QUANTUM EXPANSION OF SOLITON SOLUTIONS

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DISCLAIMER

The usual perturbative field theory, while useful as a guide to some aspects of hadronic physics, does not provide a computational framework for discussing strong interaction phenomena. The quanta obtained from quantizing free fields, which are the lowest approximation in perturbation theory, are simply too far removed from real hadrons to serve as a reasonable first approximation. Recently there has been renewed interest in making quite a different connection between hadronic physics and field theory. In this approach[1] one attempts to relate the particle-like solutions of classical non-linear field equations with physical hadrons. Of course in order to make such a connection, these classical particle-like solutions must be carried over into quantum field theory and a computational scheme must be provided. It is such a method[2, 3] for giving quantum mechanical meaning to particle-like (or soliton) classical solutions which I would like to describe. The method was developed in collaboration with T. D. Lee.

Let us begin by considering a Lagrange density depending on an N-component real scalar field ϕ^i

$$\mathcal{L} = -\tfrac{1}{2} \sum_i \left(\frac{\partial \phi^i}{\partial x_\mu}\right)^2 - g^{-2} V(g\phi^i) \quad . \tag{1}$$

If the potential $g^{-2}V(g\phi^i)$ is expanded in a power series in ϕ^i about its minimum, the parameter g will enter only terms of cubic or higher power in ϕ^i; thus g plays the role of a coupling constant. Assume that

$$\left[\phi^i(\vec{r}, t)\right]_{cl} \equiv g^{-1} \sigma^i(\vec{r}, t, z_1^0 \cdots z_K^0) \tag{2}$$

represents a family of equal energy solutions, depending on the K parameters $z_1^0 \cdots z_K^0$, for which we would like to find a quantum mechanical interpretation. The function $\sigma^i(\vec{r}, t, z_1^0 \cdots z_K^0)$ obeys

$$\frac{\partial^2 \sigma^i}{\partial x_\mu^2} - \frac{\partial V(\sigma)}{\partial \sigma^i} = 0 \tag{3}$$

and hence is chosen independent of g. The parameters $z_1^0 \cdots z_K^0$ will always be chosen so that translation of our solution in space or time can be accomplished by changing the parameters $z_1^0 \cdots z_K^0$. Therefore, there will exist values $z_k(t, z_1^0 \cdots z_K^0)$ satisfying

$$\sigma^i[\vec{r}, t, z_1^0 \cdots z_K^0] = \sigma^i[\vec{r}, 0, z_1(t, z_1^0 \cdots z_K^0) \cdots z_K(t, z_1^0 \cdots z_K^0)] \quad . \quad (4)$$

If we wish to discuss a single soliton in D dimension space, then we will introduce D parameters $z_1 \cdots z_D$ corresponding to the position of the center of mass of the soliton. In many cases, the classical solution for a single soliton at rest is time independent and no further parameters are needed. Even if such a soliton is moving, the only time dependence corresponds to simple translation of the center of mass so that again only D parameters are necessary. For example, in the case of a single moving soliton solution to the 1-dimensional sine-Gordon equation,

$$-\frac{\partial^2 \sigma}{\partial t^2} + \frac{\partial^2 \sigma}{\partial x^2} - \mu^2 \sin \sigma = 0 \quad , \qquad (5)$$

we choose

$$\sigma(x, z) = 4 \tan^{-1}\left[e^{\gamma_\upsilon (x-z)\mu^2}\right] \quad , \qquad (6)$$

depending on the single parameter z. This is a solution to Eq. (5) if $z(t, z^0) = z^0 + ut$ where $\upsilon = \sqrt{1-\gamma_\upsilon^{-2}}$ is the soliton velocity. However, we intend also to include more complicated situations. For a charged soliton the classical solution will have an additional time-dependent phase factor[4] while other periodic solutions, as was seen in the sine-Gordon case,[5] may correspond to soliton-antisoliton bound states. Similarly, it is possible to discuss soliton-soliton scattering if for $\sigma^i(\vec{r}, t; z_1^0 \cdots z_K^0)$ we use a classical solution containing two moving solitons. In this case at least $2D$ parameters z_k are introduced corresponding asymptotically to the center-of-mass positions of the two solitons.

