conditions.

This electron model is presently being refined to include not only the usual heat flux, but also the various thermoelectric effects associated with electron-ion collisions.

Finally, this entire system is closed by means of Maxwell's equations for the fields. The longitudinal part of the electric field is dealt with by means of the parallel component of Eq. (9), together with the quasi-neutrality assumption (i.e., Poisson's equation), valid because the frequencies of interest lie well below the plasma frequencies and because the wavelengths of interest are much longer than a Debye length. The transverse part of the electric field obeys the usual induction equation, namely,

$$\nabla\times\vec{E} = -\,\partial\vec{B}/\partial t \quad . \qquad (10)$$

The magnetic field is given by Ampere's law,

$$\nabla\times\vec{B} = \mu_0\,\vec{J} \quad , \qquad (11)$$

in which the displacement current has been neglected. The latter is a low-frequency approximation that is valid when the characteristic velocities are much less than the speed of light and when the energy storage is primarily magnetic. Ampere's law is then connected back to the particles by a current density expression,

$$\vec{J} = e \left[\int \vec{v} f d^3v - n\vec{u}\right]$$

$$= e \int (v-u) f d^3v \quad , \qquad (12)$$

where quasi-neutrality has been used.

If one crosses Eq. (11) with B, and makes use of Eqs. (12) and (9), one obtains the basic equation of the Freidberg Vlasov-fluid model, analogous to the equation of motion of ideal MHD, namely

$$(\nabla\times\vec{B}) \times \vec{B} = \mu_o e \int (\vec{E} + \frac{\nabla P_e}{ne} + \vec{v}\times\vec{B}) f\, d^3v \quad 1 \; . \qquad (13)$$

III. EQUILIBRIUM

The equilibrium Vlasov equation for the ions, obtainable from Eq. (8), reads

$$\vec{v}\cdot\nabla f_o + \frac{e}{m_i}(\vec{E}_o + \vec{v}\times\vec{B}_o)\cdot\nabla_{\vec{v}} f_o = 0 \quad . \qquad (14)$$

We choose an isotropic solution of Eq. (14), of the form

$$f_o = f_o\left[\frac{1}{2} m_i v^2 + e\phi_o(\vec{r})\right] \equiv f_o(\epsilon)$$

in which $\vec{E}_o = - \nabla\phi_o$. (15)

Since the ions thus carry no current in equilibrium, the first velocity moment of Eq. (14) yields

$$\nabla P_{io} = n_o e\vec{E}_o \; , \qquad (16)$$

where the ion pressure tensor, $\overleftrightarrow{P}_{io} = \int m_i \vec{v}\vec{v} f_o d\vec{v}^3$, is easily shown to be a scalar given by $P_{io} = (1/3) \int m_i v^2 f_o d^3v$.

Substitution of Eq. (16) into the equilibrium form of Eq. (9) yields

$$\nabla P_{eo} + \nabla P_{io} = - n_o e\vec{u}_o \times \vec{B}_o \equiv \vec{J}_o \times \vec{B}_o \; , \qquad (17)$$

which is identical to the equilibrium pressure-balance equation of ideal MHD.

Henceforth, we suppose f_o is Maxwellian,

$$f_o(\epsilon) = n_{oo} (\alpha_i/\pi)^{\frac{3}{2}} \exp(-\epsilon/T_{io}) \quad , \qquad (18)$$

with a uniform equilibrium temperature T_{io}. In Eq. (18), $\alpha_i = m_i/2T_{io}$, and n_{oo} is the particle density on the set of points at which $\phi_o(\vec{r}) = 0$.

For the electrons, we have $P_{eo} = n_o T_{eo}$, and we assume the electron equilibrium temperature, T_{eo}, to be also uniform. We then define a useful symbol, τ, by

$$\tau \equiv T_{eo}/T_{io} \quad , \qquad (19)$$

which proves to be a fundamental parameter throughout.

Equation (17) now reads

$$(T_{eo}+T_{io})\nabla n_o = - n_o e\vec{u}_o \times \vec{B}_o \; , \qquad (20)$$

which shows that ∇n_o is perpendicular to both $\vec{u}_o$ and $\vec{B}_o$. This has the immediate consequences that $\vec{u}_o \cdot \nabla n_o = 0$ and $\vec{B}_o \cdot \nabla n_o = 0$. Then, from the electron continuity equation of the equilibrium,

$$\nabla\cdot(n_o\vec{u}_o) = 0 = \vec{u}_o\cdot\nabla n_o + n_o\nabla\cdot\vec{u}_o = n_o\nabla\cdot\vec{u}_o .$$

These three basic properties of the assumed equilibrium will prove useful later.

$$\vec{u}_o\cdot\nabla n_o = 0 \; ; \quad \nabla\cdot\vec{u}_o = 0 \; ; \quad \vec{B}_o\cdot\nabla n_o = 0. \qquad (21)$$

Finally, we note for consistency that the equilibrium form of Eq. (13) is

$$\mu_o\vec{J}_o \times \vec{B}_o = \mu_o(n_o e\vec{E}_o + \nabla P_{eo}).$$

Substitution of Eq. (16) herein again yields Eq. (17).

IV. PERTURBATIONS

As shown by Freidberg,[1] there exists a convenient choice of gauge to aid in simplifying the equations governing the perturbations from the equilibrium configuration. This still proves to be the case in the presence of electron pressure. So we shall first derive the special gauge condition, and then use it in the adiabatic electron fluid model so as to relate the electron pressure perturbation to the other perturbation quantities of interest. Subsequently, all of these results will be employed to obtain the final form of the equations governing small perturbations from equilibrium.

