Electron Polarization Operators*

D. M. Fradkin and R. H. Good, Jr.

Institute for Atomic Research and Department of Physics

Iowa State University, Ames, Iowa

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I. INTRODUCTION

1. Preliminary Remarks

A variety of operators has been used in the past to characterize the polarization of a free Dirac particle and to remove the degeneracy that remains after momentum and charge have been specified. For example, Darwin[1] used the intrinsic angular momentum operator $\underline{\sigma}$, Mott[2] used the intrinsic magnetic moment operator $\beta\underline{\sigma}$, and Tolhoek and deGroot[3] considered a linear combination of the two $(1 - \beta)\underline{\sigma}$. In the important application of Mott scattering the polarization is described in terms of two wave functions, which in the non-relativistic limit are eigenfunctions of the spin.

However, it has been realized comparatively recently that the polarization can be discussed fluently in terms of a four-vector operator which was first introduced by Bargmann and Wigner[4] as the generators of the little group, a subgroup of the group of Lorentz transformations. It was also found that the polarization could be treated in terms of a related three-vector operator, first introduced by Stech,[5] which for an electron is $\underline{\sigma}$ in the direction of the momentum and $\beta\underline{\sigma}$ otherwise. The properties of these operators were developed especially by Michel and Wightman,[6] Tolhoek,[3] Bouchiat and Michel,[7] and Werle.[8] Some aspects of these operators that make them useful are: 1) they commute with the Hamiltonian and so correspond to an intrinsic property that doesn't change with time; 2) as shown in detail in Sec. 6, any plane wave state of an electron or positron is a completely polarized state.

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(T.R.H.)

The functions used in Mott scattering are eigenfunctions of a component of the three-vector operator so this operator already has a place in that theory. Calculations in the sense of this operator have been made for internal conversion electrons by Becker and Rose,[9] and for beta decay electrons and positrons by Jackson, Treiman and Wyld,[10] by Ebel and Feldman,[11] and by Good and Rose.[12]

It seems now that the understanding of the basic properties of these operators is complete and that a resume of their properties might be of some value. The purpose of this paper is to give a consistent account of the theory of this type of electron polarization, with a few elementary examples. The three-vector polarization operator simplifies calculations involving plane wave states, whereas the four-vector polarization operator is convenient for discussing Lorentz covariance and for taking account of external electromagnetic fields. This paper is correspondingly divided into two parts.

The problem of precession of polarization in external electromagnetic fields has been discussed in the small field limit by Tolhoek,[3] and in the classical (non-quantum) approximation by Bargmann, Michel, and Telegdi,[13] using the equations of motion of angular momentum in the rest frame. A treatment of the precession problem from first principles is given in Secs. 16 and 17.

Another treatment of the basic theory of the polarization operator and several applications are given by Rose.[14] A review of polarization phenomena and experimental techniques has been given by Page.[15]

2. Notation

Units such that $m = c = 1$ are used. Latin indices range from 1 to 3 and Greek from 1 to 4; $X_4 = it$. The symbols $A*$, $A^\dagger$, $\tilde{A}$, denote the complex conjugate, Hermitian conjugate, and transpose of any matrix A.

Abstractly, the Dirac matrices are defined by:

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2 \delta_{\mu\nu} \qquad \text{where} \qquad \gamma_\mu^\dagger = \gamma_\mu .$$

Auxiliary matrices are defined by:

$$\beta = \gamma_4 , \quad \underline{\alpha} = i\beta\underline{\gamma} , \quad \gamma_5 = \gamma_1 \gamma_2 \gamma_3 \gamma_4 \qquad \text{and}$$

$$\underline{\sigma} = i\gamma_4 \gamma_5 \underline{\gamma} = -\tfrac{1}{2} i (\underline{\gamma} \times \underline{\gamma})$$

A specific representation that will be referred to is:

$$\underline{\alpha} = \begin{pmatrix} 0 & \underline{\sigma} \\ \underline{\sigma} & 0 \end{pmatrix} , \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \quad \gamma_5 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} ,$$

where the 2 x 2 $\underline{\sigma}$ are the usual Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The electromagnetic field is described by:

$$A_\mu = (\underline{A}, A_4 = i\phi)$$

$$F_{\mu\nu} = \partial A_\nu / \partial X_\mu - \partial A_\mu / \partial X_\nu \qquad , \text{ and}$$

$$B_i = \tfrac{1}{2} \epsilon_{ijk} F_{jk} , \quad E_k = i F_{k4}$$

The Dirac equation is written as

$$H\Psi = i\hbar (\partial/\partial\tau) \Psi ,$$

where the Hamiltonian is given by:

$$H = \underline{\alpha} \cdot (\underline{p} - e\underline{A}) + \beta + e\phi$$

Here, $\underline{p}$ is $-i\hbar\underline{\nabla}$ and e is the actual charge, negative for the electron. Equivalently one may write:

$$(\gamma_\mu \pi_\mu - i)\Psi = 0 \quad ,$$

where $\pi_\mu = p_\mu - e A_\mu$ and p_μ is $-i\hbar \, \partial/\partial x_\mu$.

The charge conjugation matrix satisfies: $C^* \gamma_\mu^* C = -\gamma_4 \gamma_\mu \gamma_4$,

$C^{-1} = C^* = C^\dagger$. The charge conjugate of a wave function and an operator are:

$$\Psi^c = C^* \Psi^* \quad ; \quad \Omega^c = C^* \Omega^* C$$

3. Physical Interpretations

For the free particle, as an alternative to Dirac's hole theory, one may interpret the four solutions of the Dirac equation as actually describing electrons and positrons (instead of electrons alone). One must then assign the operators $|H|$, $(H/|H|)\underline{p}$, $(H/|H|)(\underline{x} \times \underline{p} + \frac{1}{2}\hbar\underline{\sigma})$, to be the energy, momentum, and angular momentum. The operator $|H|$ is defined in momentum space by $|H| = (p^2 + 1)^{\frac{1}{2}}$, the positive root to be taken. This point of view is carried through consistently in what follows. When equations apply non-uniformly to electrons and positrons, the upper signs apply coherently for electrons and the lower for positrons. The plane wave solutions for the free particle are therefore written as:

$$\Psi_{\pm,\lambda} = \psi_{\pm,\lambda} \exp\left[i\hbar^{-1}(\underline{p}\cdot\underline{x} - Wt)\right]$$

$$= \psi_{\pm,\lambda} \exp\left[\pm i\hbar^{-1}(\underline{q}\cdot\underline{x} - E\tau)\right] \quad,$$

where $\underline{p}$ and W are the eigenvalues of the operators $\underline{p}$ and H , and where

$\underline{q}$ and E are the eigenvalues of the momentum $(H/|H|)\,\underline{p}$ and the

energy $|H|$. Here $\underline{p}$ and W satisfy the equation:

$$p^2 - W^2 = -1 \quad.$$

The $\pm$ subscript on ψ denotes the sign of W, and λ characterizes the two-

fold polarization degeneracy. It is clear that the operator $(H/|H|)$ is +1 for

an electron state, and -1 for a positron state. Also, the energy eigenvalues

$E \doteq \pm$ W are positive.

The wave equation for the free particle is covariant with regard to charge

conjugation. Also one finds that

$$H^c = -H$$

so the operators for energy, momentum, and angular momentum are self

charge-conjugate:

$$|H|^c = |H|$$

$$\left[(H/|H|)\,\underline{p}\right]^c = (H/|H|)\,\underline{p} \quad,$$

and similarly for the angular momentum. The theory is therefore covariant

with regard to charge conjugation both for the wave equation and for the

physical assignments. Since $H^c = -H$, one sees that $(H/|H|)^c$ is

$-H/|H|$ so the charge conjugate of an electron state with momentum

$\underline{q}$ is a positron state with momentum $\underline{q}$. Finally, it is seen that

$$(\pm \underset{\sim}{\alpha} \cdot \underset{\sim}{q} + \beta) \Psi_{\pm, \lambda} = \pm E \Psi_{\pm, \lambda}$$

is the equation satisfied by the plane wave amplitudes in terms of the physical momentum $\underset{\sim}{q}$ and energy E.