We now turn to a quantum-mechanical expansion which approaches the original classical solution $\sigma^i(\vec{r}, z_1(t), \cdots, z_K(t))$ in the limit of small coupling g. First we expand the classical field $\phi(\vec{r})$ about our classical solution $\frac{1}{g}\sigma(\vec{r}, z_1 \cdots z_K)$

$$\phi^i(\vec{r}, t) = g^{-1}\sigma^i(\vec{r}, z_1 \cdots z_K) + \sum_{n=K+1}^{\infty} q_n(t)\, \psi_n^i(\vec{r}, z_1 \cdots z_K) \tag{7}$$

where $z_1 \cdots z_K$, q_{K+1}, $q_{K+2} \cdots$ are treated as coordinates and the N-component functions $\psi_n^i(\vec{r}, z_1 \cdots z_K)$ form a complete set of real functions subject to the constraints

$$\sum_{i=1}^{N} \int \psi_n^i \frac{\partial \sigma^i}{\partial z_k}\, d\tau = 0 \tag{8}$$

and the orthonormality relation

$$\sum_{i=1}^{N} \int \psi_n^i\, \psi_{n'}^i\, d\tau = \delta_{nn'} . \tag{9}$$

We propose to use the usual canonical methods to quantize the field theory described by the Lagrange density (1) using the new coordinates $z_1 \cdots z_K$, $q_{K+1} \cdots$. In terms of these coordinates the Lagrangian takes the form

$$L = \frac{1}{2}\dot{z}_k M_{kk'} \dot{z}_{k'} + \dot{z}_k M_{kn} \dot{q}_n + \dot{q}_n M_{nn'} q_{n'}$$

$$- \int d\tau \left\{ \frac{1}{2}\left[\frac{1}{g}\vec{\nabla}\sigma^i + q_n(\vec{\nabla}\psi_n^i)\right]^2 + \frac{1}{g^2} V(\sigma^i + g\, q_n \psi_n^i) \right\} \tag{10}$$

where repeated indices have been summed over: k or k' from 1 to K; n or n' from $K+1$ to ∞; and i from 1 to N. The mass matrix M is given by

$$M_{kk'} = \int d\tau \left(g^{-1}\frac{\partial \sigma^i}{\partial z_k} + q_n \frac{\partial \psi_n^i}{\partial z_k} \right)\left(g^{-1}\frac{\partial \sigma^i}{\partial z_{k'}} + q_{n'} \frac{\partial \psi_{n'}^i}{\partial z_{k'}} \right) ,$$

$$M_{kn} = M_{nk} = \int q_{n'} \frac{\partial \psi_{n'}^i}{\partial z_k}\, \psi_n^i\, d\tau \tag{11}$$

and

$$M_{nn'} = \delta_{nn'} .$$

The momenta conjugate to z_k and q_n are

$$p_k = \frac{\partial L}{\partial \dot{z}_k} = M_{kk'}\dot{z}_{k'} + M_{kn}\dot{q}_n ,$$

$$\pi_n = \frac{\partial L}{\partial \dot{q}_n} = M_{nk}\dot{z}_k + M_{nn'}\dot{q}_{n'} \tag{12}$$

The coordinates z_k, q_n and conjugate momenta p_k, π_n can all be identified as operators in the standard way and the resulting quantum-mechanical Hamiltonian is

$$H = \tfrac{1}{2} J^{-1}\Big[p_k (M^{-1})_{kk'} \, J p_{k'} + p_k (M^{-1})_{kn} \, J\pi_n + \pi_n (M^{-1})_{nk} \, J p_k$$

$$+ \pi_n (M^{-1})_{nn'} \, J \pi_{n'}\Big] + g^{-2}\int d\tau\, \bar{V}(\sigma^i + g\, q_n \varphi_n^i) \; . \tag{13}$$

