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DISPERSION THEORY AND CURRENT ALGEBRA

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ABSTRACT

Lorentz invariance and the basic assumption in dispersion theory, that the matrix element of a retarded or advanced commutator of local fields is an analytic function of the energy variable, are seen to determine the method of handling the dispersion integral, and to require the matrix element to consist of terms, each being a product of at most two poles or integral thereof. This method is used to study current-algebra commutators with the consequence that the widely employed assumption of single-pole dominance for the spin-one parts of vector or axial-vector currents is inconsistent with current-algebra. Some aspects of the $K_{\ell 3}$ form factors are also discussed.

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1. INTRODUCTION

Besides the success in soft-pion theorems,[1] current-algebra has also been investigated by assuming single-pole dominance for the spin-one parts of currents with interesting results. The relation of Kawarabayashi, Suzuki, Riazuddin and Fayyazuddin[2] is a much quoted example, and, although the validity of their derivation has been in doubt,[3] it is satisfied experimentally to a remarkable degree. Comparing this with Weinberg's sum rules,[4] the mass of the A_1 is predicted to be $\sqrt{2}$ times that of the ρ. This prediction had of course a remarkable agreement with experiment at that time. Schnitzer and Weinberg[5] further strengthened these sum rules by considering single particle intermediate states and obtained one of the two ρ-A_1-π vertices while the other cannot be determined. Since then the existence of the A_1 itself has come under some suspicion and therefore theoretical predictions about its mass and coupling constants are somewhat less meaningful. In order to capitalize on the structure of current commutators, Brown and West[6] used dispersion relations to evaluate amplitudes of the form

$$\int d^4x\, e^{iqx}\, \theta(x_0) \langle p | [A_\mu(x), V_\nu(0)] | 0 \rangle \qquad (1.1)$$

whose absorptive part was found to contain delta functions of q^2 as well as those of $(p-q)^2$. They were then faced with the problem whether to use fixed-q^2 dispersion relation or to use fixed-$(p-q)^2$ dispersion relation. They solved this dilemma by keeping an arbitrary linear combination of q^2 and $(p-q)^2$ fixed, and by imposing a consistency condition for the amplitude, concluding that the dispersion relation automatically emerged in the once-subtracted form. We will find that this is unnecessary.

The starting point of dispersion theory is to assume that the expression (1.1) is an analytical function of q_o (with $\underline{q}$ and other variables fixed)[7] in the upper half of the q_o-plane. We will see in Section 2 that this already contains enough information to solve the above-mentioned dilemma, so that no additional prescription is required. The amplitude then naturally emerges (assuming no subtractions, for simplicity) as a product of two simple poles, one in q^2 and the other in $(p-q)^2$.

Among others, one important distinction between this result and that of reference[6] is that the number of subtractions is not restricted in this approach. An effective method will be developed, which abbreviates the tedious arguments. As an exercise, we will apply this effective method in Section 3 to study the amplitude

$$\int d^4x \, e^{iqx} \theta(x_0) \langle 0 | [\pi^a(x), V_\mu^b(0)] | \pi^c(p) \rangle ,$$

only to discover that, in general, subtractions are required and that the off-mass-shell coupling $\gamma_{\rho\pi\pi}(q^2)$ enters in the combination $\gamma_{\rho\pi\pi}(q^2) / [(m_\pi^2 - q^2)(m_\rho^2 - (p-q)^2)]$.

Section 4 will be devoted to study SU(2)XSU(2) current-algebra, which will be extended to cover strangeness-changing currents in Section 5. Relations between soft-pion calculations and our method will also be discussed for the $\langle \pi | V_\mu | K \rangle$ amplitude.

2. COMMENTS ON DISPERSION RELATION

The usual basic assumption in dispersion theory is the observation that an amplitude of the form

$$M(q) = i\int d^4x\, e^{iqx}\theta(x_0)\langle\alpha|[\phi(x), \varphi(0)]|\beta\rangle \tag{2.1}$$

is an analytical function, when considered as a function of q_0, with $\underset{\sim}{q}$ fixed, in the upper half of the complex q_0-plane. Here, $|\alpha\rangle$ and $|\beta\rangle$ are arbitrary vectors in the Hilbert space of physical states, $\phi(x)$ and $\varphi(x)$ are local field operators satisfying microcausality, q and x are 4-vectors, $\theta(x_0)$ is the step function, $qx = g_{\mu\nu}x^\mu x^\nu$ and $g_{00} = -g_{11} = -g_{22} = -g_{33} = 1$. The arrival at this observation is guided by the observation that the factor $\exp(iqx)\theta(x_0)$ in (2.1) approaches zero as $x_0 \to \infty$ for $\mathrm{Im}q_0 > 0$, so that, barring abnormal behaviour of $\langle\alpha|[\phi(x),\varphi(0)]|\beta\rangle$ for large x, the integral for M(q) is well-defined for $\mathrm{Im}q_0 > 0$. In the same way, one finds that

$$M'(q) = i\int d^4x\, e^{iqx}\theta(-x_0)\langle\alpha|[\varphi(0), \phi(x)]|\beta\rangle \tag{2.2}$$

is an analytical function in the lower half of the q_0-plane, and that this function is the analytic continuation of M(q) across the real q_o-axis, so that the entire function, denoted by the H(q), exhibits the Schwartz reflection property

$$H(q_0, \underset{\sim}{q}) = H^*(q_0^*, \underset{\sim}{q}), \tag{2.3}$$

and the discontinuity across the real q_o-axis, called the absorptive part AbsM(q), is given by

$$2i\, \mathrm{Abs}\, M(q) = H(q_0 + i0, \underset{\sim}{q}) - H(q_0 - i0, \underset{\sim}{q}). \tag{2.4}$$

Therefore the absorptive part of M(q) along the q_o-axis is

$$\mathrm{Abs}\, M(q) = \frac{1}{2}\int d^4x\, e^{iqx} \langle\alpha|[\phi(x), \varphi(0)]|\beta\rangle. \tag{2.5}$$

Inserting a complete set of states between the fields $\phi(x)$ and $\varphi(0)$ in the commutator $[\phi(x), \varphi(0)]$, one is able to express AbsM(q) in terms of $\langle\alpha|\phi(0)|n\rangle$, etc:

$$\mathrm{Abs}\, M(q) = \frac{1}{2}\int d^4x\, e^{iqx}\Big[\langle\alpha|\phi(x)\sum_n |n\rangle\langle n|\varphi(0)|\beta\rangle - \langle\alpha|\varphi(0)\sum_n |n\rangle\langle n|\phi(x)|\beta\rangle\Big], \tag{2.6}$$

where n runs over the entire Hilbert space.

Purely for the purpose of illustrating the main point of this communication, let us restrict ourselves to the ideal world whose Hilbert space consists of single- and two-particle states, and whose particles are neutral and of two kinds: $|m,\underset{\sim}{p}\rangle$ and $|m',\underset{\sim}{p}\rangle$ with masses m and m' respectively (variables $\underset{\sim}{p}$ denote their momentum). Let us assume

$$\begin{aligned} \langle 0|\phi(0)|m,\underset{\sim}{p}\rangle &= \langle 0|\varphi(0)|m',\underset{\sim}{p}\rangle = (2\pi)^{-3/2}, \\ \langle 0|\phi(0)|m',\underset{\sim}{p}\rangle &= \langle 0|\varphi(0)|m,\underset{\sim}{p}\rangle = 0, \end{aligned} \tag{2.7}$$

and let the coupling between two particles of mass m with a particle of mass m' be γ while all others are zero. Then choosing $\langle\alpha|$ to be the vacuum $\langle 0|$ and $|\beta\rangle$ to be $|m, \underset{\sim}{p}\rangle$, and following Appendix A, we obtain

$$\begin{aligned} \mathrm{Abs}\,M(q,p) = \frac{1}{4}(2\pi)^{5/2}\Big[&\langle 0|\phi(0)|m,\underset{\sim}{q}\rangle\langle m,\underset{\sim}{q}|\varphi(0)|m,\underset{\sim}{p}\rangle\,\delta(q_0-\sqrt{m^2+\underset{\sim}{q}^2})/\sqrt{m^2+\underset{\sim}{q}^2} \\ &-\langle m,-\underset{\sim}{q}|\phi(0)|0\rangle\langle 0|\varphi(0)|(m,\underset{\sim}{p}),(m,-\underset{\sim}{q})\rangle\,\delta(q_0+\sqrt{m^2+\underset{\sim}{q}^2})/\sqrt{m^2+\underset{\sim}{q}^2} \\ &-\langle 0|\varphi(0)|m',\underset{\sim}{k}\rangle\langle m',\underset{\sim}{k}|\phi(0)|m,\underset{\sim}{p}\rangle\,\delta(k_0-\sqrt{m'^2+\underset{\sim}{k}^2})/\sqrt{m'^2+\underset{\sim}{k}^2} \\ &+\langle m',-\underset{\sim}{k}|\varphi(0)|0\rangle\langle 0|\phi(0)|(m,\underset{\sim}{p}),(m',-\underset{\sim}{k})\rangle\,\delta(k_0+\sqrt{m'^2+\underset{\sim}{k}^2})/\sqrt{m'^2+\underset{\sim}{k}^2}\Big] \end{aligned} \tag{2.8}$$

where $p_0 = \sqrt{m^2 + \underset{\sim}{p}^2}$, and $k_\mu = p_\mu - q_\mu$, and where we have explicitly displayed p_μ as arguments of M. These terms are represented in

Fig. 1(a)-(d), respectively.

By standard dispersion relations,[8] one shows that, for any

$\underset{\sim}{p}$, $\underset{\sim}{q}$, and $\underset{\sim}{q}'$,

$$\langle m, \underset{\sim}{q} | \varphi(0) | m, \underset{\sim}{p} \rangle = (2\pi)^{-3} \gamma / [m'^2 - (p-q)^2]$$
$$\langle m', \underset{\sim}{q}' | \phi(0) | m, \underset{\sim}{p} \rangle = (2\pi)^{-3} \gamma / [m^2 - (p-q')^2], \tag{2.9}$$

where $q_0 = \sqrt{m^2 + \underset{\sim}{q}^2}$, and $q_0' = \sqrt{m'^2 + \underset{\sim}{q}^2}$. Combining equations (2.7)-(2.9) with crossing

symmetry, one has

$$\mathrm{Abs}\, M(q,p) = \frac{\gamma}{2\sqrt{2\pi}} \left[\frac{\delta(q_0 - \sqrt{m^2 + \underset{\sim}{q}^2}) - \delta(q_0 + \sqrt{m^2 + \underset{\sim}{q}^2})}{2\sqrt{m^2 + \underset{\sim}{q}^2}\,(m'^2 - k^2)} - \frac{\delta(k_0 - \sqrt{m'^2 + \underset{\sim}{k}^2}) - \delta(k_0 + \sqrt{m'^2 + \underset{\sim}{k}^2})}{2\sqrt{m'^2 + \underset{\sim}{k}^2}\,(m^2 - q^2)} \right]. \tag{2.10}$$

We can easily convince ourselves that the combination of delta functions here is also required by Lorentz invariance, for it requires the entire function H(q,p) to be a function of the Lorentz invariants q^2 and qp. So H(q,p) is invariant under the transformation $q_\mu \to -q_\mu$ and $p_\mu \to -p_\mu$, and hence the Schwartz reflection property of H(q,p) (equation (2.3)) implies

$$\mathrm{Im}\, H(q_0 + i0, \underset{\sim}{q}, p) = -\mathrm{Im}\, H(-q_0 + i0, -\underset{\sim}{q}, -p), \tag{2.11}$$

which, in terms of M(q,p), means[9]

$$\mathrm{Abs}\, M(q,p) = -\mathrm{Abs}\, M(-q,-p). \tag{2.12}$$

This is satisfied by (2.10).

Now, assuming, for simplicity, unsubtracted dispersion relation for M(q,p) as an analytic function of q_0, we obtain

$$M(q,p) = \frac{\gamma}{\pi}\int dq_0' \frac{\mathrm{Abs}\,M(q_0',\underset{\sim}{q},p)}{q_0'-q_0}$$
$$= \frac{\gamma}{(2\pi)^{3/2}(m^2-q^2)(m'^2-k^2)} \ , \tag{2.13}$$

which exhibits the two-pole feature. Thus the basic assumption of dispersion relation, namely that M(q,p) is an analytic function of q_o in the upper half of the complex q_o-plane, dictates unambiguously the treatment on AbsM(q,p) when confronted with delta functions of both q_o and p_o-q_o.

At this point, we may note that if we had replaced the expression (2.10) of AbsM(q,p) by

$$\mathrm{Abs}\,M(q,p) = \frac{\gamma}{2\sqrt{2\pi}}\left[\frac{\delta(m^2-q^2)}{m'^2-k^2} - \frac{\delta(m'^2-k^2)}{m^2-q^2}\right] , \tag{2.14}$$

then we may still recover (2.13) if we postulate an unsubtracted dispersion relation for M(q,p) as an analytic function not of q_o, but of q^2, and assume that, for fixed qk, the discontinuity of M(q,p) across the real q^2-axis is given by (2.14) (for then k^2 is a linear function of q^2: $k^2 = m^2 - 2qk - q^2$).

In practice it is easier to use this effective technique than to go through the entire procedure every time we are faced with such a situation. In the entire paper, this effective technique will be employed.

Similar cases, where the dispersion relations in q_o require subtractions, will be treated in Appendix B. We may state the result here: Given the imaginary part of a complex function of q_o, it is always easier to find the function by inspection under the guidelines (i) that it has the correct imaginary part, and (ii) that it is Lorentz invariant, than to intergrate directly.

We may also note that in the more realistic case the Hilbert space consists of a continuum of states of various spins in addition to single particle states, and that the above consideration will be modified by intergrating over m^2 and m'^2 with suitable spectral functions.

3. A SIMPLE APPLICATION — ρ-π-π SYSTEM

Let a pion state of 4-momentum p and isospin a be denoted by $|\pi^a(p)\rangle$, and let a rho state of 4-momentum k, polarization 4-vector ε, and isospin a be denoted by $|\rho^a(k,\varepsilon)\rangle$. The off-mass-shell ρ-π-π coupling $\gamma_{\rho\pi\pi}(q^2)$ is defined by

$$\frac{i\epsilon^{abc}\,2(\varepsilon p)}{(2\pi)^3(m_\pi^2-q^2)}\gamma_{\rho\pi\pi}(q^2) = \langle\rho^b(k,\varepsilon)|\pi^a(0)|\pi^c(p)\rangle, \tag{3.1}$$

where $q_\mu = p_\mu - k_\mu$, ϵ^{abc} is the anti-symmetric tensor, $\pi^a(x)$ is the pion field, and m_π is the mass of the pion. Consider the matrix element $\langle\pi^a(q)|V_\nu^b(0)|\pi^c(p)\rangle$ of the strangeness-conserving current between two pion states. Assuming unsubtracted dispersion relation for its form factor, and assuming that the spin-one part of the unit operator $\sum_n |n\rangle\langle n|$ can be approximated by the rho-state contribution $\sum_{d,\varepsilon}\int d^3\underset{\sim}{k}\ \frac{1}{2}(m_\rho^2+\underset{\sim}{k}^2)^{-1/2}$ $|\rho^d(k,\varepsilon)\rangle\langle\rho^d(k,\varepsilon)|$, one finds, by standard dispersion technique,[10]

$$\langle\pi^a(q)|V_\nu^b(0)|\pi^c(p)\rangle = -\frac{i\epsilon^{abc}}{(2\pi)^3}m_\rho F_\rho\gamma_{\rho\pi\pi}(m_\pi^2)\frac{p_\nu+q_\nu}{m_\rho^2-k^2}, \tag{3.2}$$

where $k_\mu = p_\mu - q_\mu$, m_ρ is the mass of the rho, and F_ρ is defined by

$$\langle 0 | V_\mu^b(0) | \rho^d(k,\varepsilon)\rangle = (2\pi)^{-3/2}\,\delta^{bd}\, m_\rho F_\rho\, \varepsilon_\mu . \tag{3.3}$$

We now relax the mass-shell condition on q_μ, and generalize the amplitude (3.2) to

$$M_\nu^{abc}(q,p) \equiv i(2\pi)^{-3/2}\int d^4x\, e^{iqx}\,\theta(x_0)\langle 0 | [\pi^a(x), V_\nu^b(0)] | \pi^c(p)\rangle , \tag{3.4}$$

where, to be specific, we will only concern ourselves with the Lorentz covariant part of this amplitude. Note that for $q^2 \to m_\pi^2$,

$$(m_\pi^2 - q^2)\, M_\nu^{abc}(q,p) = \langle \pi^a(q) | V_\nu^b(0) | \pi^c(p)\rangle .$$