II. THREE-VECTOR POLARIZATION OPERATOR

4. Definition

For the free particle, the three-vector polarization operator is defined as:

$$\underset{\sim}{\Theta} = (\underset{\sim}{\sigma} \cdot \hat{\underset{\sim}{p}})(H/|H|) \hat{\underset{\sim}{p}} + \hat{\underset{\sim}{p}} \times (\beta \underset{\sim}{\sigma} \times \hat{\underset{\sim}{p}})$$

(4.1)
$$= \beta \underset{\sim}{\sigma} + (\underset{\sim}{\sigma} \cdot \hat{\underset{\sim}{p}})[(H/|H|) - \beta] \hat{\underset{\sim}{p}}$$

where $\hat{\underset{\sim}{p}}$ is defined in momentum space as $\underset{\sim}{p}/p$. Thus, for electrons/positrons, the three-vector polarization operator is $\pm \underset{\sim}{\sigma}$ in the direction of motion and $\beta \underset{\sim}{\sigma}$ perpendicular to the motion. Explicitly writing out the Hamiltonian and expanding, one may alternatively express the defining equation as:

(4.2)
$$\underset{\sim}{\Theta} = \beta \underset{\sim}{\sigma} - |H|^{-1} \gamma_5 \underset{\sim}{p} - [|H|(|H|+1)]^{-1} (\beta \underset{\sim}{\sigma} \cdot \underset{\sim}{p}) \underset{\sim}{p}$$

One finds that

(4.3)
$$\underset{\sim}{\Theta}^c = C^* \underset{\sim}{\Theta}^a C = \underset{\sim}{\Theta}$$

so the interpretation of $\underset{\sim}{\Theta}$ as the polarization operator also is covariant with respect to charge conjugation.

5. Algebraic Properties

If one introduces a right-handed orthogonal coordinate system, $\hat{\underset{\sim}{\epsilon}}_i$,

such that $\hat{\underline{\epsilon}}_i \cdot \hat{\underline{\epsilon}}_j = \delta_{ij}$, $\hat{\underline{\epsilon}}_i \times \hat{\underline{\epsilon}}_j = \epsilon_{ijk} \hat{\underline{\epsilon}}_k$, then the components of

$\underline{\Theta}$ in this system, $\Theta_i = \underline{\Theta} \cdot \hat{\underline{\epsilon}}_i$, have an algebra similar to that of the

Pauli matrices,

(5.1)
$$\Theta_i \Theta_j = \delta_{ij} + i \epsilon_{ijk} \Theta_k (H/|H|)$$

where H is the free particle Hamiltonian.

Consequently, for any unit vector, $\hat{\underline{s}}$, the above equation implies:

(5.2)
$$(\underline{\Theta} \cdot \hat{\underline{s}})^2 = 1$$

so $\underline{\Theta} \cdot \hat{\underline{s}}$ has eigenvalues ± 1. Also, it is easily verified that $\underline{\Theta}$ is

Hermitian, and that any component of $\underline{\Theta}$ commutes with the (free particle)

Hamiltonian:

(5.3)
$$[\underline{\Theta}, H] = 0$$

Therefore, $\underline{\Theta}$ corresponds to an integral of the motion, and a complete set

of eigenfunctions may be found which are simultaneously eigenfunctions of

the Hamiltonian and $\underline{\Theta} \cdot \hat{\underline{s}}$.

6. Eigenfunctions

Since $\underline{\Theta}$ commutes with the Hamiltonian, a complete set of plane wave

eigenfunctions, $\Psi_{\epsilon, \lambda}(\hat{\underline{s}}) = \psi_{\epsilon, \lambda}(\hat{\underline{s}}) \exp[i \hbar^{-1}(\underline{p} \cdot \underline{x} - \epsilon E t)]$ may be found such

that:

(6.1)
$$H \Psi_{\epsilon, \lambda}(\hat{\underline{s}}) = \epsilon E \Psi_{\epsilon, \lambda}(\hat{\underline{s}}) \qquad , \text{ and}$$

$$\underline{\Theta} \cdot \hat{\underline{s}} \Psi_{\epsilon, \lambda}(\hat{\underline{s}}) = \lambda \Psi_{\epsilon, \lambda}(\hat{\underline{s}}) ,$$

where ϵ, λ are independently ± 1. It is clear that if $\Psi(\hat{\underline{s}})$ is an eigen-function of $\underline{\Theta} \cdot \hat{\underline{s}}$ with eigenvalue +1, then $\Psi(-\hat{\underline{s}})$ is an eigenfunction with eigenvalue -1. One may therefore replace $\Psi_{\epsilon,\lambda}(\hat{\underline{s}})$ by $\Psi_{\epsilon}(\lambda\hat{\underline{s}})$ A system in an eigenstate of $\underline{\Theta} \cdot \hat{\underline{s}}$ with eigenvalue +1 is said to be polar-ized in the $\hat{\underline{s}}$ direction.

One may relate the plane wave eigenfunctions of arbitrary momentum to eigenfunctions in the rest system. Let these eigenfunctions be simultaneous eigenfunctions of the Hamiltonian and $\underline{\Theta} \cdot \hat{\underline{s}}$ in the rest system, so

$$(6.2) \qquad \Psi_{\epsilon}^{o\dagger}(\hat{\underline{s}}) \, \Psi_{\epsilon}^{o}(\hat{\underline{s}}) = 1 \qquad ,$$

$$(6.3) \qquad \beta \, \Psi_{\epsilon}^{o}(\hat{\underline{s}}) = \epsilon \, \Psi_{\epsilon}^{o}(\hat{\underline{s}}) \qquad , \text{ and}$$

$$(6.4) \qquad \beta \, \underline{\sigma} \cdot \hat{\underline{s}} \, \Psi_{\epsilon}^{o}(\hat{\underline{s}}) = \Psi_{\epsilon}^{o}(\hat{\underline{s}})$$

Defining the projection operators, P_{ϵ}, by

$$(6.5) \qquad P_{\epsilon} = \tfrac{1}{2}\left[1 + \epsilon (H/E) \right]$$
$$= (2E)^{-1}(E + \epsilon \, \underline{\alpha} \cdot \underline{p} + \epsilon \beta)$$

where H, E, $\underline{p}$, are the eigenvalues in the laboratory system, one finds, using Eqs. (6.3), (6.4), and (4.1), that:

$$(6.6) \qquad H \left(P_{\epsilon} \Psi_{\epsilon}^{o}(\hat{\underline{s}}) \right) = \epsilon E \left(P_{\epsilon} \Psi_{\epsilon}^{o}(\hat{\underline{s}}) \right) \qquad , \text{ and}$$

$$(6.7) \qquad \underline{\Theta} \cdot \hat{\underline{s}} \left(P_{\epsilon} \Psi_{\epsilon}^{o}(\hat{\underline{s}}) \right) = \underline{\Theta} \cdot \hat{\underline{s}} \, P_{\epsilon}(\beta \underline{\sigma} \cdot \hat{\underline{s}}) \Psi_{\epsilon}^{o}(\hat{\underline{s}}) = \left(P_{\epsilon} \Psi_{\epsilon}^{o}(\hat{\underline{s}}) \right) .$$