IV-A. CHOICE OF GAUGE

Dot the electron momentum equation (9) with $\vec{B}$.

$$\left(\frac{\nabla P_e}{ne} + \vec{E}\right)\cdot \vec{B} = 0.$$

Linearize this. (The subscript "1" denotes a small perturbation quantity.)

$$\left(\frac{\nabla P_{e1}}{n_o e} - \frac{\nabla P_{eo}}{n_o e}\frac{n_1}{n_o} + \vec{E}_1\right)\cdot\vec{B}_o + \left(\frac{\nabla P_{eo}}{n_o e} + \vec{E}_o\right)\cdot\vec{B}_1 = 0 .$$

Note that $(\nabla P_{eo})\cdot\vec{B}_o = 0$ from Eq. (21).

For the same reason, n_o can be taken inside the gradient operator of the ∇P_{e1} term. Then we have

$$\left[\nabla\left(\frac{P_{e1}}{n_o e}\right) + \vec{E}_1\right]\cdot\vec{B}_o + \vec{F}_o\cdot\vec{B}_1 = 0 , \qquad (22)$$

where we defined

$$\vec{F}_o \equiv \vec{E}_o + \frac{\nabla P_{eo}}{n_o e} = -\vec{u}_o\times\vec{B}_o . \qquad (23)$$

The second equation of (23) results from the equilibrium form of Eq. (9).

It is important to note that, due to the uniform equilibrium electron temperature,

$$\nabla\times\vec{F}_o = 0 . \qquad (24)$$

Now one expresses the fields in terms of potentials, namely,

$$\vec{E}_1 = -\frac{\partial\vec{A}_1}{\partial t} - \nabla\phi_1 . \qquad (25)$$

$$\vec{B}_1 = \nabla\times\vec{A}_1 . \qquad (26)$$

Then $\vec{E}_1$ and $\vec{B}_1$ satisfy the induction equation (10), and $\vec{B}_1$ satisfies $\nabla\cdot\vec{B}_1 = 0$, as desired.

Substitution of Eqs. (25) and (26) into (22) provides

$$\vec{B}_o\cdot\nabla\left(\frac{P_{e1}}{n_o e} - \phi_1\right) - \frac{\partial}{\partial t}(\vec{A}_1\cdot\vec{B}_o) + \vec{F}_o\cdot\nabla\times\vec{A}_1 = 0 . \qquad (27)$$

But

$$\vec{F}_o\cdot\nabla\times\vec{A}_1 = \nabla\cdot\left(\vec{A}_1\times\vec{F}_o\right) = \nabla\cdot\left[\vec{A}_1\times(\vec{B}_o\times\vec{u}_o)\right]$$

$$= \nabla\cdot\left[(\vec{A}_1\cdot\vec{u}_o)\vec{B}_o\right] - \nabla\cdot\left[\left(\vec{A}_1\cdot\vec{B}_o\right)\vec{u}_o\right]$$

$$= \vec{B}_o\cdot\nabla(\vec{A}_1\cdot\vec{u}_o) - \vec{u}_o\cdot\nabla(\vec{A}_1\cdot\vec{B}_o) \qquad (28)$$

in which we have made use of Eqs. (23), (24), (21), and $\nabla\cdot\vec{B}_o = 0$.

Substitution of Eq. (28) into Eq. (27) gives

$$\vec{B}_o\cdot\nabla\left(\frac{P_{e1}}{n_o e} - \phi_1 + \vec{A}_1\cdot\vec{u}_o\right) = \left(\frac{\partial}{\partial t} + \vec{u}_o\cdot\nabla\right)(\vec{A}_1\cdot\vec{B}_o) . \qquad (29)$$

We choose to work in a gauge such that

$$\vec{A}_1\cdot\vec{u}_o = \phi_1 - P_{e1}/n_o e . \qquad (30)$$

It then follows that for the perturbations of interest to us,

$$\vec{A}_1\cdot\vec{B}_o = 0 . \qquad (31)$$

In more detail, we have an equation obtained from setting the RHS of Eq. (29) to zero, of the form

$$(\partial/\partial t+\vec{u}_o\cdot\nabla)\psi_1 = 0 \quad ,$$

where ψ_1 is a perturbation quantity. As we have no interest whatsoever in that very special class of perturbations that moves rigidly with velocity $\vec{u}_o$, we must set $\psi_1 = 0$.

This orthogonality of $\vec{A}_1$ to $\vec{B}_o$ allows the introduction of a quantity $\vec{\xi}_\perp$, analogous to the MHD displacement, by means of

$$\vec{A}_1 \equiv \vec{\xi}_\perp \times \vec{B}_o \quad . \tag{32}$$

Substitution of Eq. (32) into the gauge condition of Eq. (30) yields, with the help of Eq. (23),

$$\vec{\xi}_\perp\cdot\vec{F}_o = \phi_1 - \frac{P_{e1}}{n_o e} \quad . \tag{33}$$

IV-B. THE ADIABATIC ELECTRON FLUID MODEL

For an electron fluid moving in electric and magnetic fields without viscosity, ohmic heating, electron-ion energy exchange, or heat conduction, the macroscopic energy equation can be reduced to the form

$$\left(\frac{\partial}{\partial t} + \vec{u}\cdot\nabla\right) P_e + \gamma P_e \nabla\cdot\vec{u} = 0 \quad . \tag{34}$$

The equation of continuity for the electron mass density, $\rho_e \equiv m_e n$, reads

$$\left(\frac{\partial}{\partial t} + \vec{u}\cdot \nabla\right)\rho_e + \rho_e\nabla\cdot\vec{u} = 0 \quad . \tag{35}$$

From these two equations, one obtains

$$\left(\frac{\partial}{\partial t} + \vec{u}\cdot \nabla\right)P_e = \gamma\,\frac{P_e}{\rho_e}\left(\frac{\partial}{\partial t} + \vec{u}\cdot \nabla\right)\rho_e \quad , \tag{36}$$

which is equivalent to the adiabatic law in the form

$$\left(\frac{\partial}{\partial t} + \vec{u}\cdot \nabla\right)\left(P_e\rho_e^{-\gamma}\right) = 0 \quad . \tag{37}$$

When Eq. (36) is linearized, and use is made of the condition $\vec{u}_o\cdot\nabla\rho_{eo} = 0$ and of the spatial uniformity of the equilibrium electron temperature, $\nabla P_{eo}/\rho_{eo}$ = const., then one obtains

$$\left(\frac{\partial}{\partial t} + \vec{u}_o\cdot \nabla\right)\left(P_{e1} - \frac{\gamma P_{eo}}{\rho_{eo}}\rho_{e1}\right) = (\gamma-1)\vec{u}_1\cdot\nabla P_{eo} \quad . \tag{38}$$

We note that the condition, $\gamma = 1$, implies $P_{e1} = T_{eo}n_{e1}$, i.e., no perturbations can arise in the electron temperature when $\gamma = 1$.