Invoking the assumption and arguments of the previous Section, we then find

$$\text{Abs}\, M_\nu^{abc}(q,p) = \pi\,\delta(m_\pi^2 - q^2)\langle \pi^a(q) | V_\nu^b(0) | \pi^c(p)\rangle$$

$$- \pi m_\rho F_\rho\, \delta(m_\rho^2 - k^2) \sum_\varepsilon \varepsilon_\nu \langle \rho^b(k,\varepsilon) | \pi^a(0) | \pi^c(p)\rangle . \tag{3.5}$$

With the help of (3.1) and (3.2),

$$\text{Abs}\, M_\nu^{abc}(q,p) = \frac{\pi i \epsilon^{abc}}{(2\pi)^3} m_\rho F_\rho \Big[-\frac{\delta(m_\pi^2 - q^2)}{m_\rho^2 - k^2}\,\gamma_{\rho\pi\pi}(m_\pi^2)(p_\nu + q_\nu)$$

$$- \frac{\delta(m_\rho^2 - k^2)}{m_\pi^2 - q^2}\,\gamma_{\rho\pi\pi}(q^2)\Big(-g_{\nu\beta} + \frac{k_\nu k_\beta}{m_\rho^2}\Big) 2p^\beta \Big] . \tag{3.6}$$

In terms of the form factors $f_\pm'(q^2,k^2)$ defined by

$$M_\nu^{abc}(q,p) = i(2\pi)^{-3}\epsilon^{abc}\big[f_+'(q^2,k^2)(p_\nu + q_\nu) + f_-'(q^2,k^2)(p_\nu - q_\nu)\big] , \tag{3.7}$$

the absorptive part is

$$\text{Abs}\, M_\nu^{abc}(q,p) = i(2\pi)^{-3}\epsilon^{abc}\big[\text{Im} f_+'(q^2,k^2)(p_\nu + q_\nu)$$

$$+ \text{Im} f_-'(q^2,k^2)(p_\nu - q_\nu)\big] . \tag{3.8}$$

Comparison of this with (3.6) gives the imaginary part of the form factors:

$$\mathrm{Im}\, f_+'(q^2,k^2) = -m_\rho F_\rho \gamma_{\rho\pi\pi}(q^2)\left[\frac{\delta(m_\pi^2-q^2)}{m_\rho^2-k^2} - \frac{\delta(m_\rho^2-k^2)}{m_\pi^2-q^2}\right] \tag{3.9}$$

$$\mathrm{Im}\, f_-'(q^2,k^2) = \frac{F_\rho}{m_\rho}\gamma_{\rho\pi\pi}(q^2)(m_\pi^2-q^2)\left[\frac{\delta(m_\pi^2-q^2)}{m_\rho^2-k^2} - \frac{\delta(m_\rho^2-k^2)}{m_\pi^2-q^2}\right]. \tag{3.10}$$

Appendix C shows that $\gamma_{\rho\pi\pi}(q^2)$ is, under the same general assumption of dispersion theory, a polynomial in q^2, so that (3.9) and (3.10) are in the form required by Appendix B, and, hence, the form factors are

$$f_+'(q^2,k^2) = -\frac{m_\rho F_\rho \gamma_{\rho\pi\pi}(q^2)}{(m_\pi^2-q^2)(m_\rho^2-k^2)} \tag{3.11}$$

$$f_-'(q^2,k^2) = \frac{F_\rho}{m_\rho}\,\frac{\gamma_{\rho\pi\pi}(q^2)}{m_\rho^2-k^2}. \tag{3.12}$$

Substituting them in (3.7), we have

$$M_\nu^{abc}(q,p) = \frac{i\epsilon^{abc}}{(2\pi)^3}\,\frac{m_\rho F_\rho \gamma_{\rho\pi\pi}(q^2)}{(m_\pi^2-q^2)(m_\rho^2-k^2)}\left(-g_{\nu\beta}+\frac{k_\nu k_\beta}{m_\rho^2}\right)(p^\beta+q^\beta). \tag{3.13}$$

Let us assume exact SU(2) symmetry, that is,

$$\delta(x_0)\left[V_0^a(x),\pi^b(0)\right] = i\epsilon^{abc}\pi^c(0)\delta^4(x), \tag{3.14}$$

and

$$\partial^\mu V_\mu(x) = 0. \tag{3.15}$$

Then the Ward identity reads[11]

$$k^\nu M_\nu{}^{abc}(q,p) = -i(2\pi)^{-3}\,\epsilon^{abc}, \qquad (3.16)$$

which, together with (3.13) implies that the off-mass-shell coupling $\gamma_{\rho\pi\pi}(q^2)$ is independent of q^2 and that it is given by

$$\gamma_{\rho\pi\pi}(q^2) = -m_\rho/F_\rho\,. \qquad (3.17)$$

4. SU(2)XSU(2) CURRENT ALGEBRA AND THE A-ρ-π SYSTEM

Let us consider the matrix element of a retarded or advanced commutator of two currents between the vacuum and a physical state. Employing Ward identities and the method developed above, we may study the context of current algebra. In this section, the assumptions of current algebra that will be involved are

$$\delta(x_0)\left[V_0^a(x), A_\mu^b(0)\right] = i\epsilon^{abc} A_\mu^c(0)\,\delta^4(x), \tag{4.1}$$

$$\delta(x_0)\left[A_0^a(x), V_\mu^b(0)\right] = i\epsilon^{abc} A_\mu^c(0)\,\delta^4(x), \tag{4.2}$$

$$\delta(x_0)\left[A_0^a(x), A_\mu^b(0)\right] = i\epsilon^{abc} V_\mu^c(0)\,\delta^4(x). \tag{4.3}$$

We have omittedthe Schwinger[12] terms, because we will only evaluate the Lorentz covariant part of retarded or advanced commutators, and, following Bjorken,[13] they do not contribute to the divergences of these commutators. We also assume strict SU(2), that is

$$\partial^\mu V_\mu^a(x) = 0. \tag{4.4}$$

In evaluating absorptive parts, we expect to encounter matrix elements $\langle 0|V_\mu^a|n\rangle$ of vector currents between the vacuum and physical states $|n\rangle$. For these to be non-zero, $|n\rangle$ must have isospin one, and even

G-parity. In an angular momentum expansion of such states, we find that, because V_μ^a is a divergenceless 4-vector, only the spin one states contribute. We will label such states simply by $|\rho^b(k,\varepsilon,\sigma)\rangle$ to indicate that its isospin index is b, its momentum is k, its polarization 4-vector is ε and the square of its mass is σ. All other internal variables and summation or integration thereof will be omitted for clarity. Thus we define their coupling $F_\rho(\sigma)$ with the vector current V_μ^a by, in analogy to the rho-states of the previous Section,

$$\langle 0|\, V_\mu^a(0)\,|\rho^b(k,\varepsilon,\sigma)\rangle \equiv (2\pi)^{-3/2}\delta^{ab}\sqrt{\sigma}\,F_\rho(\sigma)\varepsilon_\mu. \tag{4.5}$$

Similarly, for the axial-vector currents, we have

$$\langle 0|A_\mu^a(0)|a^b(k,\varepsilon,\sigma)\rangle \equiv (2\pi)^{-3/2}\delta^{ab}\sqrt{\sigma}\,F_a(\sigma)\varepsilon_\mu, \tag{4.6}$$

where, again omitting unessential internal variables, $|a^b(k,\varepsilon,\sigma)\rangle$ is a spin-one state of isospin one, isospin index b, odd G-parity 4-momentum k, polarization ε, and square of the mass σ. In addition to these spin-one states, there are, in principle, spin-zero states which are annihilated into the vacuum by the axial-vector current. From now on let us assume that, out of these states, only the pion states are dominant and define F_π by

$$\langle 0|A_\mu^a(0)|\pi^b(p)\rangle = i(2\pi)^{-3/2}F_\pi p_\mu. \tag{4.7}$$

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Equations (3.1), (3.2), (3.13), and (3.17) are now generalized to

$$\frac{i\epsilon^{abc}\, 2(\varepsilon p)}{(2\pi)^3(m_\pi^{\,2}-q^2)}\,\gamma_{\rho\pi\pi}(\sigma) \equiv \langle \rho^b(k,\varepsilon,\sigma)|\,\pi^a(0)\,|\pi^c(p)\rangle, \tag{4.8}$$

$$\langle \pi^a(q)|V_\nu^b(0)|\pi^c(p)\rangle = -\frac{i\epsilon^{abc}}{(2\pi)^3}\int \frac{d\sigma\sqrt{\sigma}\,F_\rho(\sigma)}{\sigma-k^2}\,\gamma_{\rho\pi\pi}(\sigma)(p_\nu+q_\nu), \tag{4.9}$$

$$M_\nu^{abc}(q,p) = \frac{i\epsilon^{abc}}{(2\pi)^3(m_\pi^2-q^2)}\int \frac{d\sigma\sqrt{\sigma}\,F_\rho(\sigma)}{\sigma-k^2}\,\gamma_{\rho\pi\pi}(\sigma)\left(-g_{\nu\beta}+\frac{k_\nu k_\beta}{\sigma}\right)(p^\beta+q^\beta), \tag{4.10}$$

$$\int \frac{d\sigma\, F_\rho(\sigma)}{\sqrt{\sigma}}\,\gamma_{\rho\pi\pi}(\sigma) = -1, \tag{4.11}$$

where

$$k_\mu = p_\mu - q_\mu, \tag{4.12}$$

and where the q^2-dependance of $\gamma_{\rho\pi\pi}(\sigma)$ has been omitted, since in later calculations only physical values (where $q^2 = m_\pi^2$) are of interest.

We will now proceed to study $\langle a^a(q,\eta,\sigma')|V_\nu^b(0)|\pi^c(p)\rangle$. Defining the a-ρ-π couplings $\gamma^S_{\rho a\pi}(\sigma,\sigma')$ and $\gamma^D_{\rho a\pi}(\sigma,\sigma')$ by

$$\langle a^a(q,\eta,\sigma')|\,\pi^c(0)\,|\,\rho^b(k',\varepsilon,\sigma)\rangle \equiv \epsilon^{abc}(2\pi)^{-3}[m_\pi^2-(p-k')^2]^{-1}\Big[\gamma^S_{\rho a\pi}(\sigma,\sigma')(\varepsilon\eta) + 2\,\gamma^D_{\rho a\pi}(\sigma,\sigma')(k'\eta)(\varepsilon q)\Big], \tag{4.13}$$

we find that

$$\langle a^a(q,\eta,\sigma')|\,V_\nu^{\,b}(0)\,|\pi^c(p)\rangle$$
$$= \int \frac{d\sigma\sqrt{\sigma}}{\sigma-k^2} F_\rho(\sigma)\left(-g_{\nu\beta}+\frac{k_\nu k_\beta}{\sigma}\right)\Big[\gamma^S_{\rho a\pi}(\sigma,\sigma')\,\eta^\beta$$
$$- \gamma^D_{\rho a\pi}(\sigma,\sigma')(k\eta)(p^\beta+q^\beta)\Big]. \qquad (4.14)$$

We have included here only graphs of the type shown in Fig. 2(a), where the strong interaction vertices are connected. Graphs like Fig. 2(b) have been omitted from this matrix element for the present, and their effects will be discussed later. Since $V_\nu^{\,b}$ is divergenceless, these couplings must satisfy

$$\int \frac{d\sigma}{\sqrt{\sigma}} F_\rho(\sigma)\Big[\gamma^S_{\rho a\pi}(\sigma,\sigma') + (\sigma'-m_\pi^2)\,\gamma^D_{\rho a\pi}(\sigma,\sigma')\Big] = 0. \qquad (4.15)$$

In like fashion, we find

$$\langle\rho^b(k,\varepsilon,\sigma)|A_\mu^{\,a}(0)|\pi^c(p)\rangle = \frac{\epsilon^{abc}}{(2\pi)^3}\Bigg\{-\frac{2F_\pi\,\gamma_{\rho\pi\pi}(\sigma)}{m_\pi^2-q^2}(\varepsilon q)q_\mu \qquad (4.16)$$
$$+\int\frac{d\sigma'\sqrt{\sigma'}}{\sigma'-q^2}F_a(\sigma')\left(-g_{\mu\alpha}+\frac{q_\mu q_\alpha}{\sigma'}\right)\Big[\gamma^S_{\rho a\pi}(\sigma,\sigma')\varepsilon^\alpha-\gamma^D_{\rho a\pi}(\sigma,\sigma')(\varepsilon q)(p^\alpha+k^\alpha)\Big]\Bigg\},$$

and

$$\langle\rho^b(k,\varepsilon,\sigma)|\,\partial^\mu A_\mu^{\,a}(0)\,|\pi^c(p)\rangle$$
$$= \frac{i\epsilon^{abc}}{(2\pi)^3}\,\frac{2m_\pi^2F_\pi}{m_\pi^2-q^2}\,\gamma_{\rho\pi\pi}(\sigma)(\varepsilon q), \qquad (4.17)$$

and, by comparing them,

$$2F_\pi \gamma_{\rho\pi\pi}(\sigma) = \int \frac{d\sigma'}{\sqrt{\sigma'}} F_a(\sigma')\left[\gamma^S_{\rho a\pi}(\sigma,\sigma') + (\sigma - m_\pi^2)\gamma^D_{\rho a\pi}(\sigma,\sigma')\right]. \quad (4.18)$$

Now we are ready to compute the following advanced commutator:

$$M_{\mu\nu}{}^{abc}(q,p) \equiv i\int d^4x\, e^{iqx}\theta(x_0)\langle 0|[A^a_\mu(x), V^b_\nu(0)]|\pi^c(p)\rangle, \quad (4.19)$$

whose absorptive part is, according to Section 2,

$$\begin{aligned}
\mathrm{Abs}\, M_{\mu\nu}{}^{abc}(q,p) &= \frac{1}{2}\int d^4x\, e^{iqx}\langle 0|[A^a_\mu(x), V^b_\nu(0)]|\pi^c(p)\rangle \\
&= \frac{1}{2}(2\pi)^4\Big[\delta(m_\pi^2 - q^2)\, iF_\pi q_\mu \langle \pi^a(q)|V^b_\nu(0)|\pi^c(p)\rangle \\
&\quad + \int d\sigma'\delta(\sigma' - q^2)\sqrt{\sigma'}F_a(\sigma')\sum_\eta \eta_\mu \langle a^a(q,\eta,\sigma')|V^b_\nu(0)|\pi^c(p)\rangle \\
&\quad - \int d\sigma\,\delta(\sigma - k^2)\sqrt{\sigma}F_\rho(\sigma)\sum_\varepsilon \varepsilon_\nu\langle \rho^b(k,\varepsilon,\sigma)|A^a_\mu(0)|\pi^c(p)\rangle \\
&= \frac{\epsilon^{abc}}{2\sqrt{2\pi}}\Big[\delta(m_\pi^2 - q^2)F_\pi q_\mu \int \frac{d\sigma\sqrt{\sigma}}{\sigma - k^2}F_\rho(\sigma)\gamma_{\rho\pi\pi}(\sigma)(p_\nu + q_\nu) \\
&\quad + \int d\sigma'\delta(\sigma' - q^2)\sqrt{\sigma'}F_a(\sigma')\left(-g_{\mu\alpha} + \frac{q_\mu q_\alpha}{\sigma'}\right)\int \frac{d\sigma\sqrt{\sigma}}{\sigma - k^2}F_\rho(\sigma)\left(-g_{\nu\beta} + \frac{k_\nu k_\beta}{\sigma}\right) \\
&\quad \times\left[\gamma^S_{\rho a\pi}(\sigma,\sigma')g^{\alpha\beta} - \gamma^D_{\rho a\pi}(\sigma,\sigma')k^\alpha(p^\beta + q^\beta)\right] \\
&\quad - \int d\sigma\,\delta(\sigma - k^2)\sqrt{\sigma}F_\rho(\sigma)\left(-g_{\nu\beta} + \frac{k_\nu k_\beta}{\sigma}\right)\Big\{-\frac{2F_\pi\gamma_{\rho\pi\pi}(\sigma)}{m_\pi^2 - q^2}q_\mu q^\beta \\
&\quad + \int \frac{d\sigma'\sqrt{\sigma'}}{\sigma' - q^2}F_a(\sigma')\left(-g_{\mu\alpha} + \frac{q_\mu q_\alpha}{\sigma'}\right)\left[\gamma^S_{\rho a\pi}(\sigma,\sigma')g^{\alpha\beta} - \gamma^D_{\rho a\pi}(\sigma,\sigma')(p^\alpha + k^\alpha)q^\beta\right]\Big\}\Big].
\end{aligned} \quad (4.20)$$

By slightly rearranging the terms, we easily check that the imaginary parts of the form factors[14] entering into $M_{\mu\nu}^{abc}(q,p)$ has the form of Equation (B.6) of Appendix B. Hence, by Appendix B,

$$M_{\mu\nu}^{abc}(q,p) = \frac{\epsilon^{abc}}{(2\pi)^{3/2}} \int \frac{d\sigma\sqrt{\sigma}}{\sigma - k^2} F_\rho(\sigma)\left(-g_{\nu\beta} + \frac{k_\nu k_\beta}{\sigma}\right)$$

$$\times \left\{ -\frac{F_\pi \gamma_{\rho\pi\pi}(\sigma)}{m_\pi^2 - q^2} q_\mu (p^\beta + q^\beta) + \int \frac{d\sigma'\sqrt{\sigma'}}{\sigma' - q^2} F_a(\sigma') \right. \tag{4.21}$$

$$\left. \times \left(-g_{\mu\alpha} + \frac{q_\mu q_\alpha}{\sigma'}\right)\left[\gamma^S_{\rho a\pi}(\sigma,\sigma') g^{\alpha\beta} - \gamma^D_{\rho a\pi}(\sigma,\sigma')(2k^\alpha q^\beta + k^\alpha k^\beta + q^\alpha q^\beta)\right]\right\},$$

and it is represented by Fig. 3. Had we included graphs like Fig. 2(b) into our amplitude, we would then find that the triangle graphs (Fig. 4) must also be included in $M_{\mu\nu}^{abc}(q,p)$. These graphs do not possess the two-pole feature[15] exhibited in (4.21), and they, among other things, contribute to the anomalous part of anomalous Ward identity.[16] Since later applications will be centered around single-particle or resonance intermediate states, these triangle graphs will not appear.