Also, is is known that

$$P_\epsilon^\dagger = P_\epsilon \quad , \quad P_\epsilon P_\epsilon = P_\epsilon$$

so

$$\left(P_\epsilon \Psi_\epsilon^o\right)^\dagger \left(P_\epsilon \Psi_\epsilon^o\right) = \Psi_\epsilon^{o\dagger} P_\epsilon \Psi_\epsilon^o = (2E)^{-1}(E+1)$$

where Eqs. (6.2), (6.3), and the fact that $\Psi_\epsilon^{o\dagger} \underset{\sim}{\alpha} \cdot \underset{\sim}{p} \Psi_\epsilon^o = 0$, which is easily proved from Eq. (6.3), is used. Therefore, the wave function given by

(6.8)
$$\Psi_\epsilon(\hat{\underline{s}}) = \left[2E/(E+1)\right]^{\frac{1}{2}} P_\epsilon \Psi_\epsilon^o(\hat{\underline{s}})$$

satisfies:

(6.9)
$$\Psi_\epsilon^\dagger(\hat{\underline{s}}) \, \Psi_\epsilon(\hat{\underline{s}}) = 1$$

(6.10)
$$H \, \Psi_\epsilon(\hat{\underline{s}}) = \epsilon E \, \Psi_\epsilon(\hat{\underline{s}}) \quad , \text{ and}$$

(6.11)
$$\underset{\sim}{\sigma} \cdot \hat{\underline{s}} \, \Psi_\epsilon(\hat{\underline{s}}) = \Psi_\epsilon(\hat{\underline{s}})$$

These functions $\Psi_\epsilon(\hat{\underline{s}})$ actually are proportional to the rest-system functions $\Psi_\epsilon^o(\hat{\underline{s}})$ Lorentz transformed to the laboratory frame. The wave function amplitudes, Ψ^o in the rest system and Ψ in the laboratory system, are related by

$$\Psi = \Lambda \Psi^o \quad ,$$

where

$$\Lambda^{-1} \gamma_\mu \Lambda = a_{\mu\rho} \gamma_\rho$$

and the transformation coefficients are

$$a_{11} = a_{44} = E \quad , \text{ and}$$

$$a_{14} = -a_{41} = -i \, q$$

the X -axis having been chosen in the $\hat{q}$ direction.

In this case the transformation matrix is found to be

$$\Lambda = [2(E+1)]^{-\frac{1}{2}} (E + \alpha_x \hat{q} + 1)$$

When this is applied to the function, $\psi_\epsilon^0(\hat{\underline{s}})$, one can replace $\hat{q}$ by $\epsilon \hat{p}$ and 1 by $\epsilon \beta$ so that

$$\begin{aligned}
\psi &= \Lambda \psi_\epsilon^0(\hat{\underline{s}}) \\
&= [2(E+1)]^{-\frac{1}{2}} (E + \epsilon \alpha_x \hat{p} + \epsilon \beta) \psi_\epsilon^0(\hat{\underline{s}}) \\
&= E^{\frac{1}{2}} [2E/(E+1)]^{\frac{1}{2}} P_\epsilon \psi_\epsilon^0(\hat{\underline{s}})
\end{aligned}$$

and this proves the assertion. One sees that if a particle has polarization $\hat{\underline{s}}$ in the laboratory system then it has the same polarization $\hat{\underline{s}}$ in the rest system. In other words, the polarization of an electron beam is the same no matter from which Lorentz frame the beam is viewed.

The explicit plane wave eigenfunctions satisfying Eqs. (6.2), (6.3), and (6.4), in the specific representation of Sec. 2, are:

$$\psi_+^0(\hat{\underline{s}}) = \begin{pmatrix} \cos\frac{1}{2}\theta\, e^{-\frac{1}{2}i\phi} \\ \sin\frac{1}{2}\theta\, e^{\frac{1}{2}i\phi} \\ 0 \\ 0 \end{pmatrix} e^{-i\tau/\hbar}$$

and

(6.12)

$$\psi_-^0(\hat{\underline{s}}) = \begin{pmatrix} 0 \\ 0 \\ -\sin\frac{1}{2}\theta\, e^{-\frac{1}{2}i\phi} \\ \cos\frac{1}{2}\theta\, e^{\frac{1}{2}i\phi} \end{pmatrix} e^{i\tau/\hbar}$$

where θ , ϕ are the polar, azimuthal angles of $\hat{\underline{s}}$. From Eq. (6.8),

the corresponding arbitrary Lorentz frame eigenfunctions are:

$$\Psi_+(\mathbf{p}, \hat{s}) = \left[\cos\tfrac{1}{2}\theta\, e^{-\tfrac{1}{2}i\phi} U_{+\frac{1}{2}}(\mathbf{p}) + \sin\tfrac{1}{2}\theta\, e^{\tfrac{1}{2}i\phi} U_{-\frac{1}{2}}(\mathbf{p})\right] e^{i(\mathbf{p}\cdot\mathbf{x} - Et)/\hbar}$$

and

(6.13)
$$\Psi_-(\mathbf{p}, \hat{s}) = \left[\cos\tfrac{1}{2}\theta\, e^{\tfrac{1}{2}i\phi} V_{-\frac{1}{2}}(\mathbf{p}) - \sin\tfrac{1}{2}\theta\, e^{-\tfrac{1}{2}i\phi} V_{+\frac{1}{2}}(\mathbf{p})\right] e^{i(\mathbf{p}\cdot\mathbf{x} + Et)/\hbar}$$

,

where $U_{\pm\frac{1}{2}}$, $V_{\pm\frac{1}{2}}$ are the functions:

$$U_{\pm\frac{1}{2}}(\mathbf{p}) = [2E(E+1)]^{-\frac{1}{2}} \begin{pmatrix} (E+1)\,\chi_{\pm\frac{1}{2}} \\ \sigma\cdot\mathbf{p}\,\chi_{\pm\frac{1}{2}} \end{pmatrix}$$

and

(6.14)

$$V_{\pm\frac{1}{2}}(\mathbf{p}) = [2E(E+1)]^{-\frac{1}{2}} \begin{pmatrix} -\sigma\cdot\mathbf{p}\,\chi_{\pm\frac{1}{2}} \\ (E+1)\,\chi_{\pm\frac{1}{2}} \end{pmatrix}$$

,

and $\chi_{\pm\frac{1}{2}}$ are the familar "spin-up", "spin-down" functions of non-relativistic theory,

(6.15)
$$\chi_{+\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad \chi_{-\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

.

The solutions given in Heitler[16] are $\Psi_\epsilon(p_z\hat{e}_3, \lambda\hat{e}_3)$ in the present notation, where Heitler's $E \gtrless 0$ corresponds to $\epsilon = \pm 1$ here, and where Heitler's $\uparrow$ and $\downarrow$ correspond to $\epsilon\lambda = 1$ and $\epsilon\lambda = -1$ here. The functions $\Psi_\pm$ of Eq. (6.13) are a similarity transformation from those of deGroot and Tolhoek.[17] They write A and B in place of $\cos\tfrac{1}{2}\theta\, e^{-\tfrac{1}{2}i\phi}$ and $\sin\tfrac{1}{2}\theta\, e^{\tfrac{1}{2}i\phi}$, and express the functions in terms of the physical momentum. Since they define the direction of polarization θ, ϕ by

(6.16)
$$B/A = \tan\tfrac{1}{2}\theta\, e^{i\phi}$$

,

it is clear that their direction of polarization coincides with the vector $\hat{\underline{s}}$ used here.