The RHS of Eq. (38) is now transformed as follows.

$$\vec{u}_1\cdot\nabla P_{eo} = \vec{u}_1\cdot T_{eo}\nabla n_o = \vec{u}_1\cdot\tau\nabla P_{io} = \vec{u}_1\cdot\frac{\tau}{1+\tau}(1+\tau)\nabla P_{io}$$

$$= \vec{u}_1\cdot\frac{\tau}{1+\tau}\left(\nabla P_{io} + \nabla P_{eo}\right) = \frac{\tau}{1+\tau}\vec{u}_1\cdot\vec{J}_o\times\vec{B}_o$$

$$= -\frac{\tau}{1+\tau}\vec{u}_1\cdot n_o e\vec{u}_o\times\vec{B}_o = +\frac{\tau}{1+\tau}n_o e\vec{u}_1\cdot\vec{B}_o\times\vec{u}_o$$

$$= \frac{\tau}{1+\tau}n_o e\vec{u}_1\times\vec{B}_o\cdot\vec{u}_o \quad , \tag{39}$$

where we have used equilibrium pressure balance, Eq. (17).

The linearized version of the electron momentum equation (9) for small perturbations reads

$$n_o e\vec{u}_1\times\vec{B}_o = -[\nabla P_{e1} + n_o e(\vec{E}_1 + \vec{u}_o\times\vec{B}_1)$$

$$+ n_1 e(\vec{E}_o + \vec{u}_o\times\vec{B}_o)] \quad . \tag{40}$$

If one scalar multiplies this by $\vec{B}_o$, one can recover Eq. (29). Substitution of Eq. (40) into Eq. (39) produces

$$\vec{u}_1\cdot\nabla P_{eo} = -\frac{\tau}{1+\tau}\vec{u}_o\cdot(\nabla P_{e1} + n_o e\vec{E}_1) \tag{41}$$

when we use the result of Eq. (16) and (21) that $\vec{u}_o\cdot\vec{E}_o = 0$. Since $\vec{u}_o\cdot\nabla n_o = 0$, Eq. (41) can also be written

$$\vec{u}_1\cdot\nabla P_{eo} = -\frac{\tau}{1+\tau}n_o e\vec{u}_o\cdot\left[\nabla\left(\frac{P_{e1}}{n_o e}\right) + \vec{E}_1\right] \quad . \tag{42}$$

Replacement of $\vec{E}_1$ by means of Eq. (25), and use of the gauge condition in Eq. (30) then yields

$$\vec{u}_1 \cdot \nabla P_{eo} = -\frac{\tau}{1+\tau} n_o e \left(\frac{\partial}{\partial t} + \vec{u}_o \cdot \nabla\right)\left(\frac{P_{e1}}{n_o e} - \phi_1\right) \quad (43)$$

Substitution of Eq. (43) into Eq. (38) produces, after division by $n_o e$,

$$\left(\frac{\partial}{\partial t} + \vec{u}_o \cdot \nabla\right)\left(\frac{P_{e1}}{n_o e} - \frac{\gamma T_{eo}}{e}\frac{n_{e1}}{n_o}\right) =$$

$$-(\gamma-1)\frac{\tau}{1+\tau}\left(\frac{\partial}{\partial t} + \vec{u}_o \cdot \nabla\right)\left(\frac{P_{e1}}{n_o e} - \phi_1\right), \quad (44)$$

where we have again used the property $\vec{u}_o \cdot \nabla n_o = 0$.

Since the very special class of rigidly moving perturbations that constitute nonvanishing solutions of the equation $(\partial/\partial t + \vec{u}_o \cdot \nabla)\psi_1 = 0$ are of no consequence, this operator must be deleted from each side of Eq. (44). Then, when we observe the gauge result in Eq. (33), Eq. (44) becomes

$$\frac{P_{e1}}{n_o e} - \frac{\gamma T_{eo}}{e}\frac{n_{e1}}{n_o} = (\gamma-1)\frac{\tau}{1+\tau}\vec{\xi}_1 \cdot F_o \quad . \quad (45)$$

Thus, the troublesome unknown perturbation $\vec{u}_1$ has been removed from the problem so that the electron pressure perturbation is now finally expressed in terms of our fundamental perturbation quantities $\vec{\xi}_1$ and

$$n_{e1} = n_1 = \int f_1 d^3v \ , \quad (46)$$

where $f_1(\vec{r},\vec{v},t)$ is the perturbation of the ion distribution function and we have used quasi-neutrality.

IV-C THE SOLUTION FOR f_1

The linearized Eq. (8) reads

$$\left[\frac{\partial}{\partial t} + \vec{v}\cdot\nabla + \frac{e}{m_i}(\vec{E}_o + \vec{v}\times\vec{B}_o)\cdot\nabla_{\vec{v}}\right] f_1 = -\frac{e}{m_i} E_1 \cdot \nabla_{\vec{v}} f_o \quad (47)$$

in which we noted that $\vec{v}\times\vec{B}_1 \cdot \nabla_{\vec{v}} f_o$ vanishes for our choice (15) of the solution f_o.

