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BROOKHAVEN NATIONAL LABORATORY
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INFORMAL REPORT

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[1] Department of Civil Engineering, The University of Akron, Akron, Ohio, Under Contract with BNL Structural Analysis Group.

[2] Department of Applied Science, Brookhaven National Laboratory, Upton, New York.

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INTRODUCTION

During recent years, there is an increasing trend towards the development of nonlinear finite element analysis method. Such analysis is needed for the determination of ultimate load carrying capacity of a complex structure or for the design of a structure subjected to extreme loads. Structural nonlinearities normally arise from two different sources: material and geometric nonlinearities. The material nonlinearity to be considered herein is the commonly used elastic-plastic material, for which the constitutive equations depend on the current deformation state of the structure. The geometric nonlinearity being considered is due to both large displacement and large strain.

There are many finite element programs that are presently available for solving nonlinear structural problems by considering various types of nonlinearities. A comprehensive survey of these programs is discussed in reference [1]. This report concerns, however, with the nonlinear analysis of a three-dimensional continuum, for which both the elastic-plastic material and large deformation are of primary interest.

Elastic-plastic analysis of three-dimensional continuums has been reported previously for problems with small deformations [2-4]. Using the Lagrangian description, a general theory for the three-dimensional finite element analysis including both the material and geometric nonlinearities was presented by Hibbit, Marcal and Rice [5], Sharifi and Yates [6]. A formulation of the similar problem with an Eulerian description was given by McMeeking and Rice [7]. However, detailed formulation of the stiffness equation for a three-dimensional isoparametric element has not become available. The purpose of this report is to follow Hibbitt and Sharifi's work and present the formulation of the stiffness equation for an 8-21 node isoparametric element with elastic-plastic material and large deformation. The formulation has been implemented in a nonlinear finite element program for the analysis of three-dimensional continuums. To demonstrate the utility of the formulation, a thick-walled cylinder was analyzed and the results are compared favorably with known solution.

BASIC EQUATIONS

To solve nonlinear problems with elastic-plastic material and large deformation, an incremental solution approach is adopted. Using the Lagrangian description, all dependent variables are defined in terms of an undeformed Cartesian reference frame, X_i. For discussion purposes, the following notations are introduced: Let u_i be the displacement vector of a material point referring to the undeformed state; S_{ij}, the second Piola-Kirchoff stress tensor; and E_{ij}, the Green strain tensor which is related to the displacement components by

$$E_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i} + u_{k,i} u_{k,j}) \tag{1}$$

where $u_{i,j} = \frac{\partial u_i}{\partial X_j}$. Now consider a typical load increment n. At the beginning of the load increment, the initial displacements, stresses and strains are assumed to be known as u_i^0, S_{ij}^0, and E_{ij}^0, respectively. Suppose that the continuum is subjected to an incremental surface traction, ΔT_i, an incremental change of state will result, i.e., Δu_i, ΔS_{ij} and ΔE_{ij}. Thus the total displacements, stresses and strains at the end of the n-th loading increment are given respectively by

$$u_i = u_i^0 + \Delta u_i \tag{2}$$

$$S'_{ij} = S_{ij}^{0'} + \Delta S_{ij} \tag{3}$$

$$E_{ij} = E_{ij}^0 + \Delta E_{ij} \tag{4}$$

From Eqs. (1) and (2), one can derive the expression for the incremental Green strain tensor, i.e.

$$\Delta E_{ij} = \Delta \varepsilon_{ij} + \Delta \eta_{ij} \tag{5}$$

where $\Delta\varepsilon_{ij}$ is the linear part and $\Delta\eta_{ij}$ is the nonlinear part of ΔE_{ij}. They are evaluated from

$$\Delta \varepsilon_{ij} = \frac{1}{2} (u^0_{i,j} + u^0_{j,i} + u^0_{k,i}\ \Delta u_{k,j} + \Delta u_{k,i}\ u^0_{k,j}) \tag{6}$$

$$\Delta \eta_{ij} = \frac{1}{2}\ \Delta u_{k,i}\ \Delta u_{k,j} \tag{7}$$

The elastic-plastic stress-strain equations adopted herein are the incremental form based on the von Mises yield criterion with either isotropic or kinematic hardening. Then the increment of Kirchoff stress is related to the increment of Green strain by

$$\Delta S_{ij} = D_{ijrs}\ \Delta E_{rs} \tag{8}$$

where D_{ijrs} represent either the elastic constants or the elastic-plastic constants, which are derived from the current yield surface. Since the derivation of D_{ijrs} can be found in references [5,6], therefore its formulation will not be repeated herein. The definition of D_{ijrs} is given in the Appendix.