In general, if the system is in a state described by the wave function $\Psi_\epsilon(\hat{\underline{s}})$, then the expectation value of $\underline{\Theta}$ is $\hat{\underline{s}}$. Symbolically,

(6.17)
$$\Psi_\epsilon^\dagger(\hat{\underline{s}}) \, \underline{\Theta} \, \Psi_\epsilon(\hat{\underline{s}}) = \hat{\underline{s}}$$

This may be easily proved, since from Eq. (5.1),

$$(\underline{\Theta} \cdot \hat{\underline{s}})\underline{\Theta} + \underline{\Theta}(\underline{\Theta} \cdot \hat{\underline{s}}) = 2\hat{\underline{s}}$$

and operation with $\Psi_\epsilon^\dagger(\hat{\underline{s}})$, $\Psi_\epsilon(\hat{\underline{s}})$ on the left and right yields the result by virtue of Eq. (6.11).

Also, any plane wave electron or positron state is a state completely polarized in some direction $\hat{\underline{s}}$. To show this, consider the expansion of an electron state χ in terms of $\Psi_+(\hat{\underline{e}}_3)$, $\Psi_+(-\hat{\underline{e}}_3)$, which for convenience are studied in the representation of Eq. (6.13). Then

(6.18)
$$\chi(\underline{p}) = a\, e^{i\alpha} U_{-\frac{1}{2}}(\underline{p}) + b\, e^{i\beta} U_{+\frac{1}{2}}(\underline{p})$$

where a, b, α, β, are real numbers, and $a^2 + b^2 = 1$. If one takes

$$a\, e^{i\alpha} = e^{\frac{1}{2}i(\alpha+\beta)} \sin\tfrac{1}{2}\theta\; e^{\frac{1}{2}i\phi}$$
$$b\, e^{i\beta} = e^{\frac{1}{2}i(\alpha+\beta)} \cos\tfrac{1}{2}\theta\; e^{-\frac{1}{2}i\phi} \qquad \text{and}$$

then this has a solution, θ, ϕ where $\sin\tfrac{1}{2}\theta = a$, $\cos\tfrac{1}{2}\theta = b$, and $\phi = \alpha - \beta$. Consequently,

(6.19)
$$\chi(\underline{p}) = e^{\frac{1}{2}i(\alpha+\beta)} \Psi_+(\underline{p}, \hat{\underline{s}})$$

and since wave functions are only defined to within a phase factor, the assertion is proved.

7. Foldy-Wouthuysen Representation

The three-vector polarization operator assumes an especially simple form in the Foldy-Wouthuysen [18] representation and many of its properties become evident in that representation. In the specific representation of Sec. 2 the free particle Hamiltonian contains even and odd operators--odd operators being matrix operators that mix the upper and lower two component spaces of the wave function (e. g. γ_5 , $\underline{\alpha}$), even operators being those which do not effect this mixing (e. g. β , $\underline{\sigma}$). The purpose of the FW transformation is to obtain a representation in which the Hamiltonian is an even operator, so that electron and positron solutions are cleanly separated into the aforementioned two-component spaces.

Any operator, A, in the FW representation is:

$$(7.1) \qquad A_{FW} = e^{iS} A e^{-iS}$$

where the desired unitary transformation is explicitly:

$$(7.2) \qquad e^{\pm iS} = [2|H|(|H|+1)]^{-\frac{1}{2}} \beta [\beta(|H|+1) \pm \underline{\alpha}\cdot\underline{p}]$$

Performing the indicated transformation, one obtains:

$$(7.3) \qquad \underline{\Theta}_{FW} = \beta \underline{\sigma} \quad ; \quad H_{FW} = \beta |H|$$

The upper/lower two-component spaces in the FW representation are associated with the Pauli non-relativistic two-component theory of the electron/positron. One sees, therefore, that the Pauli theory equivalent of the three-vector polarization operator is $\pm\underline{\sigma}$ ($\beta = \pm 1$ for electrons/positrons). It is also

seen that the algebraic properties of the three-vector polarization operator follow easily from Eq. (7.3).

8. Density Matrix

In scattering problems, where incoming and outgoing particles are treated asymptotically as free particle wave states, it is useful to have an expression for the statistical density matrix[19] as a function of the average polarization $\underset{\sim}{P}$ of the considered ensemble of particles.

In general, the electron/positron density matrix for an ensemble of single particle systems with definite energy E and momentum $\underset{\sim}{q}$ is given by

$$(8.1) \qquad \rho_{\pm} = \sum_{\lambda=1,-1} P_{\pm\lambda}\, \Psi_{\pm}(\lambda \hat{\underset{\sim}{s}})\, \Psi_{\pm}^{\dagger}(\lambda \hat{\underset{\sim}{s}})$$

where $P_{\pm\lambda}$ is the probability the particle is in polarization state $\lambda \hat{\underset{\sim}{s}}$.

Using Eq. (6.8), one may write

$$\rho_{\pm} = [2E/(E+1)]\, P_{\pm} \left(\sum_{\lambda=1,-1} P_{\pm\lambda}\, \Psi_{\pm}^{o}(\lambda \hat{\underset{\sim}{s}})\, \Psi_{\pm}^{o\dagger}(\lambda \hat{\underset{\sim}{s}}) \right) P_{\pm}$$

which, in consequence of Eq. (6.3), is equivalent to

$$(8.2) \qquad \rho_{\pm} = E\,[2(E+1)]^{-1}\, P_{\pm}\, (1\pm\beta) \left(\sum P_{\pm\lambda}\, \Psi_{\pm}^{o}(\lambda \hat{\underset{\sim}{s}})\, \Psi_{\pm}^{o\dagger}(\lambda \hat{\underset{\sim}{s}}) \right) (1\pm\beta)\, P_{\pm}$$

In the specific representation of the Dirac matrices given in Sec. 2,
$(1\pm\beta) \sum P_{\pm\lambda}\, \Psi_{\pm}^{o}(\lambda \hat{\underset{\sim}{s}})\, \Psi_{\pm}^{o\dagger}(\lambda \hat{\underset{\sim}{s}})$ is of the form

$$\begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 0 \\ 0 & X \end{pmatrix}$$

for upper and lower signs, respectively. Here, X is a 2 x 2 matrix and

therefore may be written as

$$(1 \pm \beta)(A_\pm + \underline{B}_\pm \cdot \underline{\sigma})$$

where $A_\pm$ and $\underline{B}_\pm$ are still to be determined. This gives

(8.3)
$$\rho_\pm = E\left[2(E+1)\right]^{-1} P_\pm (1 \pm \beta)(A_\pm + \underline{B}_\pm \cdot \underline{\sigma})(1 \pm \beta) P_\pm$$

Finally $A_\pm$ and $\underline{B}_\pm$ are evaluated from the relations

$$\text{Tr}(\rho_\pm) = 1$$

$$\text{Tr}(\rho_\pm \Theta_\pm) = \underline{P} , \qquad |\underline{P}| \leq 1$$

which yield the result that $A_\pm = \frac{1}{2}$, $\underline{B}_\pm = \pm \frac{1}{2}\underline{P}$. Consequently,

the plane wave density matrix is:

(8.4)
$$\rho_\pm = E\left[4(E+1)\right]^{-1} P_\pm (1 \pm \beta)(1 \pm \underline{P} \cdot \underline{\sigma})(1 \pm \beta) P_\pm$$

$$= (4E)^{-1}\Big[E \pm \beta + \underline{\alpha} \cdot \underline{q} \pm \underline{P} \cdot \underline{\sigma} \mp \gamma_5 \underline{P} \cdot \underline{q}$$

$$+ E(\underline{P} \cdot \beta \underline{\sigma}) + i\beta \underline{\alpha} \cdot (\underline{P} \times \underline{q}) \pm (E+1)^{-1}(\underline{P} \cdot \underline{q})(\underline{\sigma} \cdot \underline{q})$$

$$- (E+1)^{-1}(\underline{P} \cdot \underline{q})(\beta \underline{\sigma} \cdot \underline{q}) \Big]$$

The expression for the density matrix containing the projection operators

was given by Mühlschlagel and Köppe;[20] the expanded form by Tolhoek and

deGroot.[3]

9. Covariant Description

Michel and Wightman[6] introduced the operator

(9.1)
$$\Theta_{MW} = i \gamma_5 \gamma_\mu n_\mu$$

to describe the polarization of a plane wave state of a free particle. Here n_μ is defined to be a four-vector with components ($\hat{\underline{s}}$, O) in the rest system of the particle. [It is clear that $n_\mu n_\mu$ is unity and that $q_\mu n_\mu$ is zero since q_μ is (O , i) in the rest system.] This operator is equivalent to $\underline{\Theta} \cdot \hat{\underline{s}}$ as shown below.