The solution of Eq. (47) that is unstable [$\mathrm{Im}(\omega) > 0$] and vanishes in the distant past can be written

$$f_1 = -\frac{e}{m_i}\int_{-\infty}^{t} (\vec{E}_1 \cdot \nabla_{\vec{v}} f_o)' \, dt' \ , \quad (48)$$

where the integration is over the ion trajectories in the equilibrium fields $\vec{E}_o$ and $\vec{B}_o$.
But

$$\nabla_{\vec{v}} f_o = (\partial f_o/\partial\epsilon)(m_i \vec{v}) \quad (49)$$

and $\epsilon \equiv [(1/2\ m_i v^2 + e\phi_o]$ is a constant of the motion along the unperturbed orbit, so Eq. (48) becomes

$$f_1 = -\frac{\partial fo}{\partial \epsilon} e \int_{-\infty}^{t} (\vec{E}_1 \cdot \vec{v})' \, dt' \ . \quad (50)$$

Next $\vec{E}_1$ is expressed by the potentials $\vec{A}_1$ and ϕ_1. We then have

$$f_1 = -e\frac{\partial fo}{\partial\epsilon}\int^{t}\left(-\vec{v}\cdot\frac{\partial\vec{A}_1}{\partial t} - \vec{v}\cdot\nabla\phi_1\right)' dt'$$

$$= -e\frac{\partial fo}{\partial\epsilon}\int^{t}\left(i\omega\vec{v}\cdot\vec{A}_1 - \frac{d\phi_1}{dt} + \frac{\partial\phi_1}{\partial t}\right)' dt'$$

$$= \phi_1 e\frac{\partial fo}{\partial\epsilon} - i\omega e\frac{\partial fo}{\partial\epsilon}\int^{t}(\vec{v}\cdot\vec{A}_1 - \phi_1)' \, dt' \quad (51)$$

It is now convenient to define a new perturbation potential π_1 by

$$\pi_1 \equiv P_{e1}/n_o e \ . \quad (52)$$

Moreover, the combination $(\phi_1 - \pi_1)$ has, so far, proven to be significant, so we shall introduce this combination in Eq. (51).

$$f_1 = (\phi_1 - \pi_1)e\frac{\partial fo}{\partial\epsilon} + e\frac{\partial fo}{\partial\epsilon}\int^{t}\left(\frac{d\pi_1}{dt}\right)' dt'$$

$$+ i\omega e\frac{\partial fo}{\partial\epsilon}\int^{t}(\phi_1 - \vec{v}\cdot\vec{A}_1)' \, dt'$$

$$= (\phi_1 - \pi_1)e\frac{\partial f_o}{\partial\epsilon} + e\frac{\partial f_o}{\partial\epsilon}\int^{t}(\vec{v}\cdot\nabla\pi_1)' \, dt'$$

$$+ i\omega e \frac{\partial f_o}{\partial \epsilon} \int^t (\phi_1 - \pi_1 - \vec{v}\cdot\vec{A}_1)'\, dt' \; . \qquad (53)$$

Next, we introduce the gauge result in Eq. (33) and note that $\vec{v}\cdot\vec{A}_1 = -\vec{v}\times\vec{B}_o\cdot\vec{\xi}_1$. Then Eq. (53) becomes

$$f_1 = \vec{\xi}_\perp\cdot\vec{F}_o e \frac{\partial f_o}{\partial \epsilon} + e\frac{\partial f_o}{\partial \epsilon}\int^t (\vec{v}\cdot\nabla\,\pi_1)'\, dt'$$

$$+ i\omega e \frac{\partial f_o}{\partial \epsilon} \int^t \left[\vec{\xi}_\perp\cdot(\vec{F}_o + \vec{v}\times\vec{B}_o)\right]' dt' \; . \qquad (54)$$

For convenience, we shall define orbit integrals,

$$S_\nabla \equiv e\int^t (v\cdot\nabla\pi_1)'\, dt' \qquad (55a)$$

$$S_1 \equiv e\int^t \left[\vec{\xi}_\perp\cdot(\vec{F}_o+\vec{v}\times\vec{B}_o)\right]' dt' \qquad (55b)$$

and write f_1 as

$$f_1 = \vec{\xi}\cdot\vec{F}_o e\frac{\partial f_o}{\partial \epsilon} + \frac{\partial f_o}{\partial \epsilon} S_\nabla + i\omega\frac{\partial f_o}{\partial \epsilon} S_1 \; . \qquad (56)$$

According to Eq. (45), π_1 is to be determined by

$$\pi_1 = \frac{\gamma T_{eo}}{e}\frac{n_1}{n_o} + (\gamma-1)\frac{\tau}{1+\tau}\vec{\xi}_\perp\cdot\vec{F}_o \; . \qquad (57)$$

This expression, derived solely from the electron model, becomes useful provided an independent equation can be devoloped for n_1. This is done in the following section.

IV-D THE EQUATIONS FOR n_1 AND π_1

At this point, we utilize the Maxwellian form of f_o as expressed in Eq. (18). Then

$$\frac{\partial f_o}{\partial \epsilon} = -f_o/T_{io} \; . \qquad (58)$$

Integration of Eq. (56) over velocity space then yields

$$n_1 = -\frac{1}{T_{io}}\left\{\vec{\xi}_\perp\cdot\vec{F}_o e n_o + \int d^3v f_o S_\nabla\right.$$

$$\left. + i\omega\int d^3v f_o S_1\right\} \; . \qquad (59)$$

Define

$$\hat{f}_o \equiv f_o/n_o \; . \qquad (60)$$

and rewrite S_∇ from Eq. (55a) as

$$S_\nabla = e\pi_1 + i\omega e\int_{-\infty}^t \pi_1'\, dt' \; . \qquad (61)$$

Then Eq. (59) becomes

$$\frac{n_1}{n_o} = -\frac{1}{T_{io}}\left\{\vec{\xi}\cdot\vec{F}_o e + e\int d^3v\hat{f}_o\pi_1\right.$$

$$\left. + i\omega e\int d^3v\hat{f}_o\int^t \pi_1'\, dt' + i\omega\int d^3v\hat{f}_o S_1\right\} \; . \qquad (62)$$