where

$$\bar{V}(\xi) \equiv V(\xi) + \tfrac{1}{2}(\vec{\nabla}\xi)^2 \; . \tag{14}$$

The Jacobian factor J,

$$J = \left[\det\,(M_{kk'} - M_{kn} M_{nk'})\right]^{1/2} , \tag{15}$$

is introduced so that the resulting ordering of non-commuting factors defines an operator H equal to the usual quantum Hamiltonian

$$H = \int d\tau \left\{ \frac{1}{2}\left[\Pi^i(\vec{r})\right]^2 + \frac{1}{g^2}\bar{V}(g\,\varphi(\vec{r}))\right\} \tag{16}$$

where $\Pi^i(\vec{r})$ is the momentum operator conjugate to the local field $\varphi^i(\vec{r})$.

In order to develop a systematic perturbation expansion of the eigenstates of H in powers of g we must first deal with the terms of order g^{-2} and g^{-1} which appear in H . We exploit the close connection between g and Planck's constant h by writing an arbitrary quantum-mechanical state Ψ_α in the WKB form

$$\Psi_\alpha(z_k, q_n) = e^{iS(z_k)/g^2}\, \chi_\alpha(z_k, q_n) \; . \tag{17}$$

Here quantum states are written as wave functions depending on the variables z_k and q_n. If the function $\chi_\alpha(z_k, q_n)$ is assumed to be regular at $g = 0$, then the terms of order g^{-2} in the product $H\Psi_\alpha$ reduce to a diagonal form $\frac{1}{g^2}\mathcal{E}\,\Psi_\alpha$ if $S(z_1 \cdots z_K)$ is chosen to satisfy the Hamilton-Jacobi equation

$$\frac{1}{2}\,\frac{\partial S}{\partial z_k}\,(M_0)^{-1}_{kk'}\,\frac{\partial S}{\partial z_{k'}} + \int \bar{V}(\sigma^i(\vec{r}, z_1 \cdots z_K))\,d\tau \;=\; \mathcal{E} \tag{18}$$

where

$$(M_0)_{kk'} = \int d\tau\,\frac{\partial \sigma^i}{\partial z_k}\,\frac{\partial \sigma^i}{\partial z_{k'}} \quad . \tag{19}$$

Furthermore, the terms of order g^{-1} can be completely eliminated from the product $H\Psi$, if the solution S is chosen to generate precisely the classical time dependence $z_k(t, z_1^0 \cdots z_K^0)$ specified by Eq. (4), i.e.

$$\frac{\partial S}{\partial z_k}\,(z_k(t, z_1^0 \cdots z_K^0)) \;=\; (M_0)_{kk'}\,\frac{d}{dt}\,z_{k'}(t, z_1^0 \cdots z_K^0) \quad . \tag{20}$$

These equations can be integrated, completely determining S up to an additive constant, provided they are consistent with the identity

$$\frac{\partial}{\partial z_k}\left(\frac{\partial S}{\partial z_{k'}}\right) = \frac{\partial}{\partial z_{k'}}\left(\frac{\partial S}{\partial z_k}\right) \quad . \tag{21}$$

This requirement will be obeyed, at least for most cases of interest if we make a suitable choice of parametrization of the classical solution. For example, if σ is a classical solution describing the scattering of ℓ solitons in D dimensional space, then Lagrange's equations can be shown to imply the consistency of Eqs (20) and (21) provided we introduce parameters z_k which develop in time as

$$z_k(t, z_1^0 \cdots z_K^0) = v_k\,t + z_k^0 \tag{22}$$

and which correspond asymptotically to the $\ell \times D$ center-of-mass coordinates of the solitons.