The components of n_μ in the laboratory system are

$$\underline{n} = \hat{\underline{s}} + (E+1)^{-1}(\underline{q} \cdot \hat{\underline{s}})\, \underline{q}$$

(9.2)

$$n_4 = i\, \underline{q} \cdot \hat{\underline{s}}$$

As long as n_μ directly multiplies a plane wave solution of the Dirac equation one can replace it by $n_{\mu\,op}$ defined by

$$\underline{n}_{op} = \hat{\underline{s}} + (|H|+1)^{-1}(\underline{p} \cdot \hat{\underline{s}})\, \underline{p}$$

(9.3)

$$n_{4\,op} = i\, \underline{p} \cdot \hat{\underline{s}}\ (H/|H|)$$

Here $\underline{p}$ is $-i\hbar \underline{\nabla}$ and these operators have the properties

$$n_{\mu\,op}\, n_{\mu\,op} = 1$$

(9.4)

$$\underline{n}_{op} \cdot \underline{p} + i\, n_{4\,op}\, H$$

A direct consequence of Eq. (4.2) is that

(9.5)
$$i\, \gamma_5 \gamma_\mu\, n_{\mu\,op} = \underline{\Theta} \cdot \hat{\underline{s}}$$

One sees then that Θ_{MW} , defined for a plane wave state, is equivalent to

$\underline{\Theta} \cdot \hat{\underline{s}}$ when operating on the state function.

III. FOUR-VECTOR POLARIZATION OPERATOR

10. Definition

For the free particle, the four-vector polarization operator T_μ is defined to be

$$\underline{T} = \gamma_5 (i\underline{\gamma} - \underline{p})$$

$$= \beta \underline{\sigma} - \gamma_5 \underline{p}$$

(10.1)

$$T_4 = \gamma_5 (i\gamma_4 - iH)$$

$$= i \underline{\sigma} \cdot \underline{p}$$

This is closely related to the operator

(10.2)
$$T_{\mu BW} = -\tfrac{1}{2} \epsilon_{\mu\rho\pi\nu} \gamma_\rho \gamma_\pi p_\nu$$

which was first discussed by Bargmann and Wigner.[4] In fact, as a consequence of the relation

(10.3)
$$\gamma_\mu \gamma_\nu p_\nu = p_\mu - \tfrac{1}{2} \epsilon_{\mu\rho\pi\nu} \gamma_5 \gamma_\rho \gamma_\pi p_\nu \quad ,$$

one finds that

(10.4)
$$T_{\mu BW} = \gamma_5 (i\gamma_\mu - p_\mu) + \gamma_5 \gamma_\mu (\gamma_\nu p_\nu - i) \quad .$$

Therefore, when applied to solutions of the Dirac equation, the operators are equivalent.

11. Generators of the Little Group

The components T_μ are the generators of the little group, the subgroup of homogeneous Lorentz transformations that leave the four-vector p_μ of a plane wave state unchanged. This was pointed out in the general case of arbitrary spin and mass by Bargmann and Wigner.[4]

To see this in detail here, consider the infinitesimal Lorentz transformation

$$(11.1) \qquad x'_\mu = a_{\mu\nu} x_\nu = (\delta_{\mu\nu} + \xi_{\mu\nu}) x_\nu \qquad ,$$

where $\xi_{\mu\nu} = -\xi_{\nu\mu}$ are infinitesimals. The corresponding wave function transformation is

$$(11.2) \qquad \Psi'(x') = \Lambda \Psi(x) \qquad ,$$

where infinitesimally

$$(11.3) \qquad \Lambda = 1 + \frac{1}{4} \xi_{\sigma\rho} \gamma_\sigma \gamma_\rho \qquad .$$

Substituting Eqs. (11.1) and (11.3) into (11.2), and expanding $\Psi(x_\mu - \xi_{\mu\nu} x_\nu)$ about $\Psi(x_\mu)$ in a Taylor's series, one obtains (to first order in infinitesimals)

$$(11.4) \qquad \Psi'(x) = \left[1 + \xi_{\sigma\rho}\left(\frac{1}{4} \gamma_\sigma \gamma_\rho - x_\rho \, \partial/\partial x_\sigma\right) \right] \Psi(x) .$$

For a plane wave state of a free particle of specified four-vector p_μ ,

the wave function has the form

$$(11.5) \qquad \Psi(x) = \psi(p_\nu) \, e^{i p_\mu x_\mu / \hbar}$$

If only those homogeneous proper Lorentz transformations that leave p_μ unchanged are considered, then

$$(11.6) \qquad \xi_{\mu\nu} \, p_\nu = 0 \qquad ,$$

so that

$$\Psi'(x) = (1 + \tfrac{1}{4} \xi_{\sigma\rho} \gamma_\sigma \gamma_\rho) \Psi(x)$$

Using Eqs. (10.2) and (11.6), it may easily be verified that:

$$\varepsilon_{\nu\mu\rho\lambda} \xi_{\mu\rho} T_\lambda \Psi(x) = \xi_{\sigma\rho} \gamma_\sigma \gamma_\rho \, p_\nu \Psi(x) \, .$$

Thus, for the eigenvalue $p_\nu \neq 0$, one obtains

$$(11.7) \qquad \Psi'(x) = [1 + (4 p_\nu)^{-1} \varepsilon_{\nu\mu\rho\lambda} \xi_{\mu\rho} T_\lambda] \Psi(x) \quad ; \text{no sum on } \nu \, .$$

Eq. (11.6) implies that only three infinitesimal parameters are independent, which, for a given ν , may be taken as $\varepsilon_{\nu\mu\rho\lambda} \xi_{\mu\rho}$. Therefore, the operators T_λ are the generators of the little group.