To see the context of current-algebra, in particular, of Equations (4.1) and (4.2), we find that the Ward identity is

$$iq^{\mu} M_{\mu\nu}{}^{abc}(q,p) + D_{\nu}{}^{abc}(q,p) = i(2\pi)^{-3/2} \epsilon^{abc} F_{\pi} p_{\nu} , \quad (4.22)$$

where we have used (4.7) and where

$$D_{\nu}{}^{abc}(q,p) \equiv i \int d^4x \, e^{iqx} \theta(x_0) \langle 0 | [\partial^{\mu} A_{\mu}^{a}(x), V_{\nu}^{b}(0)] | \pi^{c}(p) \rangle . \quad (4.23)$$

In the same manner, we express $D_{\nu}^{abc}(q,p)$ in terms of the various constants as

$$D_{\nu}{}^{abc}(q,p) = \frac{i\epsilon^{abc}}{(2\pi)^{3/2}} \frac{m_{\pi}^{2} F_{\pi}}{m_{\pi}^{2} - q^{2}} q_{\mu} \int \frac{d\sigma \sqrt{\sigma}}{\sigma - k^{2}} F_{\rho}(\sigma) \left(-g_{\nu\beta} + \frac{k_{\nu} k_{\beta}}{\sigma} \right) (p^{\beta} + q^{\beta}) , \quad (4.24)$$

so that the Ward identity implies

$$\int \frac{d\sigma \sqrt{\sigma}}{\sigma - k^{2}} F_{\rho}(\sigma) \left(-g_{\nu\beta} + \frac{k_{\nu} k_{\beta}}{\sigma} \right) \Big\{ F_{\pi} \gamma_{\rho\pi\pi}(\sigma) (p^{\beta} + q^{\beta})$$

$$- \int \frac{d\sigma'}{\sqrt{\sigma'}} F_{a}(\sigma') \Big[\gamma^{S}_{\rho a \pi}(\sigma,\sigma') q^{\beta} - \gamma^{D}_{\rho a \pi}(\sigma,\sigma') \big((m_{\pi}^{2} - k^{2}) q^{\beta} + (qk) k^{\beta} \big) \Big] \Big\} = F_{\pi} p_{\nu} .$$

This, with the help of (4.18), reduces to

$$\int \frac{d\sigma \, d\sigma'}{\sqrt{\sigma'}} F_{\rho}(\sigma) F_{a}(\sigma') \Big\{ -\frac{1}{2\sqrt{\sigma}} \Big[\gamma^{S}_{\rho a \pi}(\sigma,\sigma') + (\sigma - m_{\pi}^{2}) \gamma^{D}_{\rho a \pi}(\sigma,\sigma') \Big] k_{\nu}$$

$$- \sqrt{\sigma} \, \gamma^{D}_{\rho a \pi}(\sigma,\sigma') q_{\nu} \Big\} = F_{\pi} p_{\nu} ,$$

which, because of (4.15), further reduces to

$$\int \frac{d\sigma \, d\sigma'}{\sqrt{\sigma'}} F_{\rho}(\sigma) F_{a}(\sigma') \gamma^{D}_{\rho a \pi}(\sigma,\sigma') \left(\frac{\sigma + \sigma'}{2\sqrt{\sigma}} k_{\nu} - \sqrt{\sigma} \, p_{\nu} \right) = F_{\pi} p_{\nu} .$$

Therefore, we obtain the following two sum rules:

$$\int d\sigma d\sigma' \sqrt{\frac{\sigma}{\sigma'}} F_\rho(\sigma) F_a(\sigma') \gamma^D_{\rho a \pi}(\sigma,\sigma') = -F_\pi, \tag{4.25}$$

and

$$\int \frac{d\sigma d\sigma'}{\sqrt{\sigma\sigma'}} F_\rho(\sigma) F_a(\sigma') \gamma^D_{\rho a \pi}(\sigma,\sigma')(\sigma+\sigma') = 0. \tag{4.26}$$

In addition to (4.22), there is another Ward identity (remember that $\partial^\mu V^a_\mu = 0$):

$$i k^\nu M^{abc}_{\mu\nu}(q,p) = -i(2\pi)^{-3/2}\epsilon^{abc} F_\pi p_\mu, \tag{4.27}$$

but this, again with the help of (4.15) and (4.18), reduces to (4.26)

and

$$\int d\sigma d\sigma' \sqrt{\frac{\sigma'}{\sigma}} F_\rho(\sigma) F_a(\sigma') \gamma^D_{\rho a \pi}(\sigma,\sigma') = F_\pi . \tag{4.28}$$

This equation does not represent a new relation as it follows from (4.25) and (4.26).

(II) CURRENTS BETWEEN $\mathcal{A}$ AND VACUUM

We may extend the above study to the amplitude

$$N^{abc}_{\mu\nu}(q,p,\eta,\sigma') \equiv i\int d^4x\, e^{iqx}\theta(x_0)\langle 0|[A^a_\mu(x), V^b_\nu(0)]|\mathcal{A}^c(p,\eta,\sigma')\rangle \tag{4.29}$$

whose absorptive part, by the now familiar method, is

$$\begin{aligned}\mathrm{Abs}\,N^{abc}_{\mu\nu}(q,p,\eta,\sigma') &= \frac{1}{2}\int d^4x\, e^{iqx}\langle 0|[A^a_\mu(x), V^b_\nu(0)]|\mathcal{A}^c(p,\eta,\sigma')\rangle \\ &= \frac{1}{2}(2\pi)^{5/2}\Big[\delta(m_\pi^2-q^2)\,iF_\pi q_\mu\langle\pi^a(q)|V^b_\nu(0)|\mathcal{A}^c(p,\eta,\sigma')\rangle \\ &+ \int d\sigma''\,\delta(\sigma''-q^2)\sqrt{\sigma''}\,F_{\mathcal{A}}(\sigma'')\sum_\zeta \zeta_\mu\langle\mathcal{A}^a(q,\zeta,\sigma'')|V_\nu^b(0)|\mathcal{A}^c(p,\eta,\sigma')\rangle \\ &- \int d\sigma\,\delta(\sigma-k^2)\sqrt{\sigma}\,F_\rho(\sigma)\sum_\varepsilon \varepsilon_\nu\langle\rho^b(k,\varepsilon,\sigma)|A^a_\mu(0)|\mathcal{A}^c(p,\eta,\sigma')\rangle. \end{aligned} \tag{4.30}$$

We already have the expression for $\langle\pi^a(q)|V^b_\nu(0)|\mathcal{A}^c(p,\eta,\sigma')\rangle$

(Equation (4.14)).An attempt to express $\langle\mathcal{A}^a|V^b_\nu|\mathcal{A}^c\rangle$ and $\langle\rho^b|A^a_\mu|\mathcal{A}^c\rangle$ in terms of couplings runs into the problem that, for those $\mathcal{A}$ or ρ states that are not single stable particle states, it is extremely difficult, if not impossible, to write a dispersion relation for these matrix elements. The obstacle lies in the fact that these states do not have their respective local field operators. Nevertheless, let us assume that we can define "effective" couplings $\gamma_{\rho\mathcal{A}\mathcal{A}}(\sigma,\sigma';\sigma'')$ and $\gamma'_{\rho\mathcal{A}\mathcal{A}}(\sigma,\sigma';\sigma'')$ by having an "effective" strong interaction Hamiltonian:

$$\langle a^a(q,\zeta,\sigma''),\rho^b(k,\varepsilon,\sigma)|H^{eff}(0)|a^c(p,\eta,\sigma')\rangle$$
$$\equiv -\frac{i\epsilon^{abc}}{(2\pi)^{3/2}}\Big[2\gamma_{\rho aa}(\sigma,\sigma',\sigma'')(\varepsilon\eta)(k\zeta)$$
$$+\gamma'_{\rho aa}(\sigma,\sigma',\sigma'')(\zeta\eta)(\varepsilon q)\Big], \quad (4.31)$$

and evaluate the matrix elements $<a^a|V_\nu^{\ b}|a^c>$, $<\rho^b|A_\mu^{\ a}|a^c>$ and $<\rho^b|\partial^\mu A_\mu^{\ a}|a^c>$ from Fig. 5:

$$\langle a^a(q,\zeta,\sigma'')|V_\nu^{\ b}(0)|a^c(p,\eta,\sigma')\rangle$$
$$= -\frac{i\epsilon^{abc}}{(2\pi)^3}\int\frac{d\sigma\sqrt{\sigma}}{\sigma-k^2}F_\rho(\sigma)\left(-g_{\nu\beta}+\frac{k_\nu k_\beta}{\sigma}\right)$$
$$\times\Big\{2\gamma_{\rho aa}(\sigma,\sigma',\sigma'')\big[(\zeta k)\eta^\beta-(\eta k)\zeta^\beta\big] \quad (4.32)$$
$$+\gamma'_{\rho aa}(\sigma,\sigma',\sigma'')(\zeta\eta)(p^\beta+q^\beta)\Big\},$$

$$\langle \rho^b(k,\varepsilon,\sigma)|\, A_\mu^a(0)\,|a^c(p,\eta,\sigma')\rangle$$

$$= -\frac{i\epsilon^{abc}}{(2\pi)^3}\left\{\frac{F_\pi q_\mu}{m_\pi^2-q^2}\left[\gamma^S_{\rho a\pi}(\sigma,\sigma')(\varepsilon\eta)+2\gamma^D_{\rho a\pi}(\sigma,\sigma')(\eta k)(\varepsilon p)\right]\right.$$

$$+\int\frac{d\sigma''\sqrt{\sigma''}}{\sigma''-q^2}F_a(\sigma'')\left(-g_{\mu\alpha}+\frac{q_\mu q_\alpha}{\sigma''}\right) \qquad (4.33)$$

$$\left.\times\left[\gamma_{\rho aa}(\sigma,\sigma',\sigma'')((\varepsilon\eta)(p^\alpha+k^\alpha)-2(\eta k)\varepsilon^\alpha)-2\gamma'_{\rho aa}(\sigma,\sigma',\sigma'')(\varepsilon p)\eta^\alpha\right]\right\},$$

and

$$\langle \rho^b(k,\varepsilon,\sigma)|\,\partial^\mu A_\mu^a(0)\,|a^c(p,\eta,\sigma')\rangle$$

$$= -\frac{\epsilon^{abc}}{(2\pi)^3}\,\frac{m_\pi^2 F_\pi}{m_\pi^2-q^2}\left[\gamma^S_{\rho a\pi}(\sigma,\sigma')(\varepsilon\eta)+2\gamma^D_{\rho a\pi}(\sigma,\sigma')(\eta k)(\varepsilon p)\right]. \qquad (4.34)$$

Crossing symmetry on (4.31) requires $\gamma_{\rho aa}(\sigma,\sigma',\sigma'')$ and $\gamma'_{\rho aa}(\sigma,\sigma',\sigma'')$ to be symmetric under $\sigma'\longleftrightarrow\sigma''$.

Multiplying (4.32) by k^ν and (4.33) by q^μ, we obtain respectively

$$\int\frac{d\sigma}{\sqrt{\sigma}}F_\rho(\sigma)\gamma'_{\rho aa}(\sigma,\sigma',\sigma'')(\sigma'-\sigma'')=0 \qquad (4.35)$$

and the pair

$$F_\pi \gamma^S_{\rho a\pi}(\sigma,\sigma') = (\sigma-\sigma')\int \frac{d\sigma''}{\sqrt{\sigma''}} F_a(\sigma'')\gamma_{\rho a a}(\sigma,\sigma',\sigma'') \tag{4.36}$$

$$F_\pi \gamma^D_{\rho a\pi}(\sigma,\sigma') = \int \frac{d\sigma''}{\sqrt{\sigma''}} F_a(\sigma'')\left[\gamma_{\rho a a}(\sigma,\sigma',\sigma'') + \gamma'_{\rho a a}(\sigma,\sigma',\sigma'')\right]. \tag{4.37}$$

Substituting (4.14), (4.32), and (4.33) into (4.30), and employing the effective technique, we find simply

$$\begin{aligned} N^{abc}_{\mu\nu}(q,p,\eta,\sigma') = &- \frac{i\epsilon^{abc}}{(2\pi)^{3/2}} \int \frac{d\sigma\sqrt{\sigma}\,F_\rho(\sigma)}{\sigma-k^2}\left(-g_{\nu\beta}+\frac{k_\nu k_\beta}{\sigma}\right) \\ &\times\left\{ \frac{F_\pi q_\mu}{m_\pi^2-q^2}\left[\gamma^S_{\rho a\pi}(\sigma,\sigma')\eta^\beta + (\eta k)\gamma^D_{\rho a\pi}(\sigma,\sigma')(p^\beta+q^\beta)\right]\right. \\ &+ \int \frac{d\sigma''\sqrt{\sigma''}}{\sigma''-q^2} F_a(\sigma'')\left(-g_{\mu\alpha}+\frac{q_\mu q_\alpha}{\sigma''}\right)\left[\gamma_{\rho a a}(\sigma,\sigma',\sigma'')\left((p^\alpha+k^\alpha)\eta^\beta-2(\eta k)g^{\alpha\beta}\right)\right. \\ &\left.\left.+\gamma'_{\rho a a}(\sigma,\sigma',\sigma'')\eta^\alpha(p^\beta+q^\beta)\right]\right\}, \end{aligned} \tag{4.38}$$

whose representation is shown in Fig. 6(a). The closely related matrix element, Fig. 6(b), is similarly determined

$$E^{abc}_\nu(q,p,\eta,\sigma') \equiv i\int d^4x\, e^{iqx}\theta(x_0)\langle 0|\left[\partial^\mu A^a_\mu(x), V^b_\nu(0)\right]|\rho^c(p,\eta,\sigma')\rangle \tag{4.39}$$

$$= \frac{\epsilon^{abc}}{(2\pi)^{3/2}} \frac{m_\pi^2 F_\pi}{m_\pi^2 - q^2} \int \frac{d\sigma \sqrt{\sigma}}{\sigma - k^2} F_\rho(\sigma)\left(-g_{\nu\beta} + \frac{k_\nu k_\beta}{\sigma}\right)$$

$$\times \left[\gamma^{S}_{\rho a\pi}(\sigma,\sigma')\eta^\beta + (\eta k)\gamma^{D}_{\rho a\pi}(\sigma,\sigma')(p^\beta + q^\beta)\right]. \qquad (4.40)$$