For simplicity, all subsequent formulations are restricted to one element. Following the development in [5,6,8], the stiffness equation of an element is derived from a linearized virtual work principle

$$\int_V D_{ijrs} \; \Delta\varepsilon_{rs} \; \delta(\Delta\varepsilon_{ij}) \, dv + \int S^o_{ij} \; \delta(\Delta\eta_{ij}) \, dv$$

$$= \int_A \Delta T_i \; \delta(\Delta u_i) \, dA - \int_V S^o_{ij} \; \delta(\Delta\varepsilon_{ij}) \, dV \qquad (9)$$

where V and A denote the volume and area of the undeformed element, respectively.

FINITE ELEMENT FORMULATION

Three-dimensional isoparametric element consisting of 8-21 nodes as shown in Fig. 1 is considered in the present derivation. In each element, the undeformed coordinates X_i and displacements u_i of any material point are interpolated by [9]

$$X_i = \sum_{m=1}^{M} h_m \; X_i^m \qquad (10)$$

$$u_i = \sum_{m=1}^{M} h_m \; u_i^m \qquad (11)$$

where X_i^m = undeformed coordinate of the m^{th} node in the i^{th} direction,

u_i^m = displacement component of the m^{th} node in i^{th} direction.

h_m = shape functions for a three-dimensional isoparametric element as defined in reference [9].

In matrix notation, Eqs.(10) and (11) are rewritten as

$$\{X_i\} = [H]\ \{X_i^m\} \tag{12}$$

$$\{u_i\} = [H]\ \{u_i^m\} \tag{13}$$

The derivatives of displacements can be readily found from Eq. (13)

$$\{u_{i,j}\} = [B_{NL}]\{u_i^m\} \tag{14}$$

where $\{u_{i,j}\}^T = \{u_{1,1}\ \ u_{2,2}\ \ u_{3,3}\ \ u_{1,2}\ \ u_{2,3}\ \ u_{3,1}\}$ (15)

$$\{u_i\}^T = \{u_1^1\ \ u_2^1\ \ u_3^1\ \ .\ .\ .\ .\ .\ .\ u_1^N\ \ u_2^N\ \ u_3^N\} \tag{16}$$

N, Nth nodal point of the element

and

$$[B_{NL}] = \begin{bmatrix} h_{1,1} & 0 & 0 & \cdots & h_{N,1} & 0 & 0 \\ h_{1,2} & 0 & 0 & \cdots & h_{N,2} & 0 & 0 \\ h_{1,3} & 0 & 0 & \cdots & h_{N,3} & 0 & 0 \\ 0 & h_{1,1} & 0 & \cdots & 0 & h_{N,1} & 0 \\ 0 & h_{1,2} & 0 & \cdots & 0 & h_{N,2} & 0 \\ 0 & h_{1,3} & 0 & \cdots & 0 & h_{N,3} & 0 \\ 0 & 0 & h_{1,1} & \cdots & 0 & 0 & h_{N,1} \\ 0 & 0 & h_{1,2} & \cdots & 0 & 0 & h_{N,2} \\ 0 & 0 & h_{1,3} & \cdots & 0 & 0 & h_{N,3} \end{bmatrix} \tag{17}$$

$$h_{k,i} = \frac{\partial h_k}{\partial X_i}$$

The linear strain-displacement relationship is derived by using Eqs. (6) and (13), ie.

$$\{\varepsilon\} = [B_L]\{u\} \tag{18}$$

where

$$\{\varepsilon\}^T = \{\varepsilon_{11} \quad \varepsilon_{22} \quad \varepsilon_{33} \quad 2\varepsilon_{12} \quad 2\varepsilon_{23} \quad 2\varepsilon_{31}\} \tag{19}$$

$$[B_L] = [B_{L0}] + [B_{L1}] \tag{20}$$

In the above equation, the matrix B_{L0} is due to small deformation and B_{L1} is due to initial displacements as a result of large deformations. They are given by