12. Algebraic Properties

The operators T_μ satisfy the equations

$$(12.1) \qquad T_\mu T_\mu = 3 \qquad ,$$

$$(12.2) \qquad \underset{\sim}{T} \cdot \underset{\sim}{p} + i T_4 H = 0 \qquad ,$$

$$(12.3) \qquad [T_\mu, H]_- = 0$$

The operators T_i are of primary interest because Eq. (12.2) can be used to express T_4 in terms of them. Their algebra is involved with that of the operators S_i defined by

(12.4)
$$S_i = \sigma_i + \epsilon_{ijk}\, r_j\, p_k$$

In detail one finds:

$$[T_i, T_j]_- = 2i\, \epsilon_{ijk}\, S_k \quad ,$$

$$[T_i, S_j]_- = 2i\left(\epsilon_{ijk} T_k + \epsilon_{j\ell m} T_\ell\, p_m\, p_i\right) \quad ,$$

$$[S_i, S_j]_- = 2i\, \epsilon_{ijk}\left(S_k + S_\ell\, p_\ell\, p_k\right) \quad ,$$

$$[T_i, T_j]_+ = 2\left(\delta_{ij} + p_i\, p_j\right) \quad ,$$

$$[S_i, T_j]_+ = 2\, \delta_{ij}\, H \quad ,$$

$$[S_i, S_j]_+ = 2\left[\delta_{ij}(1 + p^2) - p_i\, p_j\right] \quad ,$$

$$[T_i, H]_+ = 2\left(S_i + S_j\, p_j\, p_i\right) \quad , \text{ and}$$

$$[S_i, H]_+ = 2(1 + p^2)\, T_i - 2 T_j\, p_j\, p_i \quad .$$

There is the relation

$$\underset{\sim}{T} \cdot \underset{\sim}{J} = \underset{\sim}{J} \cdot \underset{\sim}{T} = \hbar\left(K + \tfrac{1}{2} H\right) \quad ,$$

between the polarization operators, the angular momentum operator

$$\underline{J} = \underline{x} \times \underline{p} + \tfrac{1}{2}\hbar\,\underline{\sigma} \quad ,$$

and Dirac's operator[21]

$$\hbar K = \beta\left[\underline{\sigma}\cdot(\underline{x}\times\underline{p}) + \hbar\right] \quad .$$

The charge conjugated four-vector polarization operator is

$$\underline{T}^c = C^*\underline{T}^*C = \underline{T}$$

$$T_4^c = C^* T_4^* C = -T_4$$

13. Connection with the Three-Vector Operator

The relation between the operators is

$$\underline{T} = \underline{\Theta} + (|H|+1)^{-1}(\underline{\Theta}\cdot\underline{p})\underline{p} \quad ,$$

(13.1)

$$T_4 = i\,(H/|H|)\,\underline{\Theta}\cdot\underline{p} \quad ,$$

as is easily verified. The connection between T_μ and $\underline{\Theta}$ is the same as the one between $m_{\mu\,op}$ and $\hat{\underline{s}}$, Eq. (9.3). Combining Eqs. (10.1), (9.4), and (9.5), one finds that

(13.2)
$$T_\mu\, m_{\mu\,op} = \underline{\Theta}\cdot\hat{\underline{s}} \quad .$$

Therefore the wave function $\Psi_{\pm}(\hat{\underline{s}})$ describing a plane wave state polarized in the $\hat{\underline{s}}$ direction is also an eigenstate of $T_\mu\, m_{\mu\,op}$,

(13.3) $\qquad T_\mu \, n_{\mu \, op} \, \Psi_\pm(\hat{\underline{s}}) = \Psi_\pm(\hat{\underline{s}})$

To find the expected value of T_μ one observes from Eq. (5.1), that

$$[\underline{\varrho}, \underline{\varrho} \cdot \hat{\underline{s}}]_+ = 2 \hat{\underline{s}}$$

so that Eqs. (13.1) yield

(13.4) $\qquad [T_\mu, T_\nu \, n_{\nu \, op}]_+ = [T_\mu, \underline{\varrho} \cdot \hat{\underline{s}}]_+$

$$= 2 \, n_{\mu \, op}$$

The result of taking the expected value of this last equation is

(13.5) $\qquad \Psi_\pm^\dagger(\hat{\underline{s}}) \, T_\mu \, \Psi_\pm(\hat{\underline{s}}) = \Psi_\pm^\dagger(\hat{\underline{s}}) \, n_{\mu \, op} \, \Psi_\pm(\hat{\underline{s}})$

$$= n_\mu$$

where n_μ and $\hat{\underline{s}}$ are related by Eq. (9.2).

It is clear from Eqs. (6.17), (13.5), and (9.2) that the following inter-

pretation of these operators can now be made: For a plane wave state, the

three-vector polarization operator $\underline{\varrho}$ is the laboratory system operator

corresponding to the direction of polarization $\hat{\underline{s}}$ in the rest system of the

particle; the four-vector polarization operator is the laboratory system opera-

tor corresponding to the four-vector which is the Lorentz transform of

$(\hat{\underline{s}}, 0)$ from the rest system.

14. Lorentz Transformation Properties

Since T_μ commutes with the Hamiltonian the expectation values

(14.1) $\qquad \langle T_\mu \rangle = \int \Psi^\dagger T_\mu \Psi$

where Ψ is any solution of the Dirac equation and the integral extends over all space, are constant in time. The $\langle T_i \rangle$ are real and $\langle T_4 \rangle$ is pure imaginary. It is interesting to inquire into the tensor transformation properties of these quantities.

Let the expectation values be defined in a different coordinate system by

$$(14.2) \qquad \langle T_\mu \rangle' = \int \Psi'^\dagger T_\mu \Psi' \qquad .$$

It is immediately clear that for space rotations $\langle T_i \rangle$ is a vector and $\langle T_4 \rangle$ a scalar. Pure Lorentz transformations can be easily discussed infinitesimally. The transformation is

$$\underline{x}' = \underline{x} - \underline{v}\,t \qquad ,$$

$$t' = t - \underline{v}\cdot\underline{x} \qquad , \text{ and}$$

$$(14.3) \qquad \Psi' = \Psi + \underline{v}\cdot\left(t\,\underline{\nabla}\Psi + \underline{x}\,\partial\Psi/\partial t - \tfrac{1}{2}\underline{\alpha}\,\Psi\right) \qquad .$$

On substituting Eq. (14.3) into Eq. (14.2), replacing $\partial\Psi/\partial t$ by $-i\hbar H\Psi$, and simplifying, one finds

$$\langle T_i \rangle' = \langle T_i \rangle + i\,v_i\,\langle T_4 \rangle \qquad \text{and}$$

$$\langle T_4 \rangle' = \langle T_4 \rangle - i\,v_j\,\langle T_j \rangle \qquad ,$$

which are the correct rules for a Lorentz four-vector.

For the space reflection

$$\underline{x}' = -\underline{x} \quad , \quad t' = t$$

one may consider either the usual wave function transformation

$$\Psi'(x') = i\gamma_4 \Psi(x)$$

or the Wigner-Landau combined inversion

$$\Psi'(x') = i\gamma_4 C^* \Psi^*(x)$$

In either case the result is

$$\langle T_i \rangle' = \langle T_i \rangle \ , \quad \langle T_4 \rangle' = -\langle T_4 \rangle \ .$$

Finally for the time reflection

$$\underline{x}' = \underline{x} \ , \quad t' = -t \qquad ;$$

the wave function transformation rule is

$$\Psi'(x') = \gamma_4 \gamma_5 C^* \Psi^*(x)$$

and it is found that

$$\langle T_i \rangle' = -\langle T_i \rangle \ , \quad \langle T_4 \rangle' = \langle T_4 \rangle \ .$$

In summary, for the general Lorentz transformation

$$X'_\mu = a_{\mu\nu} X_\nu \qquad ,$$

the expectation values transform according to the rule

$$\langle T_\mu \rangle' = (\det a) \, a_{\mu\nu} \langle T_\nu \rangle \ .$$

The reflection properties of $\langle T_i \rangle$ are the same as those of angular momentum.