Thus, if we remove the factor $\exp(ig^{-2}S)$ from all quantum states, the transformed Hamiltonian

$$H' = e^{-iS/g^2} H e^{iS/g^2} \tag{23}$$

can be written as a power series in g

$$H' = H'(-2) + H'(0) + H'(1) + \cdots . \tag{24}$$

With our choice of S the term of order g^{-2}, $H'(-2)$, is simply a constant $\mathcal{E}/g^2$, just the energy of our classical solution. After some algebraic rearrangement $H'(0)$ can be written

$$H'(0) = \frac{u}{\sqrt{J}} p_K \sqrt{J} + \frac{1}{2}\pi_n\pi_n + \frac{1}{2} q_n F_{nn'} q_{n'} + \frac{1}{2}(\pi_n G_{nn'} q_{n'} - q_n G_{nn'} \pi_{n'}) \tag{25}$$

where

$$F_{nn'} = \int \left[(\vec{\nabla}\psi_n^i)(\vec{\nabla}\psi_{n'}^i) + \psi_n^i \frac{\partial^2 V(\sigma)}{\partial\sigma^i\,\partial\sigma^j} \psi_{n'}^j \right] d\tau + 3u^2 f_{nk}(M_0^{-1})_{kk'} f_{n'k'} ,$$

$$f_{nk}^i = \int \psi_n^i \frac{\partial^2 \sigma^i}{\partial z_k\,\partial z_K} d\tau \tag{26}$$

and

$$G_{nn'} = -G_{n'n} = -u\int \psi_n^i \frac{\partial \psi_{n'}^i}{\partial z_K} d\tau .$$

Here we have introduced the parameters z_k so that Eq. (22) is obeyed with $u_k = 0$ for $k < K$ and $u_K = u$. Such a situation results if we explicitly choose one of the parameters, z_K^0, in Eq. (2) to correspond to time translation.

The next step in our development requires diagonalization of the zeroth order Hamiltonian $H'(0)$. If our classical solution is time independent so that $u_K = 0$, then $H'(0)$ simplifies to

$$H'(0) = \frac{1}{2}\pi_n\pi_n + \frac{1}{2} q_n F_{nn'} q_{n'} \tag{27}$$

where $F_{nn'}$ can be identified in this case as the nn' matrix element of the operator

$$-\frac{1}{2}\vec{\nabla}^2\delta_{ij}+\frac{1}{2}\frac{\partial^2 V(\sigma)}{\partial\sigma^i\partial\sigma^j} \quad . \tag{28}$$

Differentiation of the classical field equation (3) with respect to z_k, reveals that when $v=0$ the K functions $\partial\sigma^i/\partial z_k$ are eigenstates of (28) with eigenvalue zero. Thus it is possible to choose the ψ_n^i orthogonal to $\partial\sigma^i/\partial z_k$ and also eigenstates of (28) with eigenvalues $\frac{1}{2}\omega_n^2$. Consequently

$$F_{nn'} = \delta_{nn'}\frac{1}{2}\omega_n^2 \tag{29}$$

so that the operator $H'(0)$ is easily diagonalized having the spectrum

$$\mathcal{E}_\alpha = \sum_n (N_n+\frac{1}{2})\,\omega_n \tag{30}$$

where the N_n's are occupation numbers $N_n = 0, 1, 2, \cdots$. The ω_n are simply the frequencies for small oscillation about the static classical solution $\sigma(r, z_1 \cdots z_K)$. For most cases of interest, the parameters $z_1 \cdots z_K$ appearing in such a family of static degenerate solutions can be interpreted as physical quantities on which the entire Hamiltonian does not depend. Thus the eigenfrequencies ω_n will not depend on the z_k and the constants $\mathcal{E}/g^2 + \mathcal{E}_\alpha$ are then eigenvalues of H accurate to order g^0. For example when expanding about a static soliton solution to the sine-Gordon equation (5), we write

$$\phi(x) = \frac{4}{g}\tan^{-1}\left[e^{(x-z)\mu^2}\right] + \sum_n q_n\psi_n(x-z) \tag{31}$$

so that changing the parameter z corresponds to a space translation under which H is invariant. Hence $H(0)$ and the frequencies ω_n will not depend on z.