$$[B_{L0}] = \begin{bmatrix} h_{1,1} & 0 & 0 & \cdots\cdots & h_{N,1} & 0 & 0 \\ 0 & h_{1,2} & 0 & \cdots\cdots & 0 & h_{N,2} & 0 \\ 0 & 0 & h_{1,3} & \cdots\cdots & 0 & 0 & h_{N,3} \\ h_{1,2} & h_{1,1} & 0 & \cdots\cdots & h_{N,2} & h_{N,1} & 0 \\ 0 & h_{1,3} & h_{1,2} & \cdots\cdots & 0 & h_{N,3} & h_{N,2} \\ h_{1,3} & 0 & h_{1,1} & \cdots\cdots & h_{N,3} & 0 & h_{N,1} \end{bmatrix} \tag{21}$$

and

$$[B_{L1}] = \begin{bmatrix} u_{1,1}\,h_{1,1} & u_{2,1}h_{1,1} & u_{3,1}\,h_{1,1} \\ u_{1,2}\,h_{1,2} & u_{2,2}h_{1,2} & u_{3,2}\,h_{1,2} \\ u_{1,3}\,h_{1,3} & u_{2,3}h_{1,3} & u_{3,3}\,h_{1,3} \\ u_{1,1}h_{1,2}+u_{1,2}h_{1,1} & h_{2,1}h_{1,2}+u_{2,2}h_{1,1} & u_{3,1}\,h_{1,2}+u_{3,2}\,h_{1,1} \\ u_{1,2}h_{1,3}+u_{1,3}h_{1,2} & u_{2,2}h_{1,3}+u_{2,3}h_{1,2} & u_{3,2}\,h_{1,3}+u_{3,3}\,h_{1,1} \\ u_{1,3}h_{1,1}+u_{1,1}h_{1,3} & u_{2,3}h_{1,1}+u_{2,1}h_{1,3} & u_{3,3}\,h_{1,1}+u_{3,1}\,h_{1,3} \end{bmatrix}$$

(continued)

$$
\left.\begin{matrix}
\cdots\ u_{1,1}h_{N,1} & u_{2,1}h_{N,1} & u_{3,1}h_{N,1} \\
\cdots\ u_{1,2}h_{N,2} & u_{2,2}h_{N,2} & u_{3,2}h_{N,2} \\
\cdots\ u_{1,3}h_{N,3} & u_{2,3}h_{N,3} & u_{3,3}h_{N,3} \\
\cdots\ u_{1,1}h_{N,2}+u_{1,2}h_{N,1} & u_{2,1}h_{N,2}+u_{2,2}h_{N,1} & u_{3,1}h_{N,2}+u_{3,2}h_{N,1} \\
\cdots\ u_{1,2}h_{N,3}+u_{1,3}h_{N,2} & u_{2,2}h_{N,3}+u_{2,3}h_{N,2} & u_{3,2}h_{N,3}+u_{3,3}h_{N,1} \\
\cdots\ u_{1,3}h_{N,1}+u_{1,1}h_{N,3} & u_{2,3}h_{N,1}+u_{2,1}h_{N,3} & u_{3,3}h_{N,1}+u_{3,1}h_{N,3}
\end{matrix}\right] \qquad (22)
$$

Combining Eqs. (7), (9), (13), (14) and (19), and after some manipulation, one may easily find the incremental element stiffness equation in the following form

$$[k^{(0)} + k^{(1)} + k^{(1)}]\ \{\Delta u\} = \{\Delta f\} + \{\Delta e\} \qquad (23)$$

As pointed out by Hibbit et al. [5], the element stiffness matrix consists of three parts: the commonly known small displacement stiffness k^0, the initial displacement stiffness $k^{(1)}$, and the initial stress stiffness $k^{(2)}$ and they are given by

$$[k^{(0)}] = \int_V [B_{L0}]^T\ [D]\ [B_{L0}]\ dV \qquad (24)$$

$$[k^{(1)}] = \int_V [B_{L1}]^T\ [D]\ [B_{L1}]\ dV \qquad (25)$$

$$k^{(2)} = \int [B_{NL}]^T\ [\hat{S}]\,[B_{NL}]\ dV \qquad (26)$$

where [D] represents either the elastic or elastic-plastic material matrix as defined in Eq. (8), and $\hat{S}$ is the 2nd Piola-Kirchoff stress matrix which is given by

$$[\hat{S}] = \begin{bmatrix} [S] & 0 & 0 \\ 0 & [S] & 0 \\ 0 & 0 & [S] \end{bmatrix} \qquad (27)$$

$$[S] = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} \qquad (28)$$

and the force vectors on the right hand side of Eq. (23) are

$$\{\Delta f\} = \int_A [H]\{\Delta T\} \; dA \qquad (29)$$

$$\{\Delta e\} = \int_V [B_L] \; \{S\} \; dV \qquad (30)$$

where Δe is a force vector resulting from large deformation,

and $\{S\}^T = \{S_{11} \;\; S_{22} \;\; S_{33} \;\; S_{12} \;\; S_{23} \;\; S_{31}\}$ (31)