15. Effect of External Fields

The four-vector polarization operator can be generalized to the case of a Dirac electron in an external electromagnetic field. The operator is then defined by

$$T_i = \gamma_5 (i \gamma_i - \pi_i)$$

$$= \beta \sigma_i - \gamma_5 \pi_i \qquad ,$$

(15.1)

$$T_4 = \gamma_5 (i \gamma_4 - i H + i e \phi)$$

$$= i \underline{\sigma} \cdot \underline{\pi}$$

This operator has the properties

(15.2)
$$T_\mu T_\mu = 3 + e \hbar \, \underline{\sigma} \cdot \underline{B} \qquad ,$$

(15.3)
$$\underline{\pi} \cdot \underline{T} + i (H - e \phi) T_4 = e \hbar \, \underline{\alpha} \cdot \underline{B} \qquad ,$$

(15.4)
$$[\underline{T}, H]_- = i e \hbar \, \underline{\sigma} \times \underline{B} - i e \hbar \, \gamma_5 (\underline{E} + \partial \underline{A} / \partial \tau) \qquad , \text{ and}$$

(15.5)
$$[T_4, H]_- = - e \hbar \, \underline{\sigma} \cdot (\underline{E} + \partial \underline{A} / \partial \tau) \qquad .$$

Consequently the Heisenberg equations of motion are found to be

$$d \underline{T} / d\tau = e (\underline{\sigma} \times \underline{B} - \gamma_5 \underline{E})$$

(15.6)

$$d T_4 / d\tau = i e \, \underline{\sigma} \cdot \underline{E}$$

These equations can be accumulated into the form

(15.7)
$$d T_\mu / d\tau = i e \, \gamma_4 \, \gamma_5 \, \gamma_\mu \, F_{\mu\nu}$$

In these three special cases there are polarization integrals of the motion:

(1) If $\underline{E}$ is zero, T_4 is an integral.

(2) If $\underline{E}$ is zero and the magnetic field has a fixed direction $\underline{\hat{B}}$, then T_4 and $\underline{T} \cdot \underline{\hat{B}}$ are integrals.

(3) If $\underline{B}$ is zero and the component of $\underline{E}$ in some fixed direction $\underline{\hat{\kappa}}$ is zero, then $\underline{T} \cdot \underline{\hat{\kappa}}$ is an integral.

16. Classical Equations of Motion

Eq. (15.7) gives the equations of motion of the four-vector polarization. One is often interested in the analogous equations of motion for the expectation value of the polarization of a particle which is localized so that the wave function is a packet. In this limit the rate of change of the polarization can be expressed in terms of the external fields and the polarization itself.

One considers a particle corresponding to a wave function which is essentially non-zero only over a small spacial extent. This packet moves through space as the particle moves through space. It is assumed that the external electromagnetic fields and potentials do not vary appreciably within the packet and may be represented by mean values, where the mean values change with time as the packet moves through space. The wave function of the physical particle has the form,

(16.1)
$$\Psi(\underline{x}, \tau) = u(\underline{x}, \tau) \exp\left[-i \hbar^{-1} \phi(\underline{x}, \tau) \right]$$

where f is real and positive. Then,

$$H\Psi = [\partial/\partial\tau\ f(\underline{x},\tau)]\Psi + i\hbar\ [\partial/\partial\tau\ u(\underline{x},\tau)]\exp\left[-i\hbar^{-1}f(\underline{x},\tau)\right]\ .$$

It will be assumed that,

(16.2)
$$\left|\ i\hbar\ \frac{\partial u(\underline{x},\tau)}{\partial\tau}\ e^{-i f(\underline{x},\tau)/\hbar}\ \right| <<< \left|\ \frac{\partial f(\underline{x},\tau)}{\partial\tau}\ \Psi\ \right|\ ,$$

so that to a good approximation,

(16.3)
$$H\Psi = [\partial/\partial\tau\ f(\underline{x},\tau)]\Psi$$

Moreover, within the small packet it will be assumed that $[\partial/\partial\tau\ f(\underline{x},\tau)]$ may be represented by a mean value, $\bar{E}(\tau)$, so that

(16.4)
$$H\Psi = \bar{E}(\tau)\Psi$$

Under these assumptions, one finds that for any Hermitian operator Q,
[potentials and fields before integrals or average values represent mean values within the packet]

$$\int\Psi^{\dagger}Q(H-e\phi)\Psi\ d\underline{x} + \int[Q(H-e\phi)\Psi]^{\dagger}\Psi d\underline{x} = 2(\bar{E}-e\phi)\int\Psi^{\dagger}Q\Psi\ d\underline{x}$$

or

(16.5)
$$\langle[Q,\ H-e\phi]_{+}\rangle = 2(\bar{E}-e\phi)\langle Q\rangle$$

It is useful to define the quantities

(16.6)
$$\bar{\gamma} = (\bar{E}-e\phi) = -i\langle\pi_4\rangle \qquad\qquad ,\ and$$

(16.7)
$$\underline{\ell} = \bar{\gamma}^{-1} \langle \underline{\pi} \rangle$$

One expects, from the correspondence principle, the relation

(16.8)
$$\bar{\gamma} = (1 - \ell^2)^{-\frac{1}{2}}$$

to be valid to a high degree of approximation, since this implies that

$$\langle \pi_\mu \rangle \langle \pi_\mu \rangle = -1$$

Using Eq. (16.6), Eq. (16.5) may be written as

(16.9)
$$\langle Q \rangle = (2\bar{\gamma})^{-1} \langle [Q, H - e\phi]_+ \rangle$$

This equation for the expectation value of a Hermitian operator when a localized wave function represents a particle is used to relate operator equations to packet observables.

By application of Eq. (16.9) and using the result

$$[H - e\phi, i\gamma_4\gamma_5\gamma_\mu]_+ = 2T_\mu$$

it is found that:

(16.10)
$$\langle i\gamma_4\gamma_5\gamma_\mu \rangle = \bar{\gamma}^{-1} \langle T_\mu \rangle$$

(16.11)
$$\langle \underline{\alpha} \rangle = \underline{\ell} \qquad \text{, and}$$

(16.12)
$$\langle \beta \rangle = \bar{\gamma}^{-1}$$

Substituting Eq. (16.10) into Eq. (15.7) and factoring the fields out, one finds the equation of motion for the polarization (average) of a particle as seen in the laboratory system, namely,

$$(16.13) \qquad d/d\tau \langle T_\mu \rangle = e \, \bar{\gamma}^{-1} F_{\mu\nu} \langle T_\nu \rangle$$

It is shown below that in the particle's rest system, $\bar{\gamma} = 1$, $\langle T_4 \rangle = 0$, so in the rest system, Eq. (16.13) is analogous to the usual classical equation of motion of (spin $\frac{1}{2}$) angular momentum, $\underline{\Sigma}$,

$$d/d\tau \, \underline{\Sigma} = e \, (\underline{\Sigma} \times \underline{B})$$

where the g-factor is 2.

Also, since

$$d/d\tau \, \bar{\gamma} = d/d\tau \langle H - e\phi \rangle = e \, \underline{E} \cdot \langle \underline{\alpha} \rangle \qquad \text{and}$$

$$d/d\tau \langle \underline{\pi} \rangle = e \, [\, (\langle \underline{\alpha} \rangle \times \underline{B}) + \underline{E} \,]$$

one finds from Eqs. (16.11), (16.6), and (16.7), that:

$$(16.14) \qquad d/d\tau \langle \underline{\pi} \rangle = e \, [\, \underline{E} + (\underline{\upsilon} \times \underline{B}) \,] \qquad ,$$

$$(16.15) \qquad d/d\tau \, \bar{\gamma} = e \, \underline{E} \cdot \underline{\upsilon} \qquad ,$$

$$(16.16) \qquad d/d\tau \, \underline{\upsilon} = e \, \bar{\gamma}^{-1} [\, \underline{E} + (\underline{\upsilon} \times \underline{B}) - (\underline{E} \cdot \underline{\upsilon}) \underline{\upsilon} \,] \qquad \text{and}$$

$$(16.17) \qquad d/d\tau \, \upsilon = e \, \bar{\gamma}^{-1} (\underline{E} \cdot \underline{\upsilon})(1 - \upsilon^2)/\upsilon = e \, (\bar{\gamma}^3 \upsilon)^{-1} (\underline{E} \cdot \underline{\upsilon}).$$

Eq. (16.14) is the expected relation from Ehrenfest's theorem, and the other equations of motion are also exactly analogous to the corresponding classical equations.