The Ward identities are

$$ik^\nu N_{\mu\nu}{}^{abc}(q,p,\eta,\sigma') = -(2\pi)^{-3/2}\epsilon^{abc}\sqrt{\sigma'}F_a(\sigma')\,\varepsilon_\mu\,, \qquad (4.41)$$

and

$$iq^\mu N_{\mu\nu}{}^{abc}(q,p,\eta,\sigma') + E_\nu{}^{abc}(q,p,\eta,\sigma')$$

$$= (2\pi)^{-3/2}\epsilon^{abc}\sqrt{\sigma'}F_a(\sigma')\,\varepsilon_\nu\,, \qquad (4.42)$$

which respectively, with the help of (4.15), (4.36), and (4.37), reduce to

$$\int d\sigma d\sigma'' \sqrt{\frac{\sigma''}{\sigma}} F_\rho(\sigma) F_a(\sigma'')\gamma'_{\rho aa}(\sigma,\sigma',\sigma'') = -\sqrt{\sigma'}F_a(\sigma') \qquad (4.43)$$

and

$$\int d\sigma d\sigma'' \sqrt{\frac{\sigma}{\sigma''}} F_\rho(\sigma) F_a(\sigma'')\gamma_{\rho aa}(\sigma,\sigma',\sigma'') = \sqrt{\sigma'}F_a(\sigma'). \qquad (4.44)$$

(III) CURRENTS BETWEEN ρ AND VACUUM

For evaluating a similar matrix element for ρ-states, we already have the necessary ingredients (4.16) and (4.33). We obtain

$$P_{\mu\nu}^{abc}(q,p,\varepsilon,\sigma) \equiv i\int d^4x\, e^{iqx}\theta(x_0)\langle 0|[A_\mu^a(x), A_\nu^b(0)]|\rho^c(p,\varepsilon,\sigma)\rangle \tag{4.45}$$

$$= \frac{i\epsilon^{abc}}{(2\pi)^{3/2}}\left[\frac{F_\pi}{m_\pi^2-q^2}q_\mu\left\{\frac{2F_\pi}{m_\pi^2-k^2}\gamma_{\rho\pi\pi}(\sigma)(\varepsilon q)k_\nu\right.\right.$$

$$\left.+\int\frac{d\sigma''\sqrt{\sigma''}}{\sigma''-k^2}F_a(\sigma'')\left(-g_{\nu\beta}+\frac{k_\nu k_\beta}{\sigma''}\right)\left[\gamma^S_{\rho a\pi}(\sigma,\sigma'')\varepsilon^\beta+(\varepsilon k)\gamma^D_{\rho a\pi}(\sigma,\sigma'')(p^\beta+q^\beta)\right]\right\}$$

$$-\int\frac{d\sigma'\sqrt{\sigma'}}{\sigma'-q^2}F_a(\sigma')\left(-g_{\mu\alpha}+\frac{q_\mu q_\alpha}{\sigma'}\right)\left\{\frac{F_\pi}{m_\pi^2-k^2}k_\nu\left[\gamma^S_{\rho a\pi}(\sigma,\sigma')\varepsilon^\alpha\right.\right.$$

$$\left.+(\varepsilon q)\gamma^D_{\rho a\pi}(\sigma,\sigma')(p^\alpha+k^\alpha)\right]+\int\frac{d\sigma''\sqrt{\sigma''}}{\sigma''-k^2}\left(-g_{\nu\beta}+\frac{k_\nu k_\beta}{\sigma''}\right)F_a(\sigma'')$$

$$\left.\left.\times\left[\gamma_{\rho aa}(\sigma,\sigma',\sigma'')(-\varepsilon^\alpha(p^\beta+q^\beta)+(p^\alpha+k^\alpha)\varepsilon^\beta)-2\gamma'_{\rho aa}(\sigma,\sigma',\sigma'')(\varepsilon q)g^{\alpha\beta}\right]\right\}\right], \tag{4.46}$$

and

$$F_\nu^{abc}(q,p,\varepsilon,\sigma) \equiv i\int d^4x\, e^{iqx}\theta(x_0)\langle 0|[\partial^\mu A_\mu^a(x), A_\nu^b(0)]|\rho^c(p,\varepsilon,\sigma)\rangle \tag{4.47}$$

$$= \frac{\epsilon^{abc}}{(2\pi)^{3/2}} \frac{m_\pi^2 F_\pi}{m_\pi^2 - q^2} \left\{ \frac{2 F_\pi \gamma_{\rho\pi\pi}(\sigma)}{m_\pi^2 - k^2} (\varepsilon q) k_\nu + \int \frac{d\sigma'' \sqrt{\sigma''}}{\sigma'' - k^2} F_a(\sigma'') \right.$$

$$\left. \times \left(-g_{\nu\beta} + \frac{k_\nu k_\beta}{\sigma''}\right) \left[\gamma^S_{\rho a\pi}(\sigma,\sigma'') \varepsilon^\beta + (\varepsilon k) \gamma^D_{\rho a\pi}(\sigma,\sigma'')(p^\beta + q^\beta) \right] \right\}. \tag{4.48}$$

The Ward identity

$$i q^\mu P_{\mu\nu}^{abc}(q,p,\varepsilon,\sigma) + F_\nu^{abc}(q,p,\varepsilon,\sigma) = (2\pi)^{-3/2} \epsilon^{abc} \sqrt{\sigma} F_\rho(\sigma) \varepsilon_\nu \tag{4.49}$$

yields, through (4.18), (4.36) and (4.37),

$$\int d\sigma' d\sigma'' \sqrt{\frac{\sigma'}{\sigma''}} F_a(\sigma') F_a(\sigma'') \gamma_{\rho aa}(\sigma,\sigma';\sigma'') = \sqrt{\sigma} F_\rho(\sigma). \tag{4.50}$$

The other Ward identity can be shown to be equivalent to (4.49) because of the following property:

$$P_{\mu\nu}^{abc}(q,p,\varepsilon,\sigma) = P_{\nu\mu}^{bac}(k,p,\varepsilon,\sigma). \tag{4.51}$$

(IV) DISCUSSION OF SUM RULES

Up to here, we have obtained a number of sum rules from current-algebra and Ward identities. We will now study their significance. From (4.28), and (4.37),

$$\int d\sigma d\sigma' d\sigma'' \sqrt{\frac{\sigma'}{\sigma\sigma''}} F_\rho(\sigma) F_a(\sigma') F_a(\sigma'') \left[\gamma_{\rho a a}(\sigma,\sigma';\sigma'') + \gamma'_{\rho a a}(\sigma,\sigma';\sigma'')\right] = F_\pi^2, \quad (4.52)$$

which enables us to obtain Weinberg's[4] first sum rule from (4.43) and (4.50):

$$\int d\sigma \left[F_\rho^2(\sigma) - F_a^2(\sigma) \right] = F_\pi^2 . \quad (4.53)$$

Weinberg's second sum rule also follows, but from (4.44) and (4.50):

$$\int d\sigma \, \sigma \left[F_\rho^2(\sigma) - F_a^2(\sigma) \right] = 0. \quad (4.54)$$

From (4.15), (4.18), (4.25), (4.28) we also derive that

$$\int \frac{d\sigma}{\sqrt{\sigma}} F_\rho(\sigma) \gamma_{\rho\pi\pi}(\sigma) = -1 . \quad (4.11)$$

This, of course, had been obtained earlier from another consideration, but it is comforting to note that our method of dispersion relation shows some internal consistency.

In current-algebra calculations, it is a common practice to assume a single ρ-state to saturate the vector current, and a single A_1-state plus the pion-state to saturate the axial-vector current. This assumption combined with our sum rules also leads to interesting results. For example, (4.43) reads

$$\gamma'_{\rho a a} = -m_\rho / F_\rho \ , \tag{4.55}$$

which except for a factor of -2 is exactly the same obtained by Schnitzer and Weinberg,[5] while (4.44) and (4.50) read

$$\gamma_{\rho a a} = \frac{m_{A1}^2}{m_\rho F_\rho} \ , \tag{4.56}$$

$$\gamma_{\rho a a} = \frac{m_\rho F_\rho}{F_{A1}^2} \ . \tag{4.57}$$

The last two equations are mutually dependent by virtue of Weinberg's second sum rule (4.54), and this expression of $\gamma_{\rho a a}$ cannot be determined by the method of Schnitzer and Weinberg.[5] One may then proceed to employ this single-pole dominance assumption and work with other sum rules such as (4.15), (4.18), (4.25), (4.28), (4.36) and (4.37), and then finds that, under this assumption, (4.25) and (4.28) lead to the following absurdity: $m_{A1}^2 = -m_\rho^2$. Faced with this difficulty, the next best assumption one may hope for is that either the vector current or the axial-vector current is single-pole dominated, but not both. In the following, we will show that even this is too optimistic.

(V) ρ AND a SPECTRA

(4.15) can be cast as a sum rule for $\gamma_{\rho aa}$ and $\gamma'_{\rho aa}$ through relations (4.36) and (4.37):

$$\int \frac{d\sigma\, d\sigma'}{\sqrt{\sigma\sigma'}} F_\rho(\sigma) F_a(\sigma') \left[(\sigma - m_\pi^2) \gamma_{\rho aa}(\sigma, \sigma', \sigma'') + (\sigma'' - m_\pi^2) \gamma'_{\rho aa}(\sigma, \sigma', \sigma'') \right] = 0, \tag{4.58}$$

which, because of (4.43) and (4.44), can further be integrated to yield

$$\int \frac{d\sigma\, d\sigma'\, d\sigma''}{\sqrt{\sigma\sigma'\sigma''}} F_\rho(\sigma) F_a(\sigma') F_a(\sigma'') \left[\gamma_{\rho aa}(\sigma, \sigma', \sigma'') + \gamma'_{\rho aa}(\sigma, \sigma', \sigma'') \right] = 0. \tag{4.59}$$

At the same time (4.25) and (4.37) imply

$$\int d\sigma\, d\sigma'\, d\sigma'' \sqrt{\frac{\sigma}{\sigma'\sigma''}}\, F_\rho(\sigma) F_a(\sigma') F_a(\sigma'') \times \left[\gamma_{\rho aa}(\sigma, \sigma', \sigma'') + \gamma'_{\rho aa}(\sigma, \sigma', \sigma'') \right] = -F_\pi^2 . \tag{4.60}$$

Now let us assume that the <u>vector current</u> is single-pole dominated and observe that the last two equations, which now read

$$\int \frac{d\sigma'\, d\sigma''}{\sqrt{\sigma'\sigma''}} F_a(\sigma') F_a(\sigma'') \left[\gamma_{\rho aa}(\sigma'; \sigma'') + \gamma'_{\rho aa}(\sigma'; \sigma'') \right] = 0$$

and

$$m_\rho F_\rho \int \frac{d\sigma'\, d\sigma''}{\sqrt{\sigma'\sigma''}} F_a(\sigma') F_a(\sigma'') \left[\gamma_{\rho aa}(\sigma', \sigma'') + \gamma'_{\rho aa}(\sigma', \sigma'') \right] = -F_\pi^2 ,$$

require the unphysical result that $F_\pi^2 = 0$. Thus this assumption is unattractive.

Under the alternate assumption of single-pole dominance for the axial-vector current, (4.52) and (4.59) now read respectively

$$F_A^2 \int \frac{d\sigma}{\sqrt{\sigma}} F_\rho(\sigma)\left[\gamma_{\rho a a}(\sigma) + \gamma'_{\rho a a}(\sigma)\right] = F_\pi^{\,2}$$

and

$$\int \frac{d\sigma}{\sqrt{\sigma}} F_\rho(\sigma)\left[\gamma_{\rho a a}(\sigma) + \gamma'_{\rho a a}(\sigma)\right] = 0\,;$$

these two equations also require the unphysical result that $F_\pi^2 = 0$, and likewise this assumption is also unphysical.

5 κ-K^*-K_A-K SYSTEM AND THE $K_{\ell 3}$ FORM FACTORS

Extending the considerations of the previous arguments to SU(3)XSU(3) current-algebra involving strangeness-changing currents, one obtains, as before, essentially the same conclusions with the exception that the analogue of (4.11), due to non-conservation of strangeness-changing vector current, must be modified. We must then consider scalar states of strangeness one and isospin half, and shall call them $|\varkappa_s(p,\mu)\rangle$ with isospinor index s, momentum p, and mass $\sqrt{\mu}$. Its spin-one "brother" will be donated by $|K_s^*(p,\varepsilon,\mu)\rangle$, where ε is its polarization vector, and the parity partner of this will be denoted by $|K^A{}_s(p,\varepsilon,\sigma)\rangle$. The coupling of κ-states with others will be defined as follows:

$$\langle K^t(q)|\pi^i(0)|\varkappa_s(p,\mu)\rangle = i(2\pi)^{-3}\frac{1}{2}(\sigma_i)_s{}^t\frac{\gamma_{\varkappa K\pi}(\mu)}{m_\pi{}^2-(p-q)^2}, \tag{5.1}$$

$$\langle K^{At}(q,\varepsilon,\sigma)|\pi^i(0)|\varkappa_s(p,\mu)\rangle = (2\pi)^{-3}\frac{1}{2}(\sigma_i)_s{}^t\frac{\gamma_{\varkappa K^A\pi}(\mu,\sigma)2(\varepsilon p)}{m_\pi{}^2-(p-q)^2}, \tag{5.2}$$

$$\langle a^i(q,\varepsilon,\sigma)|K^t(0)|\varkappa_s(p,\mu)\rangle = (2\pi)^{-3}\frac{1}{2}(\sigma_i)_s{}^t\frac{\gamma_{\varkappa aK}(\mu,\sigma)2(\varepsilon p)}{m_K{}^2-(p-q)^2}, \tag{5.3}$$

while $\gamma_{K^*K\pi}(\mu)$ is defined analogously to $\gamma_{\rho\pi\pi}(\mu)$, whereas $\gamma^{S,D}_{K^*aK}(\mu,\sigma)$ and $\gamma^{S,D}_{K^*K^A\pi}(\mu,\sigma)$ are defined analogously to $\gamma^{S,D}_{\rho a\pi}(\mu,\sigma)$. We also define $F_\kappa(\mu)$, $F_{K^*}(\mu)$ and $F_{K^A}(\sigma)$ as the coupling of the respective states with their respective currents.

The entire program in Section 4 can now be repeated with these entities, and we will not display it in detail. However we will only note the following relevant relations:

$$\langle K^t(q)|V_\nu^{\bar{K}_s}(0)|\pi^i(p)\rangle = -(2\pi)^{-3}\frac{1}{2}(\sigma_i)_s{}^t$$
$$\times\left\{k_\nu\int\frac{d\mu\,F_\kappa(\mu)}{\mu-k^2}\gamma_{\kappa K\pi}(\mu)+\int\frac{d\mu\sqrt{\mu}}{\mu-k^2}F_{K^*}(\mu)\gamma_{K^*K\pi}(\mu)\left(-g_{\nu\beta}+\frac{k_\nu k_\beta}{\mu}\right)(p^\beta+q^\beta)\right\}, \tag{5.4}$$

$$F_\pi\,\gamma_{\kappa K\pi}(\mu) = (\mu-m_K{}^2)\int\frac{d\sigma}{\sqrt{\sigma}}F_a(\sigma)\,\gamma_{\kappa aK}(\mu,\sigma), \tag{5.5}$$

$$\int d\mu\,F_\kappa(\mu)\,\gamma_{\kappa K\pi}(\mu) = (m_K{}^2-m_\pi{}^2)\int\frac{d\mu}{\sqrt{\mu}}F_{K^*}(\mu)\,\gamma_{K^*K\pi}(\mu), \tag{5.6}$$

$$2F_\pi\gamma_{K^*K\pi}(\mu) = -\int\frac{d\sigma}{\sqrt{\sigma}}F_a(\sigma)\left[\gamma^S_{K^*aK}(\mu,\sigma)+(\mu-m_{K^*}{}^2)\,\gamma^D_{K^*aK}(\mu,\sigma)\right], \tag{5.7}$$

$$2\int d\mu\,F_\kappa(\mu)\gamma_{\kappa K^A\pi}(\mu,\sigma) = \int\frac{d\mu}{\sqrt{\mu}}F_{K^*}(\mu)\left[\gamma^S_{K^*K^A\pi}(\mu,\sigma)+(\sigma-m_\pi{}^2)\gamma^D_{K^*K^A\pi}(\mu,\sigma)\right], \tag{5.8}$$

$$2\int d\mu\,F_\kappa(\mu)\gamma_{\kappa aK}(\mu,\sigma) = \int\frac{d\mu}{\sqrt{\mu}}F_{K^*}(\mu)\left[\gamma^S_{K^*aK}(\mu,\sigma)+(\sigma-m_K{}^2)\gamma^D_{K^*aK}(\mu,\sigma)\right], \tag{5.9}$$

$$F_K \gamma_{\pi K\pi}(\mu) = (\mu - m_\pi^2)\int \frac{d\sigma}{\sqrt{\sigma}} F_{K^A}(\sigma)\gamma_{\pi K^A\pi}(\mu,\sigma), \tag{5.10}$$

$$2F_K \gamma_{K^*K\pi}(\mu) = \int \frac{d\sigma}{\sqrt{\sigma}} F_{K^A}(\sigma)\left[\gamma^S_{K^*K^A\pi}(\mu,\sigma) + (\mu - m_\pi^2)\gamma^D_{K^*K^A\pi}(\mu,\sigma)\right], \tag{5.11}$$