Finally let us consider the time-dependent case. Although $H'(0)$ is still quadratic in the q_n's and π_n's, the situation is more complicated than the static case because their coefficients will in general depend on z_K and $H'(0)$ also contains a term linear in p_K. In fact, the eigenvalue condition

$$H'(0)\, X_\alpha = \mathcal{E}_\alpha X_\alpha \tag{32}$$

can be compared to a time-dependent Schroedinger equation in which z_K/v is interpreted as the time and the quadratic terms

$$H_2 = \frac{1}{2}\left[\pi_n \pi_n + q_n F_{nn'} q_{n'} + \pi_n G_{nn'} q_{n'} - q_n G_{nn'} \pi_{n'}\right] \tag{33}$$

as a time-dependent Hamiltonian. Thus the eigenstates X_α can be written

$$X_\alpha(z_k, q_n) = \frac{1}{\sqrt{J}}\, e^{i\delta p_K \cdot z_K}\, U(z_K)\, X_\alpha(z_1 \cdots z_{K-1}, 0, q_n) \quad . \tag{34}$$

Here δp_K is a constant, $X_\alpha(z_1 \cdots z_{K-1}, 0, q_n)$ is any function of z_k, $k < K$ and the q_n while $U(z_k)$ is a "time" development operator obeying

$$\begin{aligned} i v \frac{\partial}{\partial z_K} U(z_K) &= H_2 U(z_K) \quad , \\ U(0) &= I \quad . \end{aligned} \tag{35}$$

The energy eigenvalue of the state $\exp\left[iS/g^2\right] X_\alpha$ is $\mathcal{E}/g^2 + v\delta p_K$. Although Eq. (35) has no general explicit solution even for the quadratic Hamiltonian H_2, it can be solved quite easily in certain cases.

For example if the classical solution $\sigma^i(\vec{r}, t)$ changes slowly compared to the characteristic frequencies of H_2, then Eq. (35) can be solved in the usual adiabatic approximation. On the other hand, if $\sigma^i(\vec{r}, t)$ is periodic with period T, then the close connection between H_2 and the Hamiltonian describing small oscillations about the classical orbit allows us to find the operator for "time" development through one period, $U(vT)$, quite easily in terms of the stability angles β_ℓ which appear in the classical problem. In particular, $U(vT)$ is given by

$$U(vT) = e^{-i \sum_\ell (A_\ell^+ A_\ell + 1/2)\, \beta_\ell\, T} \quad . \tag{36}$$

where

$$A_\ell = a_n^\ell q_n + b_n^\ell \pi_n \tag{37}$$

with

$$i(a_n^\ell b_n^{\ell'*} - b_n^\ell a_n^{\ell'*}) = \delta_{\ell\ell'} \quad .$$

The quantities a_n^ℓ, b_n^ℓ and β_ℓ are related to the classical small oscillation problem in the following way. First parametrize small oscillations about the classical solution $z_k^{cl} = u_k t + z_k^0$, $q_n = \pi_n = 0$ by writing

$$\begin{aligned} z_k(t) &= z_k^{cl}(t) + \delta z_k(t) \quad , \quad p_k(t) = \frac{1}{g^2}\frac{\partial S}{\partial z_k}(z_k^{cl} + \delta z_k) + \delta p_k(t) \\ q_n(t) &= \delta q_n(t) \quad , \quad \pi_n(t) = \delta \pi_n(t) \quad . \end{aligned} \tag{38}$$

The quantities a_n^ℓ, b_n^ℓ represent those initial conditions

$$\delta q_n = -b_n^\ell \lambda \quad , \qquad \delta \pi_n = a_n^\ell \lambda \tag{39}$$

which change only by an overall phase $e^{-i\beta_\ell T}$ when time developed through a period

$$\begin{aligned} \delta q_n(T) &= -e^{-i\beta_\ell T} b_n \lambda \quad , \\ \delta \pi_n(T) &= e^{-i\beta_\ell T} a_n \lambda \quad . \end{aligned} \tag{40}$$

