LA-3748

324
10-17-67

LOS ALAMOS SCIENTIFIC LABORATORY
of the
University of California
LOS ALAMOS • NEW MEXICO

MASTER

Some Discrete Transforms and Their Continuous Analogs

UNITED STATES
ATOMIC ENERGY COMMISSION
CONTRACT W-7405-ENG. 36

DISCLAIMER

Printed in the United States of America. Available from
Clearinghouse for Federal Scientific and Technical Information
National Bureau of Standards, U. S. Department of Commerce
Springfield, Virginia 22151
Price: Printed Copy $3.00; Microfiche $0.65

LA-3748
UC-34, PHYSICS
TID-4500

LOS ALAMOS SCIENTIFIC LABORATORY
of the
University of California
LOS ALAMOS • NEW MEXICO

Report written: October 1, 1965

Report distributed: October 2, 1967

Some Discrete Transforms and Their Continuous Analogs

by

D. E. Nagle

THIS PAGE

WAS INTENTIONALLY

LEFT BLANK

SOME DISCRETE TRANSFORMS AND THEIR CONTINUOUS ANALOGS

by

D. E. Nagle

ABSTRACT

It is shown that the Cauchy, Taylor, and Laurent theorems, which refer to analytic functions, have discrete analogs which hold for arbitrary functions of the complex variable z, with the single restriction that the values of z divide the unit circle.

Introduction. Certain transforms for the discrete function $f(z)$ are exhibited, and their analogies to the integral transforms of Fourier, Cauchy, Taylor, and Laurent are pointed out.

We consider $f(z)$ to be defined for the points

$$z_p = \exp\,(2\pi ip/N)\ ,\quad p = 1 \dots N\ .$$

If we define

$$\alpha_q \equiv N^{-1} \sum_{p=1}^{N} f(z_p) \exp\,(-2\pi ipq/N)\ , \tag{1}$$

then

$$f(z_p) = \sum_{q=1}^{N} \alpha_q \exp\,(2\pi ipq/N)\ . \tag{2}$$

Analog of the Fourier Integrals. Although these equations are analogous to the Fourier integral theorem, no restrictions are placed on the function f; restrictions are placed only on its argument. Equations (1) and (2) were used by Fortescue[1] to treat the problem of N-phase networks.

Analog of the Cauchy and Taylor Integrals. Construct sums of the form

$$U_k = \frac{1}{2iN \sin\frac{\pi}{N}} \sum_{p=1}^{N} f(z_p)\Delta z_p \big/ (z_p)^{k+1}\ . \tag{3}$$

$k = 1 \dots N$, where

$$\Delta z_p = z_{p+\frac{1}{2}} - z_{p-\frac{1}{2}}\ .$$

These sums are the analogs of the Cauchy integrals for the quantities

$$(k!)^{-1} \frac{d^k g(z)}{dz^k}\Big|_{z=0} = \frac{1}{2\pi i} \oint \frac{g(z)dz}{z^{k+1}}\ ,$$

where $g(z)$ is analytic inside the contour of integration. Noting that

$$\Delta z_p = 2z_p i \sin\frac{\pi}{N}\ ,$$

and comparing with Eq. (1), we have

$$U_k = \alpha_k\ . \tag{4}$$

If we allow the equation

$$U_k \equiv (k!)^{-1} \frac{d^k f(z)}{dz^k} \Big|_{z=0}$$

to formally define the function f at the origin together with its first N-1 derivatives, and use Eq. (2) in the form

$$f(z_p) = \sum_{q=0}^{N-1} \alpha_q z_p^{\,q} \,, \qquad (5)$$

we have

$$f(z) = \sum_{q=0}^{N-1} \frac{1}{q!} \frac{d^q}{dz^q} f(z) \Big|_{z=0} (z)^q \,. \qquad (6)$$

Equation (6) with $z = z_p$ is the analog of Taylor's series. For $z \neq z_p$, Eq. (6) may be regarded as a polynomial fitting the points $f(z_p)$.

Analog of the Cauchy Integral About the Point $z = a$. Consider the expression

$$W \equiv (2iN \sin \tfrac{\pi}{N})^{-1} \sum_{p=1}^{N} f(z_p)\Delta z_p \Big/ (z_p - a) \,, \qquad (7)$$

which is the analog of the Cauchy integral about $z = a$, namely,

$$g(a) = \frac{1}{2\pi i} \oint g(z)dz \Big/ _{z-a} \,.$$

Noting that

$$\frac{1}{z_p - a} = z_p^{-1}(1 - a^N)^{-1} \left[1 + \frac{a}{z} + \frac{a^2}{z^2} + \ldots + \left(\frac{a}{z}\right)^{N-1}\right]$$

and comparing with Eq. (1), we have

$$W(1 - a^N) = \sum_{q=0}^{N-1} a^q \alpha_q \equiv f(a) \,, \qquad (8)$$

where f(a) is given by Eq. (6) with $z = a$. For $|a| < 1$, $N \to \infty$, the right-hand side of Eq. (7) reduces to the Cauchy integral, and the right-hand side of Eq. (8) reduces to the usual Taylor development for f(a) if the series converges uniformly. The condition that the function $f(z_p)$, which initially was arbitrary for certain points on the unit circle, behaves in the limit $N \to \infty$ like an analytic function is that the functions α_q approach zero so rapidly as to assure uniform convergence of the series, Eq. (8).

If we form the expression

$$U = \frac{1}{2iN \sin \frac{\pi}{N}} \sum_p \frac{f(z_p)\Delta z_p}{(z_p - a)^2} \qquad (9)$$

in analogy to the Cauchy integral

$$\frac{dg}{dz}\Big|_{z=a} = \frac{1}{2\pi i} \oint \frac{g(z)dz}{(z - a)^2} \,,$$

we can show by similar methods that

$$U = \frac{1}{(1 - a^N)^2} \sum_{q=1}^{N} \alpha_q a^{q-1} \left[q + (N-q)a^N\right] \,. \qquad (10)$$

In the limit $N \to \infty$ for $|a| < 1$, Eq. (10) reduces to the function obtained by a term by term differentiation of the series, Eq. (8), with respect to a, i.e., the result, Eq. (10), is the discrete analog of the Cauchy result.

Analog of the Laurent Series. In the case $|a| > 1$, Eq. (8) may be rewritten as

$$W\left(1 - \frac{1}{a^N}\right) = \sum_{N-q=1}^{N} \alpha_q \, a^{-(N-q)}$$

and

$$W\left(1 - \frac{1}{a^N}\right) = \sum_{k=1}^{N} \alpha_{N-k} \, a^{-k} \,, \qquad (11)$$

where

$$\alpha_{N-k} = \frac{\sum_{p=0}^{N-1} f(z_p) \, z_p^{\,k-1} \Delta z_p}{2iN \sin \frac{\pi}{N}} \,,$$

which is the analog of the Laurent's series for a function that is zero at infinity but may have singularities within the circle C surrounding the origin, namely,

$$g(a) = \sum_{n=1}^{\infty} \frac{a^{-n}}{2\pi i} \oint t^{n-1} g(t) \, dt \,.$$

Applications. Equations (3) through (11) were developed while investigating the behavior of cavities for linear accelerators.

REFERENCE

1. C. L. Fortescue, "Method of Symmetrical Co-ordinates Applied to the Solution of Polyphase Networks," Trans. AIEE, XXXVII-2, 1027 (1918).