$$\begin{aligned}
& i\int d^4x\, e^{iqx}\theta(x_0)\langle 0|[A_\mu^i(x), V_\nu^{K^+}(0)]|K_s(p)\rangle \\
&= -\frac{i}{2(2\pi)^{3/2}}(\sigma_i)_s{}^t\Bigg[\frac{F_\pi}{m_\pi^2 - q^2}q_\mu\Bigg\{-k_\nu\int\frac{d\mu}{\mu - k^2}F_\pi(\mu)\gamma_{\pi K\pi}(\mu) \\
&\quad + \int\frac{d\mu\sqrt{\mu}}{\mu - k^2}F_{K^*}(\mu)\gamma_{K^*K\pi}(\mu)\left(-g_{\nu\beta} + \frac{k_\nu k_\beta}{\mu}\right)(p^\beta + q^\beta)\Bigg\} \\
&\quad + \int\frac{d\sigma\sqrt{\sigma}}{\sigma - q^2}F_a(\sigma)\left(-g_{\mu\alpha} + \frac{q_\mu q_\alpha}{\sigma}\right)\Bigg\{-k_\nu\int\frac{d\mu}{\mu - k^2}F_\pi(\mu)\gamma_{\pi aK}(\mu,\sigma)(p^\alpha + k^\alpha) \\
&\quad + \int\frac{d\mu\sqrt{\mu}}{\mu - k^2}F_{K^*}(\mu)\left(-g_{\nu\beta} + \frac{k_\nu k_\beta}{\mu}\right)\Big[\gamma^S_{K^*aK}(\mu,\sigma)g^{\alpha\beta} \\
&\quad - \gamma^D_{K^*aK}(\mu,\sigma)(2k^\alpha q^\beta + k^\alpha k^\beta + q^\alpha q^\beta)\Big]\Bigg\}\Bigg],
\end{aligned} \tag{5.12}$$

and

$$i\int d^4x\, e^{iqx}\,\theta(x_0)\langle 0|[A_\mu^{Kt}(x), V_\nu^{\bar{K}_s}(0)]|\pi^i(p)\rangle$$
$$= \frac{i}{2(2\pi)^{3/2}}(\sigma_i)_s{}^t\Big[\frac{F_K}{m_K{}^2-q^2}q_\mu\Big\{-k_\nu\int\frac{d\mu}{\mu-k^2}F_\kappa(\mu)\gamma_{\kappa K\pi}(\mu)$$
$$-\int\frac{d\mu\sqrt{\mu}}{\mu-k^2}F_{K^*}(\mu)\gamma_{K^*K\pi}(\mu)\Big(-g_{\nu\beta}+\frac{k_\nu k_\beta}{\mu}\Big)(p^\beta+q^\beta)\Big\}$$
$$+\int\frac{d\sigma\sqrt{\sigma}}{\sigma-q^2}F_{K^A}(\sigma)\Big(-g_{\mu\alpha}+\frac{q_\mu q_\alpha}{\sigma}\Big)\Big\{-k_\nu\int\frac{d\mu}{\mu-k^2}F_\kappa(\mu)\gamma_{\kappa K^A\pi}(\mu,\sigma)(p^\alpha+k^\alpha)$$
$$+\int\frac{d\mu\sqrt{\mu}}{\mu-k^2}F_{K^*}(\mu)\Big(-g_{\nu\beta}+\frac{k_\nu k_\beta}{\mu}\Big)\Big[\gamma^S_{K^*K^A\pi}(\mu,\sigma)g^{\alpha\beta}$$
$$-\gamma^D_{K^*K^A\pi}(\mu,\sigma)(2k^\alpha q^\beta+k^\alpha k^\beta+q^\alpha q^\beta)\Big]\Big\}\Big]. \tag{5.13}$$

The current commutators required here are

$$\delta(x_0)[A_0^{Kt}(x), V_\mu^{\bar{K}_s}(0)] = -\tfrac{1}{2}(\sigma_i)_s{}^t A_\mu{}^i(0)\delta^4(x), \tag{5.14}$$

$$\delta(x_0)[A_\mu{}^{Kt}(x), V_0^{\bar{K}_s}(0)] = -\tfrac{1}{2}(\sigma_i)_s{}^t A_\mu{}^i(0)\delta^4(x), \tag{5.15}$$

$$\delta(x_0)[A_0{}^i(x), V_\mu^{Kt}(0)] = -\tfrac{1}{2}(\sigma_i)_s{}^t A_\mu{}^{Ks}(0)\delta^4(x), \tag{5.16}$$

and

$$\delta(x_0)[A_\mu^{\,i}(x), V_0^{\,K^t}(0)] = -\frac{1}{2}(\sigma_i)_s^{\ t}\, V_\mu^{K^s}(0)\delta^4(x). \qquad (5.17)$$

Then the Ward identities relevant to (5.12) yield the sum rules

$$\int d\mu\, d\sigma \sqrt{\frac{\sigma}{\mu}}\, F_{K*}(\mu) F_a(\sigma)\, \gamma^D_{K^* a K}(\mu,\sigma) = -F_K\,, \qquad (5.18)$$

and

$$\int d\mu\, d\sigma \sqrt{\frac{\mu}{\sigma}}\, F_{K*}(\mu) F_a(\sigma)\, \gamma^D_{K^* a K}(\mu,\sigma) = F_K\,, \qquad (5.19)$$

while those relevant to (5.13) yield

$$\int d\mu\, d\sigma \sqrt{\frac{\sigma}{\mu}}\, F_{K*}(\mu) F_{KA}(\sigma)\, \gamma^D_{K^* K^A \pi}(\mu,\sigma) = F_\pi\,, \qquad (5.20)$$

and

$$\int d\mu\, d\sigma \sqrt{\frac{\mu}{\sigma}}\, F_{K*}(\mu) F_{KA}(\sigma)\, \gamma^D_{K^* K^A \pi}(\mu,\sigma) = -F_\pi\,. \qquad (5.21)$$

Considering equations (5.5), (5.7), and (5.9), the first pair of sum rules ((5.18) and (5.19)) shows

$$\int \frac{d\mu}{\sqrt{\mu}}\, F_{K*}(\mu)\, \gamma_{K^* K \pi}(\mu) = -\frac{F_K}{F_\pi} - \int \frac{d\mu}{\mu - m_K^2}\, F_\kappa(\mu)\, \gamma_{\kappa K \pi}(\mu)\,, \qquad (5.22)$$

but the second pair of sum rules((5.20) and (5.21)),when combined with (5.8), (5.10) and (5.11), shows

$$\int \frac{d\mu}{\sqrt{\mu}}\, F_{K*}(\mu)\, \gamma_{K^* K \pi}(\mu) = -\frac{F_\pi}{F_K} + \int \frac{d\mu}{\mu - m_\pi^2}\, F_\kappa(\mu)\, \gamma_{\kappa K \pi}(\mu)\,. \qquad (5.23)$$

If we express (5.4) in terms of $K_{\ell 3}$-decay form factors $f_\pm$, defined by

$$\langle \bar{K}_s(q) | V_\nu^{K^t}(0) | \pi^i(p) \rangle$$
$$= -(2\pi)^{-3} \frac{1}{2} (\sigma_i)_s{}^t \left[f_+(k^2)(p_\nu + q_\nu) + f_-(k^2)(q_\nu - p_\nu) \right], \quad (5.24)$$

we find that

$$f_+(k^2) = -\int \frac{d\mu \sqrt{\mu}\, F_{K^*}(\mu)}{\mu - k^2} \gamma_{K^* K\pi}(\mu), \quad (5.25)$$

$$f_-(k^2) = -\int \frac{d\mu\, F_\varkappa(\mu)}{\mu - k^2} \gamma_{\varkappa K\pi}(\mu)$$
$$- (m_\pi^2 - m_K^2) \int \frac{d\mu}{\sqrt{\mu}} \frac{F_{K^*}(\mu) \gamma_{K^* K\pi}(\mu)}{\mu - k^2}. \quad (5.26)$$

Earlier we have been discouraged by assuming single-pole dominance of the spin-one part of vector and axial-vector currents. Now we will see that assuming single-pole dominance of the spin-zero part of vector current is not bad at all, for then (5.22) and (5.23) respectively read

$$f_+(0) = \frac{F_K}{F_\pi} + \frac{F_\varkappa \gamma_{\varkappa K\pi}}{m_\varkappa^2 - m_K^2}, \quad (5.27)$$

and

$$f_+(0) = \frac{F_\pi}{F_K} - \frac{F_\varkappa \gamma_{\varkappa K\pi}}{m_\varkappa^2 - m_\pi^2}. \quad (5.28)$$

From these two relations we see that (because of (5.6) and (5.25))

$$f_+(0) = \pm 1, \tag{5.29}$$

and

$$m_\kappa^{\ 2} = \frac{m_K^{\ 2} F_K \mp m_\pi^2 F_\pi}{F_K \mp F_\pi}, \tag{5.30}$$

where the upper and lower signs give two different solutions. With $|F_\pi| = 131$ MeV, $|F_K| = 149$ MeV, $m_\pi = 137$ MeV, and $M_K = 495$ MeV, the last equation determines the mass of the κ to be

$$m_\kappa \simeq \begin{cases} 1380 \text{ MeV} \\ 370 \text{ MeV}, \end{cases}$$

for the upper and lower signs respectively.

The lower sign thus leads to a κ with mass lower than the K-π threshold and we definitely do not observe a stable strange scalar at that mass. Therefore let us discard the lower sign. The upper sign certainly does not fare much better, for, though m_κ is above the K-π threshold, no κ resonance has been established in the region around 1380 MeV. However, we may regard this as an effective parameterization, and so, with the upper sign we have

$$f_+(0) = 1. \tag{5.31}$$

Thus this approximation surprisingly resembles the Ademollo-Gatto theorem[17] which states that $f_+(0)$ equals to 1 to first order of symmetry-breaking. By virtue of Cabibbo's[18] theory of weak interactions, experimental data on kaon decays yields[19]

$$|f_+(0)F_\pi/F_K| = 1/(1{\cdot}28 \pm 0{\cdot}06)\,.$$

With the above values of F_π and F_K one obtains $f_+(0) \simeq 0.95$, in reasonable agreement with (5.31). Like all single-pole dominance models, it necessarily predicts a very small but negative value of $f_-(0)$:

$$f_-(0) \simeq -0{\cdot}17.$$ [20]

Experimentally, the status of the value of $f_-(0)$ is confusing, but, overall, it seems to favour the value $\simeq -1$.[21]

Soft meson calculations on $\langle\pi|V|K\rangle$ with π soft has been found to differ appreciatively from that with K soft. Some authors maintain that extrapolation of the K-mass to zero is not as good as extrapolation of the π-mass to zero. With the above sum rules, one observes that these two different extrapolations are consistent, provided that the off-mass-shell couplings $\gamma_{K^*K\pi}$ and $\gamma_{\kappa K\pi}$ are independent of the square of the off-mass-shell momentum. For under such an assumption, the current matrix element with pion off-mass-shell,

$$R_{\mu\ ;s}^{a\ \ t}(q,p)$$
$$\equiv (2\pi)^{-3/2} i(m_\pi^2 - q^2)\int d^4x\, e^{iqx}\theta(x_0)\langle 0|[\pi^a(x), V_\mu^{\bar{K}^t}(0)]|K_s(p)\rangle, \tag{5.32}$$

and that with K off-mass-shell,

$$\bar{R}_{\mu\ ;s}^{a\ \ t}(q,p)$$
$$\equiv (2\pi)^{-3/2} i(m_K^2 - p^2)\int d^4x\, e^{-ipx}\theta(-x_0)\langle \pi^a(q)|[V_\mu^{K^t}(0), K_s(x)]|0\rangle, \tag{5.33}$$

are both given by

$$R_{\mu\ ;s}^{a\ \ t}(q,p) = \bar{R}_{\mu\ ;s}^{a\ \ t}(q,p)$$
$$= -\frac{1}{2(2\pi)^3}(\sigma_a)_s{}^t\left\{\int \frac{d\mu\sqrt{\mu}}{\mu - k^2} F_{K^*}(\mu)\gamma_{K^*K\pi}(\mu)\left(-g_{\mu\alpha} + \frac{k_\mu k_\alpha}{\mu}\right)(p^\alpha + q^\alpha)\right.$$
$$\left. - k_\mu \int \frac{d\mu}{\mu - k^2} F_\kappa(\mu)\gamma_{\kappa K\pi}(\mu)\right\}. \tag{5.34}$$

Now soft-π and soft-K calculations based on the hypothesis of partial conservation of axial-vector current (PCAC) require respectively

$$R_{\mu\ ;s}^{a\ \ t}(0,p) = -(2\pi)^{-3}\frac{1}{2}(\sigma_a)_s{}^t(F_K/F_\pi)\,p_\mu{}^{22} \tag{5.35}$$

and

$$\bar{R}_{\mu\ ;s}^{a\ \ t}(q,0) = -(2\pi)^{-3}\frac{1}{2}(\sigma_a)_s{}^t(F_\pi/F_K)\,q_\mu. \tag{5.36}$$

With help from (5.34) they state that

$$-\int \frac{d\mu}{\sqrt{\mu}} F_{K^*}(\mu)\gamma_{K^*K\pi}(\mu) - \int \frac{d\mu\, F_\chi(\mu)}{\mu - m_K^2} \gamma_{\chi K\pi}(\mu) = \frac{F_K}{F_\pi} , \tag{5.37}$$

and

$$-\int \frac{d\mu}{\sqrt{\mu}} F_{K^*}(\mu)\gamma_{K^*K\pi}(\mu) + \int \frac{d\mu\, F_\chi(\mu)}{\mu - m_\pi^2} \gamma_{\chi K\pi}(\mu) = \frac{F_\pi}{F_K} . \tag{5.38}$$

These are nothing but equations (5.22) and (5.23) derived earlier from dispersion relations and current commutators. We therefore conclude that the hypothesis of PCAC is equivalent to the assumption that (a) the pseudoscalar mesons dominate the divergences of the axial-vector currents and (b) the couplings of off-mass-shell mesons are independent of the square of the off-mass-shell meson momentum.

6. CONCLUSION

We have thus shown that analyticity in q_0 necessarily leads to amplitudes of two-pole form in cases where the retarded or advanced commutators are sandwiched between the vacuum and physical states of spins zero or one, and we have concluded that the hypothesis of single-pole dominance for spin-one parts of vector or axial-vector currents, although enjoys quite impressive successes, is inconsistent with current-algebra itself. If we now try to remedy the situation and assume two poles for vector currents and two for axial-vector currents, then there are more parameters associated with these poles than our sum rules can determine. This means that we have a large amount of freedom in which we can postulate models about these poles. In this respect, considerable interest may be found in the dual resonance model,[23] in which states of spins zero and one are populous, and which has already shown some surprises with current algebra.[24]

In our considerations, Weinberg's two sum rules[4] follow automatically. Perhaps this is not surprising because in his related work with

Schnitzer,[5] the two-pole form was extensively assumed. However, the celebrated relation of Kawarabayashi, Suzuki, Riazuddin and Fayyazuddin[2] (KSRF) does not emerge **from** our approach, and therefore the mass formula $2m_\rho^2 = m_{A1}^2$, which was obtained by Weinberg[4] when he combined the the above relation with his two sum rules, cannot be obtained here. The answer to this can be found in Appendix D, which proves the KSRF relation under the assumptions that (i) there are no low-lying A_1-like states to dominate the axial-vector current, and (ii) the effects of the continuum of A_1-like states can be represented in the amplitude $\langle \pi | A_\mu | \rho(\varepsilon) \rangle$ (where ε is the polarization) as the subtraction constant in the once-subtracted dispersion relation for the form factor that multiplies into ε_μ . Since these assumptions are incompatible with those which we have worked with, we cannot possibly combine the KSRF relation with Weinberg's sum rules. Actually this contradiction may be more comforting now than it would have been in the past, as the experimental status of the A_1 (originally thought to have a mass of 1080 MeV, and so satisfy the relation $m_{A1}^2 = 2\, m_\rho^2$ excellently) has eroded somewhat.[25]

Finally we may add that, although our prediction, that $f_+(0) = 1$ for $K_{\ell 3}$ decay, is not bad compared to experiment, we must view the underlying assumption with skepticism, for it also predicts a K-π s-wave enhancement at about 1380 MeV. Experimentally, the situation is far from clear, and no s-wave resonance has ever been definitely established, although there are some suspicious resonances[25] reported at 1080 MeV, 1110 MeV, 1160 MeV, and 1260 MeV.