Throughout this motion δp_k is chosen zero and δz_k of order g while λ is a small proportionality constant. Knowledge of $U(uT)$ is sufficient to determine the eigenvalues of $H'(0)$. Because $\sigma^i(\vec{r}, z_1 \cdots z_K)$ is periodic in z_K with period uT, the coordinates $z_1 \cdots z_K$, $q_{K+1} \cdots$ and $z_1 \cdots z_K + uT$, $q_{K+1} \cdots$ determine the same configuration of our physical system. Consequently, we must require that our wave function have the same value at the points $z_K = 0$ and $z_K = uT$. The values of the wave function at these two points are explicitly connected given $U(uT)$ and Eq. (34); the allowed energies $\mathcal{E}/g^2 + u\delta p_K$ are then fixed by the requirement that

$$i\delta p_K uT + \int_0^{uT} P_K \, dz_K - \sum_\ell (N_\ell + \tfrac{1}{2})\beta_\ell T = 2\pi n \tag{41}$$

where $p_K = \frac{\partial S}{\partial z_K}$. If applied to the breather mode of the sine-Gordon equation, this condition gives exactly the spectrum found by Dashen, Hasslacher and Neveu.[5] In a similar fashion, if we know the connection between the small oscillations about a two-soliton solution long before and long after the scattering, we can discuss soliton-soliton scattering accurate to order g^0.

The method described above allows quantum mechanical description of various classical solutions to non-linear field equations. If the Hamiltonian $H'(0)$, very closely related to the classical small oscillation problem, can be diagonalized, then the effects of terms higher order in g can be systematically calculated using ordinary perturbation theory. The method appears to be relatively simple, using the familiar canonical Hilbert space formulation of Quantum Mechanics, and has been applied to interpret quantum-mechanically both static and time-dependent classical particle-like solutions. It is our hope that this general approach will prove useful in developing a realistic quantum field theory of hadrons.

References

1. The possibility of associating elementary particles with particle-like solutions to classical non-linear field equations was first raised by J. K. Perring and T. H. R. Skyrme, Nucl. Phys. 31, 550 (1962), and D. Finkelstein and C. W. Misner, Ann. Physics 6, 230 (1959). Recent interest was stimulated by the work of H. B. Nielsen and P. Olsen, Nucl. Phys. B61, 45 (1973), G. t'Hooft, Nucl. Phys. B79, 276 (1974), and A. M. Polyakov, Landau Institute preprint.

2. This problem has attracted a great deal of attention recently. Several different approaches have been developed. Those based on various forms of functional integration techniques are R. F. Dashen, B. Hasslacher and A. Neveu, Phys. Rev. D10, 4114 and 4130 (1974),

IAS preprint COO-2220-37; J. L. Gervais and B. Sakita, CCNY preprint HEP-74/6; J. L. Gervais, A. Jevicki and B. Sakita, CCNY preprint HEP-75-2; C. Callan and D. Gross, Princeton University preprint, and R. Rajaraman and E. J. Weinberg, IAS preprint COO-2220-41; L. D. Faddeev, "Quantization of Solitons", IAS preprint. Those based on the "Kerman-Klein method" and the Green's function approach are J. Goldstone and R. Jackiw, MIT preprint, and A. Klein and F. R. Krejs, University of Pennsylvania preprint. Those based on variational techniques are W. A. Bardeen, M. S. Chanowitz, S. D. Drell and T. M. Yan, Phys. Rev. D11, 1094 (1975); P. Vinciarelli, CERN preprint TH. 1993 (1975); K. Cahill, Phys. Letters 53B, 174 (1974). See also M. Creutz, Phys. Rev. D10, 1749 (1974); M. Creutz and K. S. Soh, preprint BNL-19363. For a review of some of these methods see R. Rajaraman, IAS preprint COO-2220-47.

3. Our method, N. H. Christ and T. D. Lee, Columbia University preprint CO-2271-55, is based on the usual canonical Hilbert space formulation of Quantum Mechanics. A similar approach to the quantization of static classical solutions has been suggested by E. Tomboulis, MIT preprint. M. Creutz, preprint BNL-20121, has recently proposed independently a canonical procedure for quantizing static and some time-dependent solutions, also similar in certain respects to the work I will describe.

4. An example of our treatment of such a charged soliton is discussed by T. D. Lee in his contribution to this conference. See also E. Weinberg and R. Rajaraman (ref. 2) and M. Creutz (ref. 3).

5. R. F. Dashen, B. Hasslacher, A. Neveu, reference 2.