Note that π_1 is independent of the particular velocity $\vec{v}$ and that $\int d^3v\hat{f}_o = 1$. Then multiply Eq. (62) by $\gamma T_{eo}/e$ so as to introduce Eq. (57). (Recall that $\tau \equiv T_{eo}/T_{io}$.) The result is an equation relating π_1 to $\vec{\xi}_\perp$. This equation is found to be

$$\frac{dS_\pi}{dt} = \pi_1 = -\frac{\tau}{1+\tau}\vec{\xi}_\perp\cdot\vec{F}_o - i\omega\frac{\gamma\tau}{1+\gamma\tau}\int d^3v$$

$$\left\{[S_\pi + (1/e)S_1]\hat{f}_o\right\} \qquad (63)$$

in which we have introduced another orbit integral, S_π, by

$$S_\pi \equiv \int_{-\infty}^t \pi_1'\, dt' \; . \qquad (64)$$

IV-E. THE EQUATION FOR $\vec{\xi}_\perp$ ANALOGOUS TO THE EQUATION OF THE MOTION FOR THE MHD DISPLACEMENT

Equation (13), linearized, reads

$$(1/\mu_o)\left[(\nabla\times\vec{B}_o)\times\vec{B}_1 + (\nabla\times\vec{B}_1)\times\vec{B}_o\right]$$

$$= e\int(\vec{F}_o+\vec{v}\times\vec{B}_o)f_1 d^3v$$

$$+ e\int\left(\vec{E}_1 + \frac{\nabla P_{e1}}{n_o e} - \frac{\nabla P_{eo}}{n_o e}\frac{n_1}{n_o}\right)f_o d^3v \quad , \qquad (65)$$

in which we noted that $(\vec{v}\times\vec{B}_1)f_o(\epsilon)$ integrates to zero. We note that

$$\frac{\nabla P_{e1}}{n_o e} = \nabla\pi_1 + \pi_1\frac{\nabla n_o}{n_o} \qquad (66)$$

and, using Eq. (57),

$$\frac{\nabla P_{eo}}{n_o e}\cdot\frac{n_1}{n_o} = \frac{T_{eo}}{e}\cdot\frac{\nabla n_o}{n_o}\cdot\frac{n_1}{n_o} = \frac{1}{\gamma}\frac{\nabla n_o}{n_o}\cdot\frac{\gamma T_{eo}}{e}\cdot\frac{n_1}{n_o}$$

$$= \frac{1}{\gamma}\frac{\nabla n_o}{n_o}\left[\pi_1 - (\gamma-1)\frac{\tau}{1+\tau}\vec{\xi}_\perp\cdot\vec{F}_o\right] \quad . \qquad (67)$$

Substitution of Eqs. (56), (66), and (67) into Eq. (65) yields

$$\frac{1}{\mu_o}\left[(\nabla\times\vec{B}_o)\times\vec{B}_1 + (\nabla\times\vec{B}_1)\times\vec{B}_o\right]$$

$$= e\int(\vec{F}_o+\vec{v}\times\vec{B}_o)(\vec{\xi}_\perp\cdot\vec{F}_o e + S_\nabla + i\omega S_1)\frac{\partial f_o}{\partial\epsilon}d^3v$$

$$+ e\int\left[\vec{E}_1+\nabla\pi_1 + \frac{\gamma-1}{\gamma}\frac{\nabla n_o}{n_o}\left(\pi_1 + \frac{\tau}{1+\tau}\vec{\xi}_\perp\cdot\vec{F}_o\right)\right]f_o d^3v \ . \qquad (68)$$

Consider the RHS of Eq. (68). Since $\partial f_o/\partial\epsilon = -f_o/T_{1o}$, the $\vec{\xi}_\perp\cdot\vec{F}_o$ term of the $\partial f_o/\partial\epsilon$ integral yields

$$e\int(\vec{F}_o+\vec{v}\times\vec{B}_o)(\vec{\xi}\cdot\vec{F}_o e)\frac{\partial f_o}{\partial\epsilon}d^3v = -\frac{e^2}{T_1}\vec{F}_o\vec{F}_o\cdot\vec{\xi}_\perp n_o$$

$$= -(1+\tau)\frac{\nabla n_o}{n_o}(\vec{\xi}_\perp\cdot n_o e\vec{F}_o) \qquad (69)$$

where we used the fact that

$$n_o e\vec{F}_o = \nabla P_{eo} + \nabla P_{eo} \quad . \qquad (70)$$

The vector-potential part of the E_1-term of Eq. (68) yields

$$e\int(\vec{E}_1)_A f_o d^3v = i\omega e\vec{A}_1 n_o = i\omega e n_o\vec{\xi}_\perp\times\vec{B}_o \quad . \qquad (71)$$

The scalar-potential part of the $\vec{E}_1$-term of Eq. (68), together with the $\nabla\pi_1$-term, yields

$$e\int[(\vec{E}_1)_\phi + \nabla\pi_1]f_o d^3v = -e\nabla(\phi_1-\pi_1)n_o$$

$$= -\nabla(\xi_\perp\cdot\vec{F}_o)n_o e$$

$$= -\nabla(\vec{\xi}_\perp\cdot\vec{F}_o n_o e) + \vec{\xi}_\perp\cdot\vec{F}_o e\nabla n_o$$

$$= -\nabla(\vec{\xi}_\perp\cdot\nabla P_o) + \vec{\xi}_\perp\cdot\vec{F}_o e n_o\frac{\nabla n_o}{n_o} \quad , \qquad (72)$$

where

$$P_o \equiv P_{io} + P_{eo} \quad , \qquad (73)$$

and we have used the gauge result of Eq. (33), and the pressure-balance equation (70).