SAMPLE PROBLEM

The above formulation has been implemented into a finite element program, which is a modified version of the general purpose nonlinear finite element program NONSAP developed by Bathe, Ramm and Wilson [8]. To demonstrate the validity of the present formulation, a thick-walled cylinder subjected to internal pressure was considered and it was represented by a three-dimensional model as shown in Fig. 2. One quarter of the cylinder was used for the analysis. The analysis model consists of 40 elements: 20 layers in the radial direction and 2 segments in the circumferential direction. Each element has 12 nodes, i.e. 8 corner nodes and 4 midside nodes on the curved boundaries. This problem has previously analyzed by Hartzman [10] with a two-dimensional model and the analytical results were obtained by MacGregor [11].

The cylinder is made of aluminum 17SO and its elastic-plastic stress-strain curve is given in reference [11]. Isotropic hardening rule was used in the analysis. Shown in Fig. 2 are the plots of the internal pressure vs. the hoop strains at the inner and outer surface of the cylinder. As seen in the figure, the strain response at the outer surface obtained from the 3-D solution is identical with Hartzman's result, however it is slightly below the analytical solution by MacGregor. The second curve in Fig. 2 represents the hoop strain of the cylinder at the inner wall, which is well into the large strain range.

CONCLUSION

The stiffness formulation for a three-dimensional isoparametric element with elastic-plastic material property and large deformation was completed and the formulation has been incorporated into a non-linear finite element program for solving large size problems. The element type presented herein can be applied not only to 3-D continuums, but also to plate or shell structures, for which degenerated isoparametric elements may be used.

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APPENDIX - ELASTIC - PLASTIC CONSTITUTIVE RELATIONS

For elastic-plastic material, the constitutive relations in terms of incremental Kirchoff stresses and incremental Green strains are given by

$$\Delta S_{ij} = D_{ijrs} \ \Delta E_{rs} \tag{A-1}$$

or in matrix notation, Eq. (A-1) is rewritten in the form

$$\{\Delta S\} = [D] \ \{\Delta E\} \tag{A-2}$$

Where D represents either the elastic or elastic-plastic matrix. The elastic-plastic matrix D assumes different values depending whether the isotropic or kinematic hardening law is used.

For isotropic hardening law, the von Mises yield criterion is given by

$$f = \frac{1}{2} S'_{ij} \ S'_{ij} - \kappa^2 = 0 \tag{A-3}$$

where S'_{ij} is the deviatoric component of the Kirchoff stress tensor and κ is a measure of strain hardening which is assumed to be a function of plastic work

$$\kappa = F\,(W_p) \tag{A-4}$$

$$W_p = \int S_{ij} \ dE^P_{ij} \tag{A-5}$$

Based on the above relationship, one can easily derive the expression for the elastic-plastic matrix D,

$$[D] = [D^E] - [D^P] \tag{A-6}$$

where D^E represents the elastic matrix which is given by

$$[D^E] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ & 1-\nu & \nu & 0 & 0 & 0 \\ & & 1-\nu & 0 & 0 & 0 \\ & & & \frac{1-2\nu}{2} & 0 & 0 \\ & \text{symmetric} & & & \frac{1-2\nu}{2} & 0 \\ & & & & & \frac{1-2\nu}{E} \end{bmatrix} \quad \text{(A-7)}$$

E is the Young's modulus and ν, the Poisson's ratio. The plastic matrix, D^P is defined by

$$[D^P] = G \; \beta \; [S_2] \quad \text{(A-8)}$$

$$G = \frac{E}{2(1+\nu)}$$

where

$$\beta = \frac{1}{\kappa^2(1 + \frac{H}{G})} \quad \text{(A-9)}$$

$$H = \kappa \frac{d\kappa}{dWp} \quad \text{(A-10)}$$

and

$$[S_2] = \begin{bmatrix} S'_{11}S'_{11} & S'_{11}S'_{22} & S'_{11}S'_{33} & S'_{11}S'_{12} & S'_{11}S'_{23} & S'_{11}S'_{31} \\ & S'_{22}S'_{22} & S'_{22}S'_{33} & S'_{22}S'_{12} & S'_{22}S'_{23} & S'_{22}S'_{31} \\ & & S'_{33}S'_{33} & S'_{33}S'_{12} & S'_{33}S'_{23} & S'_{33}S'_{31} \\ & \text{Symmetric} & & S'_{12}S'_{12} & S'_{12}S'_{23} & S'_{12}S'_{31} \\ & & & & S'_{23}S'_{23} & S'_{23}S'_{31} \\ & & & & & S'_{31}S'_{31} \end{bmatrix} \quad \text{(A-11)}$$