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APPENDIX A

Consider the following commutator amplitude:

$$\text{Abs} \equiv \frac{1}{2}\int d^4x\, e^{iqx} \langle 0 | [\phi(x), \varphi(0)] | \beta(p)\rangle, \tag{A.1}$$

where $|\beta(p)\rangle$ is a single particle state of momentum p_μ. Introducing a complete set of states $|n(s)\rangle$ into the commutator, the first term is

$$\frac{1}{2}(2\pi)^4 \sum_n \int d^4s\, \delta(s^2 - m_n^2)\theta(s_0)\delta^4(s-q)\langle 0|\phi(0)|n(s)\rangle\langle n(s)|\varphi(0)|\beta(p)\rangle, \tag{A.2}$$

where m_n is the mass of $|n(s)\rangle$.

If $|n(s)\rangle$ is a state of the form

$$|n(s)\rangle = |n'(s'), \beta(p)\rangle \tag{A.3}$$

(where s = s'+p), then the last factor in (A.2) can be split into connected and disconnected parts (Figs. 7 (a) and (b) respectively):

$$\begin{aligned} &\langle n(s)|\varphi(0)|\beta(p)\rangle \\ &= \langle n(s)|\varphi(0)|\beta(p)\rangle_{conn} + \langle n'(s'), \beta(p)|\varphi(0)|\beta(p)\rangle_{disc} \\ &= \langle n(s)|\varphi(0)|\beta(p)\rangle_{conn} + \langle n'(s')|\varphi(0)|0\rangle \end{aligned} \tag{A.4}$$

If $\langle n(s)|\varphi(0)|\beta(p)\rangle$ cannot be split this way, we say its disconnected part is zero. Replacing the dummy variable n' and s' in the disconnected part by n and s, and integrating over d^4s, the first term (A.2) takes the simple form:

$$\frac{1}{4}(2\pi)^4\sum_n\Big[\langle 0|\phi(0)|n(q)\rangle\langle n(q)|\varphi(0)|\beta(p)\rangle_{conn}\,\delta(q_0-\sqrt{m_n^2+\mathbf{q}^2})\Big/\sqrt{m_n^2+\mathbf{q}^2}$$

$$+\langle n(-k)|\varphi(0)|0\rangle\langle 0|\phi(0)|n(-k),\beta(p)\rangle_{conn}\,\delta(k_0+\sqrt{m_n^2+\mathbf{k}^2})\Big/\sqrt{m_n^2+\mathbf{k}^2}\Big].$$

These are represented by Figs.8(a) and (d) respectively. Treating the second term similarly, we find it can be represented by Figs.8(b) and (c). Therefore

$$\mathrm{Abs}=\frac{1}{4}(2\pi)^4\sum_n\Big[\langle 0|\phi(0)|n(q)\rangle\langle n(q)|\varphi(0)|\beta(p)\rangle\,\delta(q_0-\sqrt{m_n^2+\mathbf{q}^2})\Big/\sqrt{m_n^2+\mathbf{q}^2}$$

$$-\langle n(-q)|\phi(0)|0\rangle\langle 0|\varphi(0)|n(-q),\beta(p)\rangle\,\delta(q_0+\sqrt{m_n^2+\mathbf{q}^2})\Big/\sqrt{m_n^2+\mathbf{q}^2}$$

$$-\langle 0|\varphi(0)|n(k)\rangle\langle n(k)|\phi(0)|\beta(p)\rangle\,\delta(k_0-\sqrt{m_n^2+\mathbf{k}^2})\Big/\sqrt{m_n^2+\mathbf{k}^2}$$

$$+\langle n(-k)|\varphi(0)|0\rangle\langle 0|\phi(0)|n(-k),\beta(p)\rangle\,\delta(k_0+\sqrt{m_n^2+\mathbf{k}^2})\Big/\sqrt{m_n^2+\mathbf{k}^2}\Big],$$

where we have dropped the subscript "conn" and understand from now on that all matrix elements refer to connected parts only.

APPENDIX B

Consider the following function as an analytic function of q_o:

$$F(q) = \frac{(p-q)^2}{m^2-q^2}, \tag{B.1}$$

where q and p are 4-vectors. The absorptive, or imaginary, part of F(q) is

$$\operatorname{Im} F(q) = \frac{1}{2i}\left[F(q_0+i0,\underset{\sim}{q}) - F(q_0-i0,\underset{\sim}{q})\right]$$

$$= \pi\left\{\left[(p_0-a)^2-(\underset{\sim}{p}-\underset{\sim}{q})^2\right]\delta(a-q_0) - \left[(p_0+a)^2-(\underset{\sim}{p}-\underset{\sim}{q})^2\right]\delta(a+q_0)\right\}/2a, \tag{B.2}$$

where $a = \sqrt{m^2+\underset{\sim}{q}^2}$.

An once-subtracted dispersion integral gives

$$K + \frac{q_0}{\pi}\int \frac{dq_0'\, \operatorname{Im} F(q_0',\underset{\sim}{q})}{q_0'(q_0'-q_0)}$$

$$= \left\{q_0^2 - 2q_0p_0 + \left[p_0^2-(\underset{\sim}{p}-\underset{\sim}{q})^2+a^2K\right]q_0^2/a^2 + a^2K\right\}/(m^2-q^2), \tag{B.3}$$

where K, the subtraction constant, is determined by Lorentz invariance of F(q) as follows. Since the denominator is Lorentz invariant, so must be the numerator. However, the leading term in momentum in the numerator is $[p_0^2 - (\underset{\sim}{p} - \underset{\sim}{q})^2 + a^2 K]\, q_0^2/a^2$ which cannot be Lorentz invariant. So this term must be zero:

$$p_0^2 - (\underset{\sim}{p} - \underset{\sim}{q})^2 + a^2 K = 0.$$

(B.3), therefore, gives

$$K + \frac{q_0}{\pi}\int \frac{dq_0' \, \mathrm{Im}\, F(q_0', \underset{\sim}{q})}{q_0'(q_0' - q)} = \frac{(p-q)^2}{m^2 - q^2},$$

which is just F(q).

Realizing that ImF(q) has the alternate form

$$\frac{1}{\pi}\mathrm{Im}\, F(q) = (p-q)^2\left[\frac{\delta(q_0 - \sqrt{m^2 + \underset{\sim}{q}^2})}{\sqrt{m^2 + \underset{\sim}{q}^2} + q_0} - \frac{\delta(q_0 + \sqrt{m^2 + \underset{\sim}{q}^2})}{\sqrt{m^2 + \underset{\sim}{q}^2} - q_0}\right],$$

every term of which is Lorentz invariant except for a denominator and a delta function, we could have obtained F(q) from it by inspection, if we had observed that the imaginary part of $(m^2 - q^2)^{-1}$ is just

$$\frac{\delta(q_0 - \sqrt{m^2 + \underset{\sim}{q}^2})}{\sqrt{m^2 + \underset{\sim}{q}^2} + q_0} - \frac{\delta(q_0 + \sqrt{m^2 + \underset{\sim}{q}^2})}{\sqrt{m^2 + \underset{\sim}{q}^2} - q_0}.$$

We can now set up the general rule that: for any function G(q), whose imaginary part along a cut on the real q_0-axis is given by

$$\frac{1}{\pi} \mathrm{Im}\, G(q) = P(q^2, (p-q)^2)$$

$$\times \left\{ \frac{1}{n^2-(p-q)^2} \left[\frac{\delta(q_0 - \sqrt{m^2+\underset{\sim}{q}^2})}{\sqrt{m^2+\underset{\sim}{q}^2}+q_0} - \frac{\delta(q_0 + \sqrt{m^2+\underset{\sim}{q}^2})}{\sqrt{m^2+\underset{\sim}{q}^2}-q_0} \right] \right.$$

$$\left. + \frac{1}{m^2-q^2} \left[\frac{\delta(p_0 - q_0 + \sqrt{n^2+(\underset{\sim}{p}-\underset{\sim}{q})^2})}{\sqrt{n^2+(\underset{\sim}{p}-\underset{\sim}{q})^2}-p_0+q_0} - \frac{\delta(p_0 - q_0 - \sqrt{n^2+(\underset{\sim}{p}-\underset{\sim}{q})^2})}{\sqrt{n^2+(\underset{\sim}{p}-\underset{\sim}{q})^2}+p_0-q_0} \right] \right\},$$

(B.4)

where $P(q^2, (p-q)^2)$ is an arbitrary finite polynomial in q^2 and $(p-q)^2$, then the function $G(q)$, consistent with Lorentz invariance, is

$$G(q) = \frac{P(q^2,(p-q)^2)}{(m^2-q^2)[n^2-(p-q)^2]} \quad . \tag{B.5}$$

We can easily check that this function has the correct imaginary part and can be obtained from a dispersion integral with a suitable number of subtractions. This rule can also be translated into the effective language, similar to the effective technique discussed in Section 2, and it reads: If the imaginary part of a function $G(q)$ is given by

$$\frac{1}{\pi}\,\mathrm{Im}\,G(q) = P(q^2,(p-q)^2)\left[\frac{\delta(m^2-q^2)}{n^2-(p-q)^2} - \frac{\delta(n^2-(p-q)^2)}{m^2-q^2}\right], \tag{B.6}$$

along a cut on the real q^2-axis for constant $(p-q)q$,

then the function $G(q)$, consistent with Lorentz invariance, is

$$G(q) = \frac{P(q^2,(p-q)^2)}{(m^2-q^2)[n^2-(p-q)^2]}. \tag{B.7}$$

In this effective technique, the function $G(p)$ is treated as an analytic function of q^2 with $(p-q)q$ fixed. It is then easy to see that, since $(p-q)^2 = p^2 - 2(p-q)q - q^2$, $G(q)$, as given by (B.7), has the correct imaginary part as given by (B.6).

APPENDIX C

With the standard reduction technique, the definition (3.1) of $\gamma_{\rho\pi\pi}(q^2)$ is modified to be

$$(2\pi)^{-3} i \epsilon^{abc} \lambda(\varepsilon k) \gamma_{\rho\pi\pi}(q^2)/(m_\pi^2 - q^2) \equiv (2\pi)^{-3/2} i (m_\pi^2 - p^2) \int d^4x \, e^{-ipx} \times \theta(x_0) \langle \rho^b(k,\varepsilon) | [\pi^a(0), \pi^c(x)] | 0 \rangle \Big|_{p^2 = m_\pi^2}, \quad \text{(C.1)}$$

where $q_\mu = p_\mu - k_\mu$. Now the assumption of dispersion relation is that the form factor involved in (C.1), that is $\gamma_{\rho\pi\pi}(q^2)/(m_\pi^2 - q^2)$, is analytic in the upper half of the q_0-plane with at most a cut along the real q_0-axis. This at once implies that $\gamma_{\rho\pi\pi}(q^2)$ has no singularity at finite q^2 except possibly along the real q_0-axis. To investigate the singularity along the real q_0-axis, the absorptive part of (C.1) is

$$\frac{m_\pi^2 - p^2}{2(2\pi)^{3/2}} \int d^4x \, e^{-ipx} \langle \rho^b(k,\varepsilon) | [\pi^a(0), \pi^c(x)] | 0 \rangle \Big|_{p^2 = m_\pi^2}$$

$$= -\frac{i\pi\epsilon^{abc}}{(2\pi)^3} \lambda(\varepsilon k) \frac{\gamma_{\rho\pi\pi}(m_\pi^2)}{2\sqrt{m_\pi^2 + \underset{\sim}{q}^2}} \left[\delta(q_0 - \sqrt{m_\pi^2 + \underset{\sim}{q}^2}) - \delta(q_0 + \sqrt{m_\pi^2 + \underset{\sim}{q}^2}) \right].$$

Thus the only discontinuity of $\gamma_{\rho\pi\pi}(q^2)/(m_\pi^2 - q^2)$ across the real q_0-axis is at $q_0 = \pm\sqrt{m_\pi^2 + \underset{\sim}{q}^2}$, and of the same form as that of the function $(m_\pi^2 - q^2)^{-1}$. $\gamma_{\rho\pi\pi}(q^2)$ thus has no discontinuity along the real q_0-axis, and, combining this with the earlier statement that it has no singularity at finite q_0 outside the real q_0-axis, we conclude that $\gamma_{\rho\pi\pi}(q^2)$ has <u>no</u> singularity at finite q_0. The assumption that the amplitude (C.1) satisfies a dispersion relation implies that $\gamma_{\rho\pi\pi}(q^2)$ is bounded by a finite polynomial for large q_0, and, since it has no singularity at finite q_0, $\gamma_{\rho\pi\pi}(q^2)$ must be a finite polynomial of q_0. Hence, by Lorentz invariance, $\gamma_{\rho\pi\pi}(q^2)$ is a finite polynomial of q^2.

APPENDIX D

Different from the spirit of the paper, we will discuss the alternate hypothesis about axial-vector currents: that there are no low-lying spin-one states which dominate the axial-vector currents, but that the overall effect of spin-one states can be described by the subtraction constant of a once-subtracted dispersion relation. For the amplitude

$$\langle \pi^a(q)|A_\mu^b(0)|\rho^c(p,\varepsilon)\rangle = (2\pi)^{-3}\epsilon^{abc}\left[f(k^2)\varepsilon_\mu + \cdots\right], \qquad (D.1)$$

(where $k_\mu = p_\mu - q_\mu$), this means that only π-states are used to dominate the axial-vector current , and that $f(k^2)$, since it is entirely a spin-one effect, satisfies an once-subtracted dispersion relation in k^2. To be specific, let us postulate un-subtracted dispersion relations for all other form factors in (D.1). Then standard technique finds that

$$\langle \pi^a(q)|A_\mu^b(0)|\rho^c(p,\varepsilon)\rangle = -\frac{\epsilon^{abc}}{(2\pi)^3}\left[\frac{2F_\pi \gamma_{\rho\pi\pi}}{m_\pi^2 - k^2}(\varepsilon k)k_\mu + K\varepsilon_\mu\right], \qquad (D.2)$$

where K is the subtraction constant, and that

$$\langle \pi^a(q)|\,\partial^\mu A_\mu^b(0)|\rho^c(p,\varepsilon)\rangle = \frac{i\epsilon^{abc}}{(2\pi)^3}\,\frac{2m_\pi^2 F_\pi}{m_\pi^2 - k^2}\,\gamma_{\rho\pi\pi}(\varepsilon k). \qquad (D.3)$$

In order that these two equations may be consistent, we determine K to be

$$K = 2F_\pi \gamma_{\rho\pi\pi}, \qquad (D.4)$$

so that (D.2) becomes

$$\langle \pi^a(q) | A_\mu^b(0) | \rho^c(p,\varepsilon) \rangle = -\frac{\epsilon^{abc}}{(2\pi)^3} \, 2 F_\pi \gamma_{\rho\pi\pi} \left[\frac{(\varepsilon k)}{m_\pi^2 - k^2} k_\mu + \varepsilon_\mu \right]. \tag{D.5}$$

One then finds that the absorptive part of $P_{\mu\nu}^{abc}(q, p, \varepsilon)$ (as defined by an equation similar to (4.45), but omitting σ here) is given by

$$\frac{1}{\pi} \mathrm{Abs}\, P_{\mu\nu}^{abc}(q,p,\varepsilon) = -\frac{i\epsilon^{abc}}{(2\pi)^{3/2}} \, 2 F_\pi^2 \gamma_{\rho\pi\pi}$$

$$\times \left\{ \delta(q^2 - m_\pi^2) q_\mu \left[\frac{(\varepsilon k)}{m_\pi^2 - k^2} k_\nu + \varepsilon_\nu \right] + \delta(k^2 - m_\pi^2) k_\nu \left[\frac{(\varepsilon q)}{m_\pi^2 - q^2} q_\mu + \varepsilon_\mu \right] \right\}, \tag{D.6}$$

and that, by the effective technique,

$$P_{\mu\nu}^{abc}(q,p,\varepsilon) = -\frac{i\epsilon^{abc}}{(2\pi)^{3/2}} \, 2 F_\pi^2 \gamma_{\rho\pi\pi}$$

$$\times \left[\frac{(\varepsilon k) q_\mu k_\nu}{(m_\pi^2 - q^2)(m_\pi^2 - k^2)} + \frac{q_\mu \varepsilon_\nu}{m_\pi^2 - q^2} - \frac{k_\nu \varepsilon_\mu}{m_\pi^2 - k^2} \right]. \tag{D.7}$$

In the same manner, one obtains, similarly to (4.47),

$$F_\nu^{abc}(q,p,\varepsilon) = -\frac{\epsilon^{abc}}{(2\pi)^3} \frac{2 m_\pi^2 F_\pi}{m_\pi^2 - q^2} \gamma_{\rho\pi\pi} \left[\frac{(\varepsilon k)}{m_\pi^2 - k^2} k_\nu + \varepsilon_\nu \right], \tag{D.8}$$

so that the Ward identity (4.49) results in the relation

$$2 F_\pi^2 \gamma_{\rho\pi\pi} = - m_\rho F_\rho . \tag{D.9}$$

Noting that (3.17) in here reads

$$m_\rho^{-1} F_\rho \gamma_{\rho\pi\pi} = -1, \tag{D.10}$$

we therefore have the KSRF relation

$$2 F_\pi^2 \gamma_{\rho\pi\pi}^2 = m_\rho^2 . \tag{D.11}$$

FOOTNOTES AND REFERENCES

[1] An excellent review on current-algebra was presented by S. Weinberg, Proceeding of the 14TH INTERNATIONAL CONFERENCE ON HIGH ENERGY PHYSICS, page 253 (CERN, 1968).