Finally, in the $\frac{\nabla n_o}{n_o}$ integral on the RHS of Eq. (68), we use Eq. (63), thus replacing the $(\pi_1, \vec{\xi}_\perp\cdot\vec{F}_o)$ combination by an expression involving the orbit integrals S_π and S_1.

Considering all of the above, Eq. (68) can be rewritten as follows.

$$\frac{1}{\mu_o}\left[(\nabla\times\vec{B}_o)\times\vec{B}_1 + (\nabla\times\vec{B}_1)\times\vec{B}_o\right] + \nabla(\vec{\xi}_\perp\cdot\nabla P_o)$$

$$+ \tau\frac{\nabla n_o}{n_o}\vec{\xi}_\perp\cdot e\vec{F}_o n_o$$

$$= e\int(\vec{F}_o + \vec{v}\times\vec{B}_o)(S_\nabla + i\omega S_1)\frac{\partial f_o}{\partial\epsilon}d^3v$$

$$+ i\omega e n_o\vec{\xi}_\perp\times\vec{B}_o$$

$$- i\omega e\frac{\gamma-1}{\gamma}\cdot\frac{\nabla n_o}{n_o}\cdot\frac{\gamma\tau}{1+\gamma\tau}\int[S_\pi + (1/e)S_1]f_o d^3v \ . \qquad (74)$$

Here, we have used a factor n_o to convert $\hat{f}_o$ back to f_o in the last term on the RHS of Eq. (74).

Next, we shall convert from the S_1-formalism to the S-formalism, as described in the introduction, Eqs. (5) and (6). This is done as follows. From Eq. (55-b), we have

$$S_1 \equiv e\int_{-\infty}^{t} [\vec{\xi}_\perp \cdot (\vec{F}_o + \vec{v} \times \vec{B}_o)]' dt'$$

$$= e\int_{-\infty}^{t} \left[\vec{\xi}_\perp \cdot \left(\frac{\nabla P_{eo}}{n_o e} + E_o + \vec{v} \times \vec{B}_o\right)\right]' dt'$$

$$= e\tau\int_{-\infty}^{t} \left(\vec{\xi}_\perp \cdot \frac{\nabla P_{io}}{n_o e}\right)' dt' + \int_{-\infty}^{t} \left(\vec{\xi}_\perp \cdot m_1 \frac{d\vec{v}}{dt}\right)' dt'$$

$$= e\tau\int_{-\infty}^{t} (\vec{\xi}_\perp \cdot \vec{E}_o)' dt' + \vec{\xi}_\perp \cdot m_1 \vec{v}$$

$$- \int_{-\infty}^{t} \left(m_1 \vec{v} \cdot \frac{d\vec{\xi}_\perp}{dt}\right)' dt' \quad , \tag{75}$$

where we have used Eq. (16).

Let us define the following orbit integrals.

$$S_E \equiv \int_{-\infty}^{t} (\vec{\xi}_\perp \cdot e\vec{E}_o)' dt' \quad . \tag{76a}$$

$$S \equiv \int_{-\infty}^{t} \left(m_1 \vec{v} \cdot \frac{d\vec{\xi}_\perp}{dt}\right)' dt' \quad . \tag{76b}$$

Then Eq. (75) becomes

$$S_1 = \tau S_E + \vec{\xi}_\perp \cdot m_1 \vec{v} - S \quad . \tag{77}$$

Upon substitution of Eq. (77) into Eq. (74), we note that the $(\vec{\xi}_\perp \cdot \vec{v})$ term makes no contribution to the $(\gamma-1)$ term of Eq. (74); whereas in the $\partial f_o/\partial \varepsilon$ integral, it makes a contribution, $-i\omega e n_o \vec{\xi}_\perp \times \vec{B}_o$, which exactly cancels another such term on the right side of Eq. (74). Now Eq. (74) can be written

$$\vec{F}_{MHD}(\vec{\xi}_\perp) + \tau \frac{\nabla n_o}{n_o} \vec{\xi}_\perp \cdot n_o e \vec{F}_o$$

$$= e \int (\vec{F}_o + \vec{v} \times \vec{B}_o)(S_\nabla + i\omega [\tau S_E - S]) \frac{\partial f_o}{\partial \varepsilon} d^3v$$

$$-i\omega \frac{\gamma-1}{\gamma} \frac{\nabla n_o}{n_o} \frac{\gamma\tau}{1+\gamma\tau} \int (eS_\pi + \tau S_E - S) f_o d^3v \tag{78}$$

where we have defined the incompressible MHD force operator by

$$\vec{F}_{MHD}(\vec{\xi}_\perp) \equiv \frac{1}{\mu_o} \left[(\nabla \times \vec{B}_o) \times \vec{B}_1 + (\nabla \times \vec{B}_1) \times \vec{B}_o\right]$$

$$+ \nabla(\vec{\xi}_\perp \cdot \nabla P_o) \quad , \tag{79}$$

and we note here that P_o is the sum of the electron and ion pressures. Furthermore,

$$\vec{B}_1 = \nabla \times \vec{A}_1 = \nabla \times (\vec{\xi}_\perp \times \vec{B}_o) \tag{80}$$

as in ideal MHD.