In the case of kinematic hardening, the von Mises yield criterion has the form

$$f = \frac{1}{2}\ (S'_{ij} - \alpha_{ij})(S'_{ij} - \alpha_{ij}) - \kappa_o^2 = 0 \qquad \text{(A-12)}$$

where $\kappa_o = \sigma_y/\sqrt{3}$, σ_y = uniaxial yield stress (A-13)

$$d\alpha_{ij} = c\ dE^P_{ij} \qquad \text{(A-14)}$$

$$c = \frac{2}{3}\ \frac{E\ E_p}{E - E_p} \qquad \text{(A-15)}$$

E_p = plastic modulus of the uniaxial stress-strain curve evaluated at the current state.

Based on the criterion in Eq. (A-12), one can derive

$$[D] = [D^E] - [D^P] \qquad \text{(A-16)}$$

where D^E is defined in Eq. (A-6) and D^P is given by

$$[D^P] = G\ \gamma[\overline{S}_2] \qquad \text{(A-17)}$$

where $$\gamma = \frac{2}{\kappa_o^2[\frac{E}{1+\nu} + c + 2c\ (\overline{S}'^2_{12} + \overline{S}'^2_{23} + \overline{S}'^2_{31})]} \qquad \text{(A-18)}$$

and the matrix $\overline{S}_2$ has the same form as defined in Eq. (A-11) except that the stress deviators are replaced by their shifted quantities, i.e.

$$\overline{S}'_{ij} = S'_{ij} - \alpha_{ij} \qquad \text{(A-19)}$$

A final remark should be in order. Since the above equations were derived from a Lagrangian description the Kirchoff stresses and Lagrangian strains were used for stress and strain measures. Then, for large deformations the elastic-plastic constant, e.g. E_p, must be determined from the uniaxial data of the same stress and strain measures. However, the uniaxial stress-strain curve is usually obtained on the basis of engineering stress and engineering strain. The conversion from the engineering stress and strain to Kirchoff stress and Lagrangian strain can be made according to

$$\text{Kirchoff stress} = \frac{\sigma_E}{1+\varepsilon_E} \tag{A-20}$$

$$\text{Lagrangian strain} = \varepsilon_E + \frac{1}{2}\varepsilon_E^2 \tag{A-21}$$

where

σ_E = Engineering stress = $\frac{P}{A_o}$

ε_E = Engineering strain = $\frac{u}{L_o}$

P = Uniaxial load

A_o = Undeformed cross-sectional area of a specimen

L_o = Undeformed length of a specimen

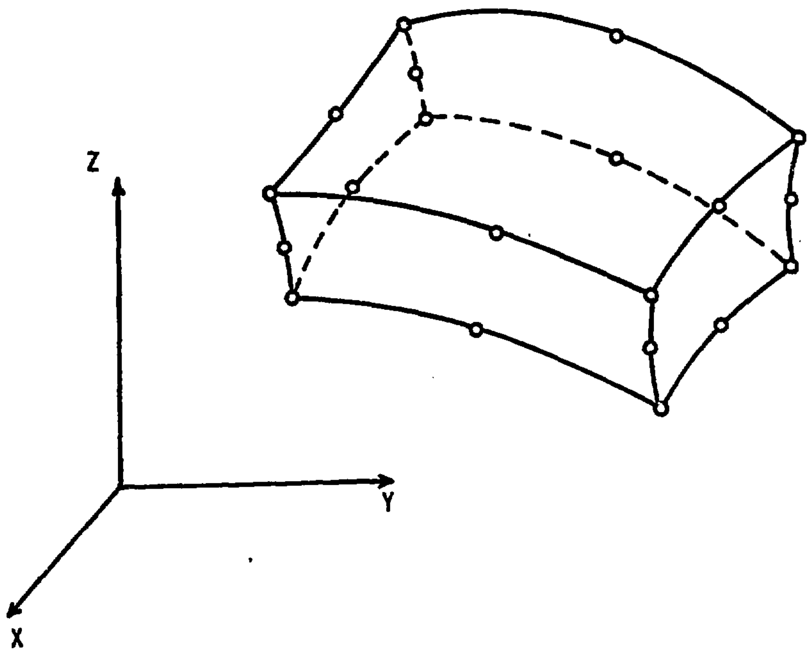

Fig. 1 A three-dimensional isoparametric element.