[2] K. Kawarabayashi and M. Suzuki, Phys. Rev. Letts. 16, 255 (1966); Riazuddin and Fayyazuddin, Phys. Rev. 147, 1071 (1966).

[3] D. A. Geffen, Phys. Rev. Letts. 19, 770 (1967).

[4] S. Weinberg, Phys. Rev. Letts. 18, 507 (1967). In this reference $\rho_V(\mu^2)$ and $\rho_A(\mu^2)$ are $\sigma F_\rho{}^2(\sigma)$ and $\sigma F_{\mathcal{a}}{}^2(\sigma)$ of this paper respectively, while its isospin-currents are twice ours.

[5] H. J. Schnitzer and S. Weinberg, Phys. Rev. 164, 1828 (1967). Like Reference 4, its isospin-currents are also twice ours.

[6] S. G. Brown and G. B. West, Phys. Rev. Letts. 19, 812 (1967), and Phys. Rev. 168, 1605 (1968).

[7] This is not strictly true when q_μ is constrained by a mass-shell

condition. The $\underset{\sim}{q}$ is not constant in length, but in direction.

8 For example, S. Gasiorowicz, Elementary Particle Physics, John Wiley & Sons Inc. (1966); and G. Baton, Dispersion Techniques in Field Theory, Benjamin (1965).

9 In general a Lorentz covariant amplitude can be expanded as Σ(Lorentz scalar function) x (Lorentz tensor) and its absorptive part is defined to be ΣIm(Lorentz scalar function) x (Lorentz tensor). Since M(q,p) is a Lorentz scalar, its absorptive part is just its imaginary part.

10 It is to be noted that, with our convention of the Lorentz metric, the sum over the three independent polarizations ε of a spin-one state of 4-momentum k and mass m is $\Sigma_{\varepsilon}\varepsilon_{\mu}\varepsilon_{\nu} = -g_{\mu\nu} + k_{\mu}k_{\nu}/m^2$.

11 In general, the proof of Ward identity is as follows:

$$
\begin{aligned}
& iq^{\mu}\int d^4x\, e^{iqx}\theta(x_0)\langle\alpha|[j_{\mu}(x),\varphi(0)]|\beta\rangle \\
&= \int d^4x\,(\partial^{\mu}e^{iqx})\theta(x_0)\langle\alpha|[j_{\mu}(x),\varphi(0)]|\beta\rangle \\
&= -\int d^4x\, e^{iqx}\delta(x_0)\langle\alpha|[j_0(x),\varphi(0)]|\beta\rangle \\
&\qquad -\int d^4x\, e^{iqx}\theta(x_0)\langle\alpha|[\partial^{\mu}j_{\mu}(x),\varphi(0)]|\beta\rangle,
\end{aligned}
$$

where we have assumed that the surface term in the partial integration is zero.

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[13] Bjorken's conjecture about the relationship between covariant and non-covariant time-ordered products (J.D. Bjorken, Phys. Rev. 148, 1467 (1966)) can be easily translated into the language of retarded or advanced commutators.

[14] Though we have not displayed $M^{abc}_{\mu\nu}(q,p)$ explicitly in terms of form factors, it is easy to convince oneself that (a) calculating the amplitude by a dispersion integral, and (b) treating any Lorentz tensor, i, and any quantity not dependent on q^2 or k^2 as constants with respect to integration, is equivalent to separating the amplitude into form factors and calculating them separately.

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FIGURE CAPTION

Fig. 1. Representations of $\int d^4x e^{iqx} \langle 0|[\phi(x),\varphi(0)]|m,\underset{\sim}{p}\rangle$.

Fig. 2. Representations of $\langle \mathcal{A}|V_\nu(0)|\pi\rangle$.

Fig. 3. Representation of $i\int d^4x\ e^{iqx}\ \theta(x_0)\ \langle 0|[A_\mu^a(x),\ V_\nu^b(0)]|\pi^c(p)\rangle$.

The central blob represents strong interaction vertices.

Fig. 4. Triangle graph of $i\int d^4x\ e^{iqx}\ \theta(x_0)\ \langle 0|[A_\mu^a(x),\ V_\nu^b(0)]|\pi^c(p)\rangle$.

Fig. 5. Representations of (a) $\langle \mathcal{A}^a(q,\zeta,\sigma'')|V_\nu^b(0)|\mathcal{A}^c(p,\eta,\sigma')\rangle$,

(b) $\langle \rho^b(k,\varepsilon,\sigma)|A_\mu^a(0)|\mathcal{A}^c(p,\eta,\sigma')\rangle$, and

(c) $\langle \rho^b(k,\varepsilon,\sigma)|\partial^\mu A_\mu^a(0)|\mathcal{A}^c(p,\eta,\sigma')\rangle$.

Fig. 6. Representations of

(a) $i\int d^4x\ e^{iqx}\ \theta(x_0)\ \langle 0|[A_\mu^a(x),\ V_\nu^b(0)]|\rho^c(p,\eta,\sigma')\rangle$, and

(b) $i\int d^4x\ e^{iqx}\ \theta(x_0)\ \langle 0|[\partial^\mu A_\mu^a(x),\ V_\nu^b(0)]|\rho^c(p,\eta,\sigma')\rangle$.

Fig. 7. (a) Connected and (b) disconnected parts of $\langle n(s)|\varphi(0)|\beta(p)\rangle$.

Fig. 8. Representations of $\int d^4x\ e^{iqx}\ \langle 0|[\phi(x),\varphi(0)]|\beta(p)\rangle$.