Next, we use Eq. (61) with Eq. (64), namely,

$$S_\nabla = e\pi_1 + i\omega e S_\pi \quad . \tag{81}$$

This is to be substituted into Eq. (78) with π_1 expressed by means of Eq. (63). The $\partial f_o/\partial \varepsilon$ term on the RHS of Eq. (78) then reads

$$e\int \left(F_o + v \times B_o\right) \left(S_\nabla + i\omega[\tau S_E - S]\right) \frac{\partial f_o}{\partial \varepsilon} d^3v$$

$$= e\int \left(F_o + v \times B_o\right) \left(e\pi_1\right) \left(-\frac{f_o}{T_{1o}}\right) d^3v$$

$$+ i\omega e\int \left(F_o + v \times B_o\right) \left(eS_\pi + \tau S_E - S\right) \frac{\partial f_o}{\partial \varepsilon} d^3v$$

$$= -\frac{e^2}{T_{1o}} \vec{F}_o \pi_1 n_o + i\omega e\int (F_o + v \times B_o)(eS_\pi + \tau S_E - S) \frac{\partial f_o}{\partial \varepsilon} d^3v \quad . \tag{82}$$

From Eq. (63) with Eq. (77), we have

$$n_o e\pi_1 = -\frac{\tau}{1+\tau} \vec{\xi}_\perp \cdot n_o e\vec{F}_o$$

$$- i\omega \frac{\gamma\tau}{1+\gamma\tau} \int d^3v \left[\left(eS_\pi + \tau S_E - S \right) f_o \right] . \quad (83)$$

Upon substitution of Eq. (83) into Eq (82) and (82) into Eq. (78), we note [with the help of Eq. (70)] that the $\vec{F}_o$ term of Eq. (83) yields a term which exactly cancels the $\nabla n_o/n_o$ term on the LHS of Eq. (78). Taking this into account, Eq. (78) can now be written as follows.

$$\vec{F}_{MHD}(\vec{\xi}_\perp) = \frac{e\vec{F}_o}{T_{io}} i\omega \frac{\gamma\tau}{1+\gamma\tau} \int d^3v \left[(eS_\pi + \tau S_E - S) f_o \right]$$

$$+ i\omega e \int d^3v \left[(\vec{F}_o + \vec{v}\times\vec{B}_o)(eS_\pi + \tau S_E - S) \frac{\partial f_o}{\partial \varepsilon} \right]$$

$$- i\omega \frac{\gamma-1}{\gamma} \frac{\nabla n_o}{n_o} \frac{\gamma\tau}{1+\gamma\tau} \int d^3v [(eS_\pi + \tau S_E - S) f_o] . \quad (84)$$

At this point, it is convenient to define a composite orbit integral, $\mathcal{A}$, by

$$\mathcal{A} \equiv S - \tau S_E - eS_\pi . \quad (85)$$

We further observe that the $\vec{F}_o$ and ∇n_o terms of Eq. (84) can be usefully combined because

$$\frac{e\vec{F}_o}{T_{io}} = \frac{n_o e\vec{F}_o}{n_o T_{io}} = \frac{\nabla P_{io} + \nabla P_{eo}}{n_o T_{io}} = \frac{\nabla n_o}{n_o}(1+\tau) . \quad (86)$$

Then, after a little algebra, Eq. (84) takes the form

$$\vec{F}_{MHD}(\vec{\xi}_\perp) = \frac{e\vec{F}_o}{T_{io}} \frac{\tau}{1+\tau} i\omega \int d^3v (-\mathcal{A} f_o)$$

$$- i\omega e \int d^3v \left[(\vec{F}_o + \vec{v}\times\vec{B}_o) \mathcal{A} \frac{\partial f_o}{\partial \varepsilon} \right] . \quad (87)$$

Finally, we observe that, for Maxwellian ions in equilibrium, $\partial f_o/\partial\varepsilon = - f_o/T_{io}$, and also note that

$$\vec{F}_o = \vec{E}_o(1+\tau) . \quad (88)$$

Then Eq. (87) simplifies further to

$$\vec{F}_{MHD}(\vec{\xi}_\perp) = - i\omega e \int d^3v \left[(\vec{E}_o + \vec{v}\times\vec{B}_o) \mathcal{A} \frac{\partial f_o}{\partial \varepsilon} \right] . \quad (89)$$

We note that this equation of motion for $\vec{\xi}_\perp$ is formally identical to Eq. (7) of the cold electron model. There is, at this stage, no explicit dependence upon (T_e/T_i) or upon the electron-fluid adiabatic index, γ. To close the system, we must now develop an equation of motion for the composite orbit integral, $\mathcal{A}(\vec{r}, \vec{v}, t)$.

IV-F. CLOSING THE SYSTEM; THE EQUATION OF MOTION FOR THE ORBIT INTEGRAL $\mathcal{A}$

We refer to Eq. (85) and take the total time derivative of $\mathcal{A}$ along an unperturbed ion orbit. Use of Eqs. (64) and (76) then produces

$$\frac{d\mathcal{A}}{dt} = \frac{dS}{dt} - \tau \frac{dS_E}{dt} - e \frac{dS_\pi}{dt} = m_i \vec{v} \cdot \frac{d\vec{\xi}_\perp}{dt}$$

$$- \tau \vec{\xi}_\perp \cdot e\vec{E}_o - e\pi_1 \quad (90)$$

for π_1, we now invoke Eq. (63), or better, Eq. (83) divided by n_o.

$$e\pi_1 = - \frac{\tau}{1+\tau} \vec{\xi}_\perp \cdot e\vec{F}_o + i\omega \frac{\gamma\tau}{1+\gamma\tau} \int d^3v (\mathcal{A} \hat{f}_o) . \quad (91)$$

Because of the result of Eq. (88), we see that the $\vec{F}_o$ term of Eq. (91) exactly cancels the $\vec{E}_o$ term of Eq. (90). Therefore, Eq. (90) becomes

$$\frac{d\mathcal{A}}{dt} = m_i \vec{v} \cdot \frac{d\vec{\xi}_\perp}{dt} - i\omega \frac{\gamma\tau}{1+\gamma\tau} \int d^3v (\mathcal{A} \hat{f}_o) . \quad (92)$$

We note that for $\tau = 0$, the result of Eq. (92) reduces to the cold-electron result in Eq. (5).