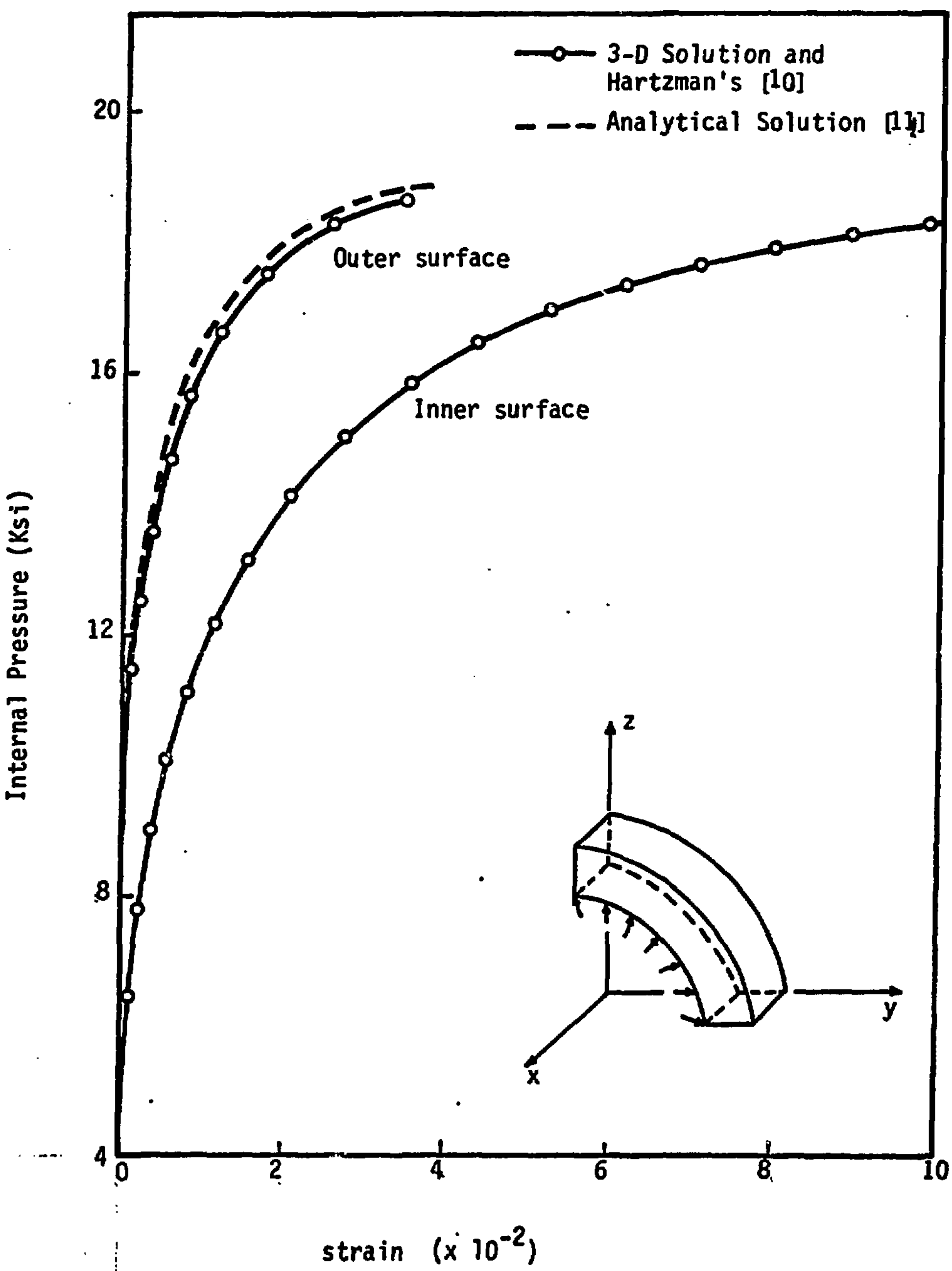

Fig. 2 Internal pressure vs. hoop strains at the inner and outer surface.