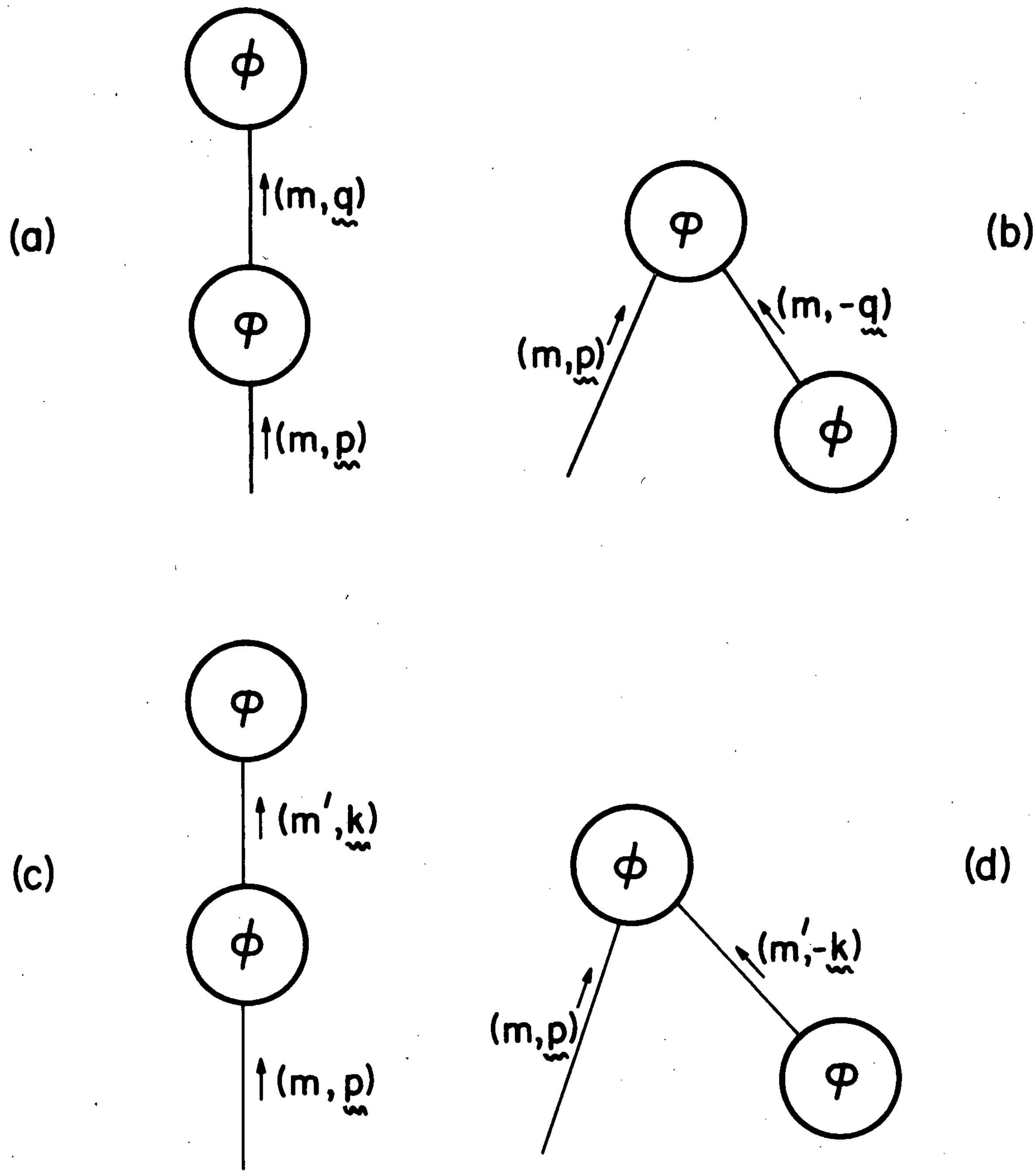


Fig. I

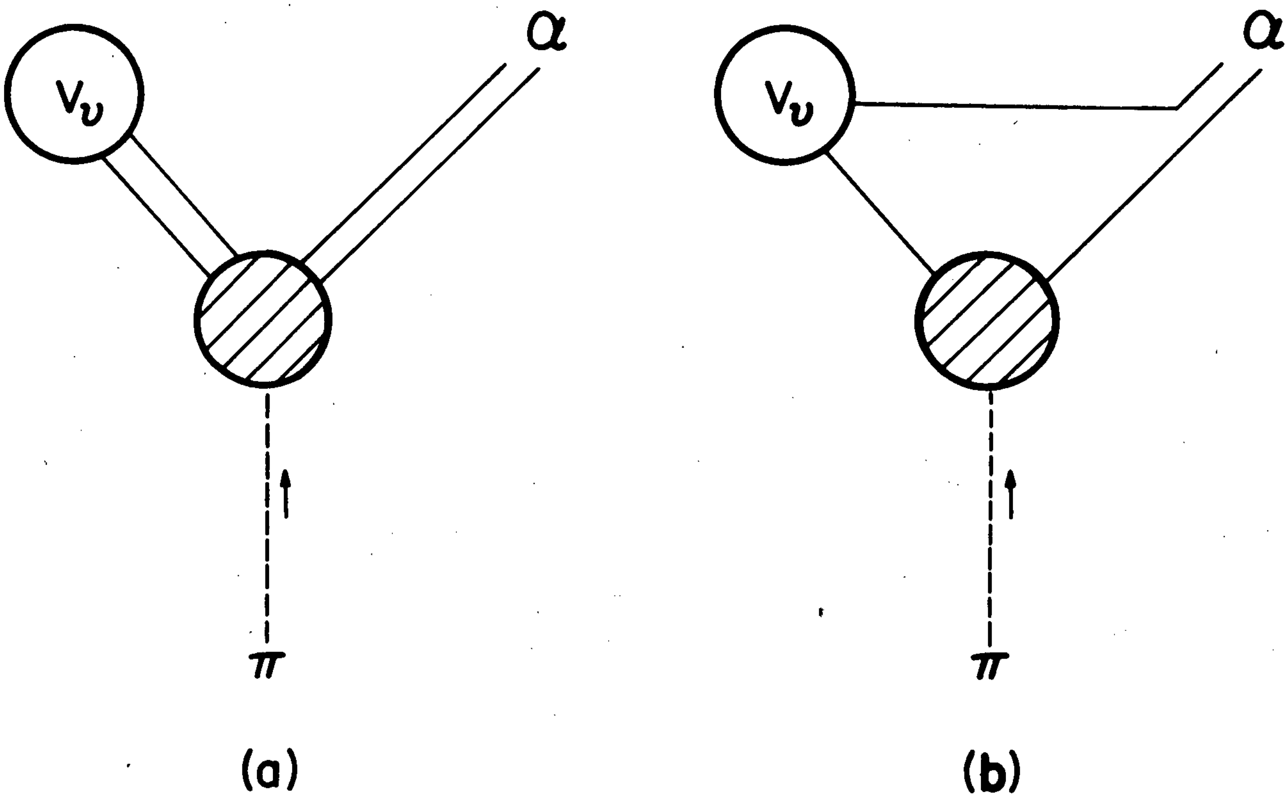


Fig.2

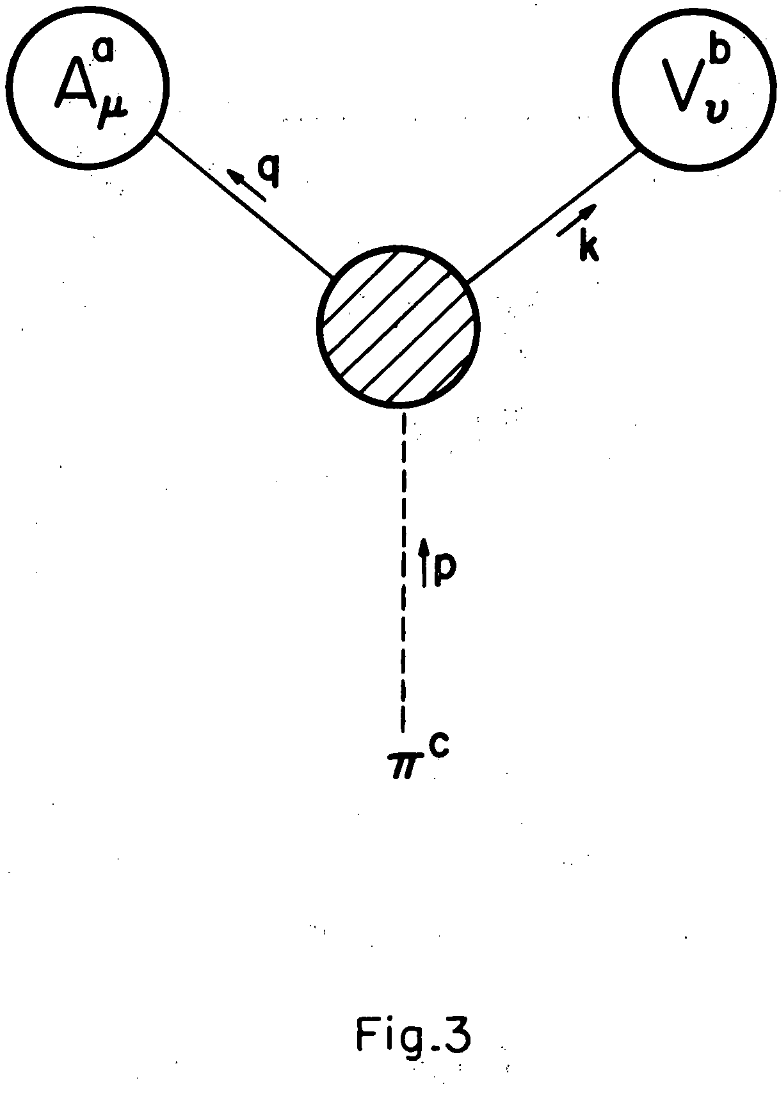


Fig.3

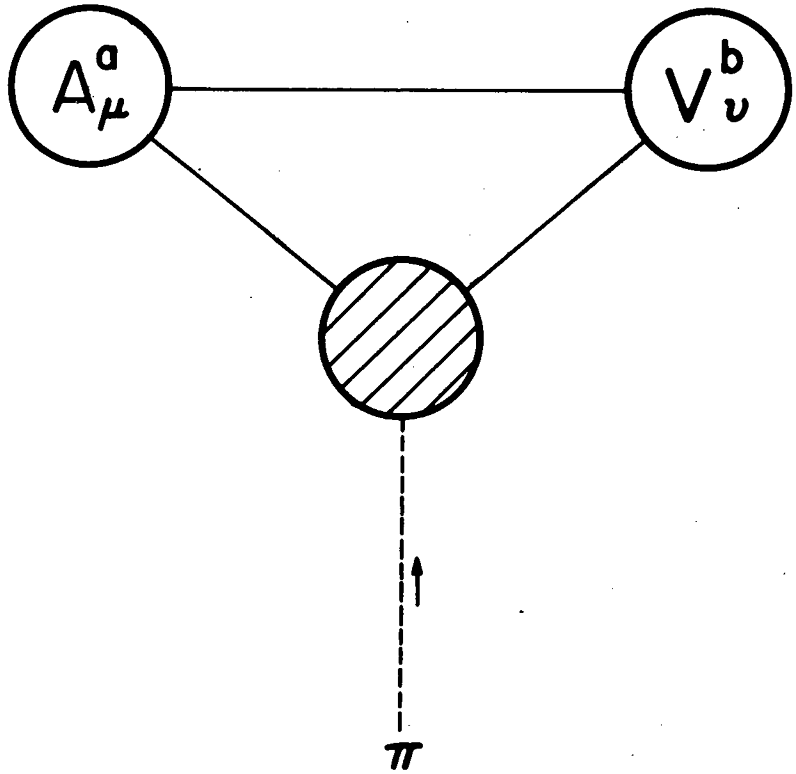
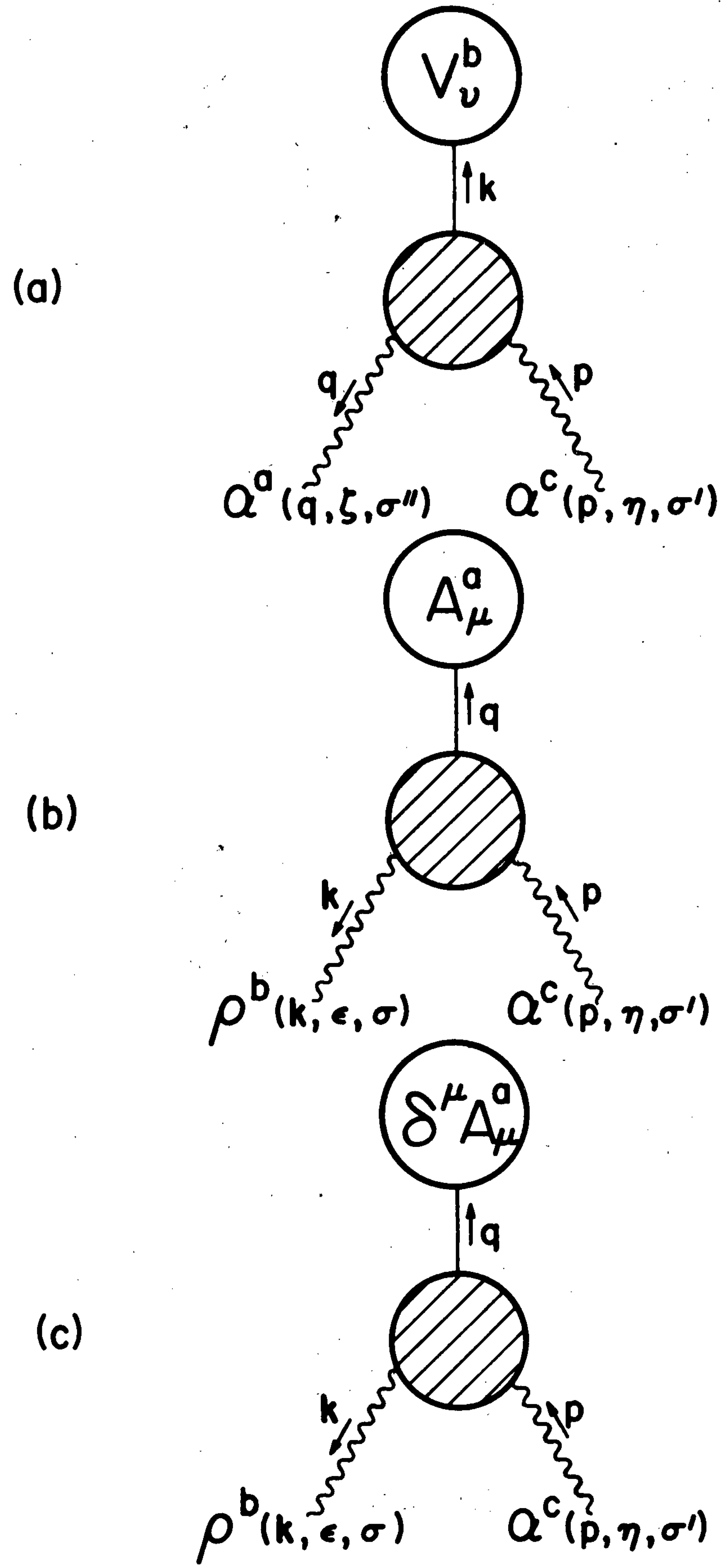
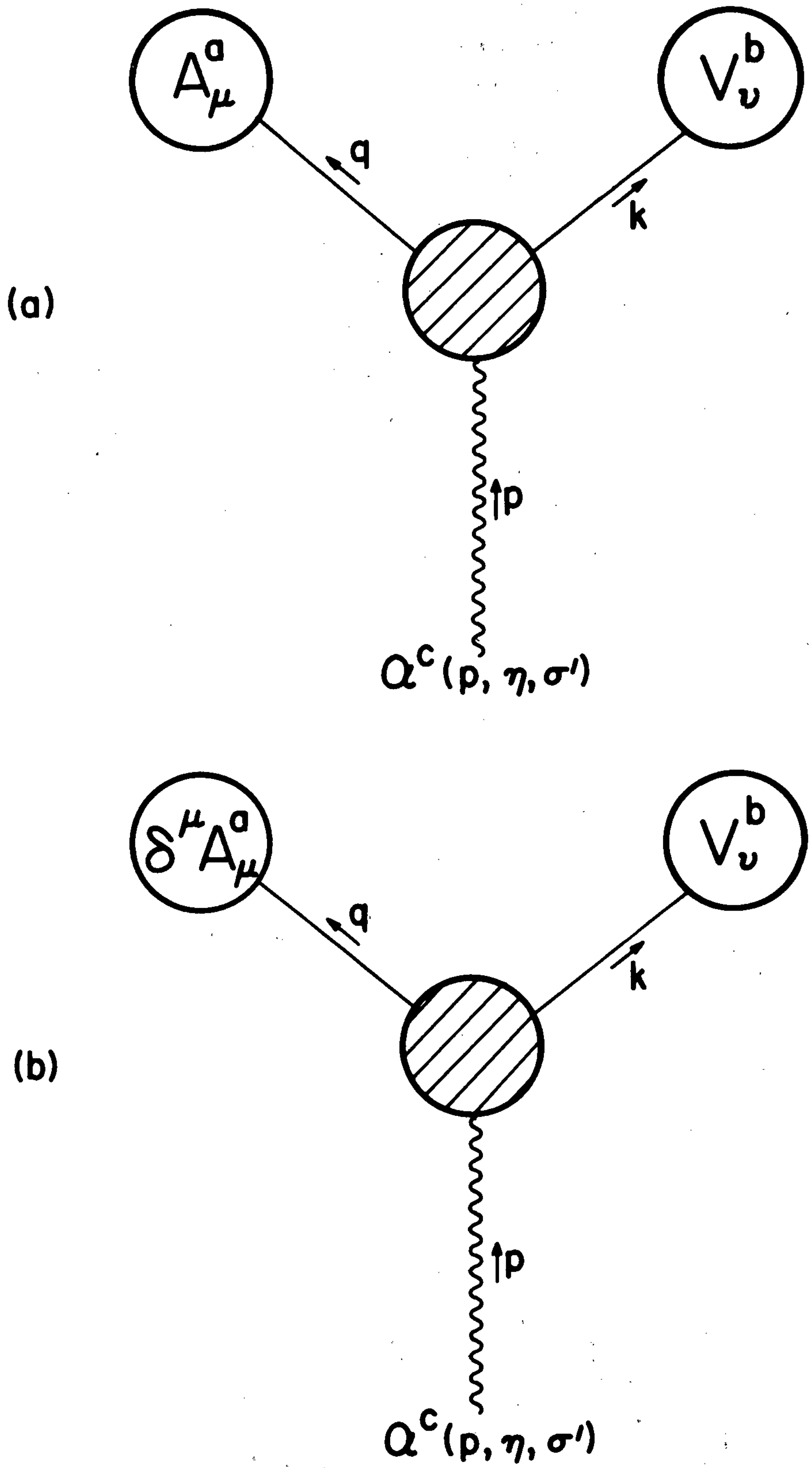


Fig.4



V^b_ν
k
(a)
q
p
$\alpha^a(q,\zeta,\sigma'')$
$\alpha^c(p,\eta,\sigma')$
A^a_μ
q
(b)
k
p
$\rho^b(k,\epsilon,\sigma)$
$\alpha^c(p,\eta,\sigma')$
$\partial^\mu A^a_\mu$
q
(c)
k
p
$\rho^b(k,\epsilon,\sigma)$
$\alpha^c(p,\eta,\sigma')$

Fig.5



A^a_μ
V^b_ν
q
k
(a)
p
$\alpha^c(p, \eta, \sigma')$
$\partial^\mu A^a_\mu$
V^b_ν
q
k
(b)
p
$\alpha^c(p, \eta, \sigma')$

Fig.6

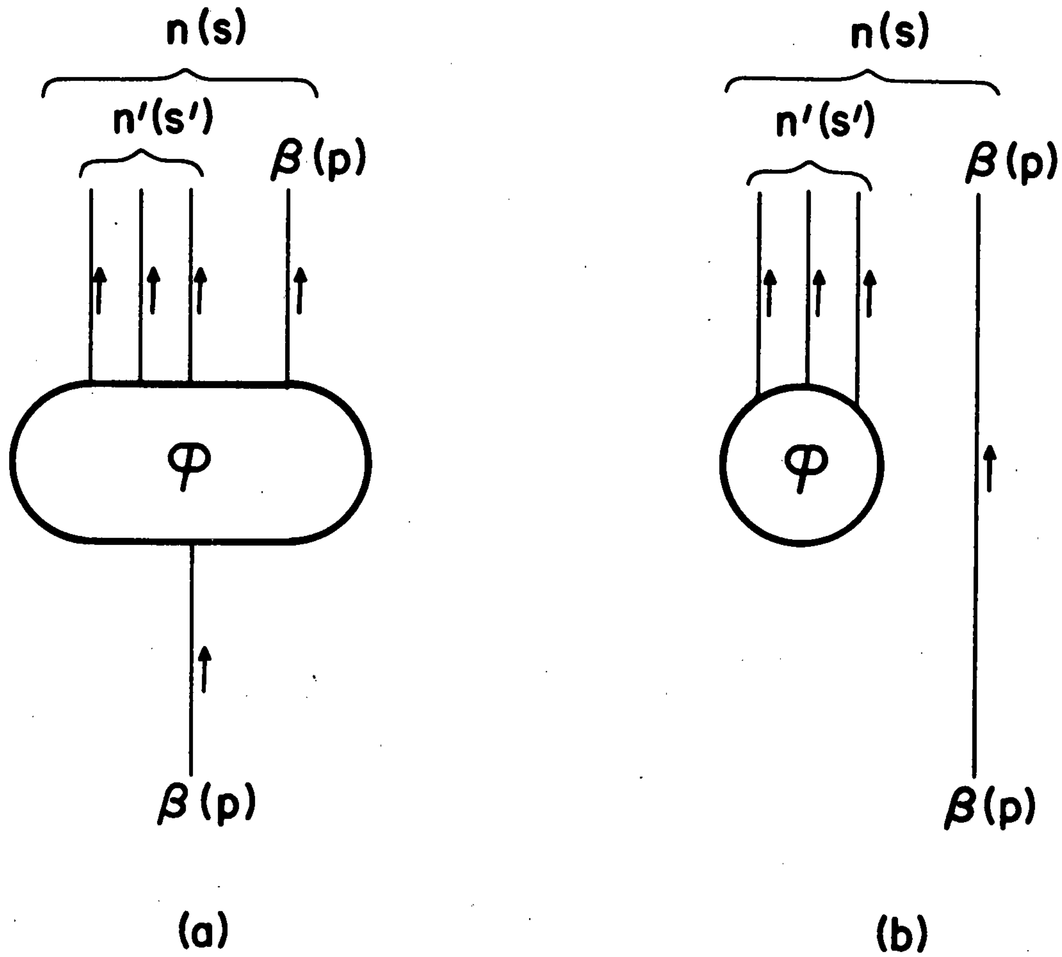


Fig.7

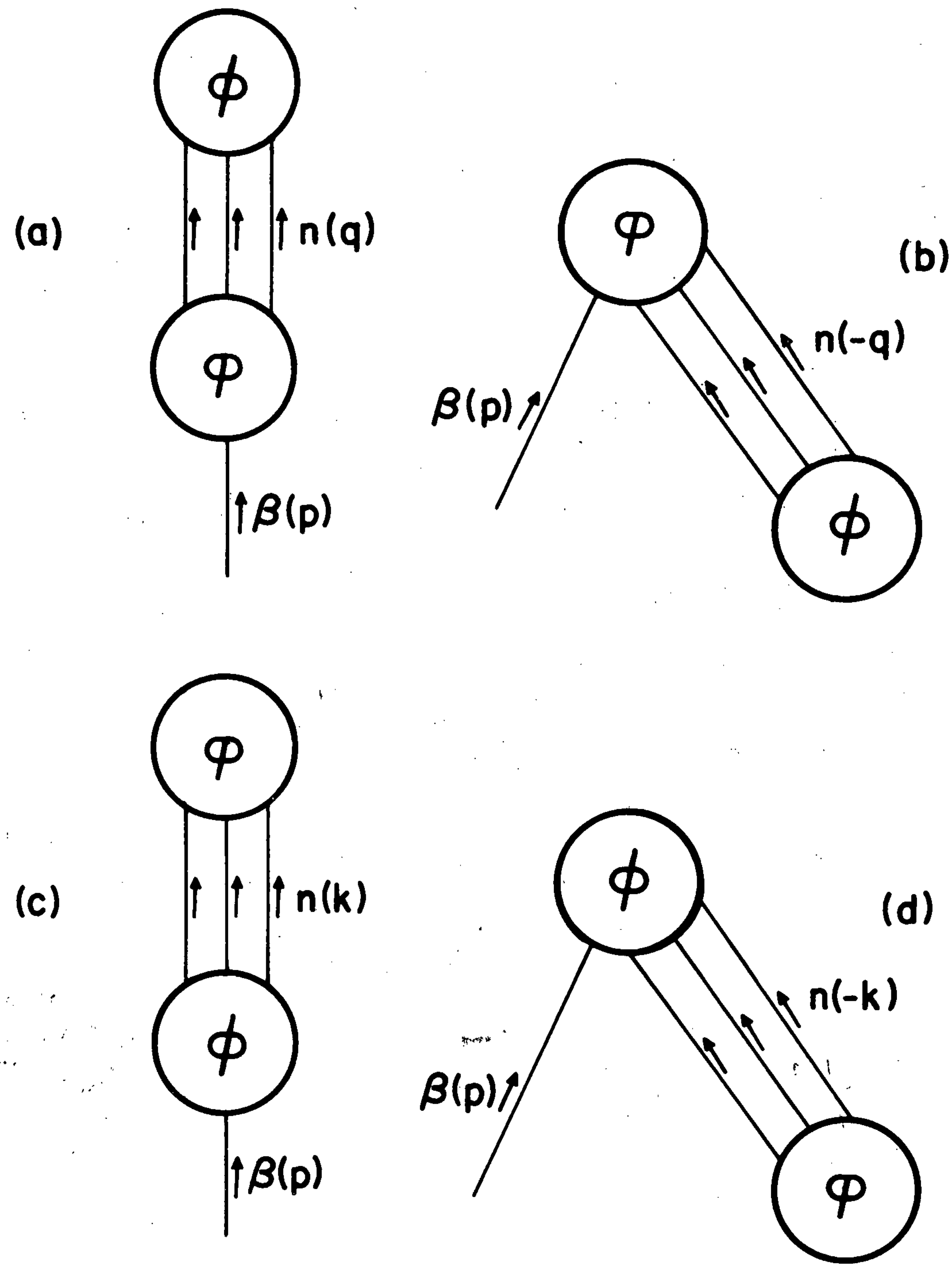


Fig.8