Thus, Eq. (89) relating $\vec{\xi}_\perp$ to $\mathcal{A}$ is now closed by Eq. (92) relating $\mathcal{A}$ to $\vec{\xi}_\perp$.

We recall that the operator, d/dt, here means

$$\frac{d}{dt} \equiv - i\omega + \vec{v}\cdot\nabla_{\vec{r}} + \frac{e}{m_i}(\vec{E}_o + \vec{v}\times\vec{B}_o)\cdot\nabla_{\vec{v}} \quad (93)$$

and we note that $\mathcal{A} = \mathcal{A}(\vec{r}, \vec{v}, t)$, and $\vec{\xi}_\perp = \vec{\xi}_\perp(\vec{r}, t)$.

V. COMPUTATIONAL METHODS

The pair of equations (89) and (92) is amenable to the same kind of numerical treatment that was found applicable to the zero temperature electron-fluid model.[5] We shall now show this explicitly.

Suppose we write Eq. (92) in the form

$$(-i\omega + [D+i\omega I])\mathscr{A} = m_i\vec{v}\cdot(-i\omega+\vec{v}\cdot\nabla)\vec{\xi}_\perp \qquad (94)$$

where D is the phase-space-convective part of the total time derivative (the Liouville operator) and I is a linear operator on $\mathscr{A}$ representing $\gamma\tau(1+\gamma\tau)^{-1}$ times the integral of Eq. (92). Together, Eqs. (89) and (94) are identical to the equations discussed by Ref. (5), except that the operator D is now replaced by [D+iωI]. Consequently, the work of Ref. (5) may be directly applied to the present model provided that the Liouville eigenfunctions and eigenvalues of Ref. (5) are reinterpreted as eigenfunctions and eigenvalues of the operator (1/i)[D+iωI]. The eigenfunctions and eigenvalues of this extended operator, [D+iωI], have a simple physical interpretation, as we now show.

Consider Eq. (63) with $\vec{\xi}_\perp \equiv 0$. It reads

$$(-i\omega + [D+i\omega I])\ S_\pi = 0 \qquad (95)$$

and is precisely the equation defining the eigenvalues (ω) and eigenfunctions of the operator [D+iωI]). Since $\vec{\xi}_\perp \equiv 0$, there are no magnetic perturbations, and an examination of the derivation that leads to Eq. (63) or Eq. (95) shows that these equations are nothing more than a statement of quasineutrality, $n_{e1} \approx n_{i1} = n_1$. Consequently, we are dealing with the long-wavelength version of Poisson's equation in the absence of magnetic perturbations. In other words, the eigenvalue problem defined by Eq. (95) represents the purely electrostatic disturbances supported by the equilibrium pinch configuration. As an example, one can consider a uniform plasma with perturbation wave vector along $\vec{B}_o$. Then Eq. (95) can be shown to yield the dispersion relation for ion-acoustic waves.

It is important to emphasize the <u>idealized</u> nature of these electrostatic modes. They do not, by themselves, exist in the real physical system any more than do the other expansion functions of Ref. (5), namely, the eigenfunctions of the operator $\vec{F}_{MHD}$. Nevertheless, both classes of expansion functions, the MHD modes, and the electrostatic modes have clear physical meanings.

To summarize, Ref. (5) has outlined a numerical approach for solving Eqs. (89) and (92) when $T_e = 0$. Our adiabatic electron gas model can be solved with the same formalism when $T_e \neq 0$. One merely replaces the expansion in Liouville eigenfunctions of Ref. (5) by an expansion in the purely electrostatic modes of the configuration.

VI. CONCLUSION

Assuming uniform equilibrium electron and ion temperatures, and treating the electrons as an adiabatic gas without macroscopic inertia, we have derived a pair of equations that govern small perturbations from the arbitrary equilibrium pinch configuration, namely,

$$\vec{F}_{MHD}(\vec{\xi}_\perp) = -\,i\omega e \int d\vec{v}^3 \left[(\vec{E}_o+\vec{v}\times\vec{B}_o)\mathscr{A}\,\partial f_o/\partial\epsilon\right]$$

$$(-i\omega+D)\,\mathscr{A} = m_i\vec{v}\cdot\ (-i\omega+\vec{v}\cdot\nabla)\vec{\xi}_\perp - i\omega\,\frac{\gamma\tau}{1+\gamma\tau}\int d\vec{v}^3[\mathscr{A}\hat{f}_o]\ .$$

Here, $\hat{f}_o \equiv f_o/n_o$ and D is the convective part of the orbit derivative in phase space.

The pressure term in $\vec{F}_{MHD}$ now contains the sum of the equilibrium electron and ion pressures, and we recall that $\tau \equiv T_{eo}/T_{io}$. The electron adiabatic index is γ.

When we set the equilibrium electron temperature to zero, the above equations reduce to those of Freidberg,[1] and Lewis and Freidberg.[5] When $T_{eo} \neq 0$, one finds a new term in the $\mathscr{A}$-equation, but the numerical methods described by Lewis and Freidberg[5] can still be applied to this more complete model.

The model of D'Ippolito and Davidson[6] also includes electron pressure effects, and also makes no small ion gyroradius expansion. However, it deals only with collisionless (Vlasov) electrons, and is therefore not suitable for treating low-frequency, long-wavelength modes for which $\omega < \nu_{ee}$, $\lambda_{ee} < \ell_\parallel$. The present fluid electron model, however, applies just when these inequalities are satisfied. Moreover, these conditions are consistent with those recently observed in the Scyllac full torus experiment.[2]

Finally, we mention that the adiabatic electron model described here has now been extended to include electron heat conduction along the magnetic field. The above formalism remains valid but γ now becomes a complex-valued, known function of position, $\Gamma(\vec{r})$. When heat conduction is (locally) small, $\Gamma \to \gamma$, and when it is large, $\Gamma \to 1$.[7]

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