Using Eqs. (16.6), (16.7), (16.13) to (16.16), it is found that:

$$(16.18) \qquad d/d\tau \, [\langle T_\mu \rangle \langle T_\mu \rangle] = 0 \qquad ,$$

$$(16.19) \qquad d/d\tau \left[\langle \pi_\mu \rangle \langle T_\mu \rangle \right] = 0 \qquad\qquad \text{and}$$

$$(16.20) \qquad d/d\tau \left[\langle \pi_\mu \rangle \langle \pi_\mu \rangle \right] = 0$$

It may be shown that $\langle \pi_\mu \rangle$ and $\langle T_\mu \rangle$ are four-vectors for continuous Lorentz transformations. The problem can be discussed infinitesimally. For infinitesimal space-time Lorentz transformation (v_i is an infinitesimal):

$$(16.21) \qquad x_i' = x_i - v_i t$$

$$(16.22) \qquad t' = t - \underline{v} \cdot \underline{x} \qquad\qquad \text{, and}$$

$$(16.23) \qquad \Psi'(\underline{x}, t) = \left[1 + v_i \left(t\, \partial/\partial x_i + x_i\, \partial/\partial t - \tfrac{1}{2}\alpha_i \right) \right] \Psi(\underline{x}, \tau)$$

For any operator, Q_μ, $\Gamma_{i\mu}$ is defined by the relation

$$(16.24) \qquad Q_\mu'(\underline{x}, \tau) = Q_\mu(\underline{x}, \tau) + v_i\, \Gamma_{i\mu}$$

Using the abbreviations,

$$\langle Q_\mu(\tau) \rangle' = \int \Psi'^\dagger(\underline{x}, \tau)\, Q_\mu'(\underline{x}, \tau)\, \Psi'(\underline{x}, \tau)\, d\underline{x}$$

$$(16.25)$$

$$\langle Q_\mu(\tau) \rangle = \int \Psi^\dagger(\underline{x}, \tau)\, Q_\mu(\underline{x}, \tau)\, \Psi(\underline{x}, t)\, d\underline{x}$$

one finds, upon substituting Eqs. (16.23), (16.24) into Eqs. (16.25), that

$$(16.26) \qquad \langle Q_\mu(\tau) \rangle' - \langle Q_\mu(\tau) \rangle =$$

$$v_i \Big\{ \tfrac{1}{2} \langle [\alpha_i, Q_\mu]_- \rangle + \langle \Gamma_{i\mu} \rangle + \hbar \langle [x_i, Q_\mu]\, p_4 - [x_4, Q_\mu]\, p_i \rangle$$

$$- \langle t\, \partial Q_\mu/\partial x_i + x_i\, \partial Q_\mu/\partial \tau \rangle + \langle x_i (-i\hbar' [Q_\mu, H]_- + \partial Q_\mu/\partial \tau) \rangle$$

In the limit of a small wave packet, one may factor an average $\langle x_i \rangle$ out of the last term, so that,

$$(16.27) \qquad \left\langle x_i \left(-i\hbar^{-1}[Q_\mu, H]_- + \partial Q_\mu / \partial \tau \right) \right\rangle \cong \langle x_i \rangle \, d/d\tau \langle Q_\mu \rangle$$

Classically, for a particle in an orbit $\overline{X} = \overline{X(t)}$, the classical four-vector $\overline{Q_\mu(t)}$ transforms according to

$$(16.28) \quad \overline{Q_\mu'(t')} = \overline{Q_\mu(t)} + \xi_{\mu\nu}\overline{Q_\nu(t)} = \overline{Q_\mu(t)} + i v_i \left(\overline{Q}_4 \delta_{\mu i} - \delta_{\mu 4} \overline{Q}_i \right)$$

Consequently, using Eqs. (16.21) and (16.22) one finds

$$(16.29) \quad \overline{Q_\mu'(t)} - \overline{Q_\mu(t)} = v_i \left[x_i \, d\overline{Q}_\mu / d\tau + i \left(\overline{Q}_4 \delta_{\mu i} - \delta_{\mu 4} \overline{Q}_i \right) \right]$$

Performing the operations indicated in Eq. (16.26), one finds that the expectation values of π_μ and T_μ transform like classical four-vectors in the limit of a small wave packet. Since any proper Lorentz transformation can be constructed from spacial rotations, for which these quantities are evidently four-vectors, and infinitesimal space-time Lorentz transformations, one sees that the expectation values of π_μ and T_μ transform like classical four-vectors when the wave function is a restricted wave packet.

As a result of the four-vector properties Eq. (16.19) implies that

$$\tilde{\gamma} \left(\underline{\ell} \cdot \langle \underline{T} \rangle + i \langle T_4 \rangle \right) \qquad = \text{constant,}$$

for all frames of reference. If one uses the specific representation of the Dirac matrices given in Sec. 2, and defines a rest system such that the expectation values of all odd operators [defined in Sec. 7] are zero, then

Eqs. (16.10), (16.11) imply that $\underline{\ell} = \langle T_4 \rangle = 0$ in such a system.

Consequently, the constant equals zero, and it is true, in all systems, that

$$(16.30) \qquad \langle T_4 \rangle = i\, \underline{\ell} \cdot \langle \underline{I} \rangle$$

In order to determine the polarization in the rest system, it is convenient to introduce the unit polarization four-vector,

$$(16.31) \qquad \hat{T}_\mu = \left(\langle T_\nu \rangle \langle T_\nu \rangle \right)^{-\frac{1}{2}} \langle T_\mu \rangle$$

Following a procedure suggested by Bargmann et al.,[13] one may resolve $\langle \underline{I} \rangle$ into components parallel and perpendicular to the momentum, i.e.,

$$(16.32) \qquad \langle \underline{I} \rangle = |\langle \underline{I} \rangle| \left(\cos\alpha\, \hat{\underline{\ell}} + \sin\alpha\, \hat{\underline{n}} \right)$$

where $\hat{\underline{\ell}} = \underline{\ell}/\ell$; $\hat{\underline{n}} \cdot \underline{\ell} = 0$; $\hat{\underline{n}} \cdot \hat{\underline{n}} = 1$. Then, from Eqs. (16.13), (16.18), (16.30), (16.31), and (16.32);

$$(16.33) \qquad d/d\tau\, \hat{T}_\mu = e\, \bar{\gamma}^{-1} F_{\mu\nu} \hat{T}_\nu$$

$$(16.34) \qquad \hat{T}_\mu = \left[1 - \ell^2 \cos^2\alpha \right]^{-\frac{1}{2}} \left(\cos\alpha\, \hat{\underline{\ell}} + \sin\alpha\, \hat{\underline{n}} \,,\; i\ell\cos\alpha \right)$$

One may equivalently write:

$$(16.35) \qquad \hat{T}_\mu = \left(\bar{\gamma}\, \hat{\underline{\ell}},\, i\ell\bar{\gamma} \right) \cos\varphi + \left(\hat{\underline{n}},\, 0 \right) \sin\varphi$$

where

$$(16.36) \qquad \begin{aligned} \cos\varphi &= \bar{\gamma}^{-1} \left(1 - \ell^2 \cos^2\alpha \right)^{-\frac{1}{2}} \cos\alpha \qquad\qquad , \text{ and} \\ \sin\varphi &= \left(1 - \ell^2 \cos^2\alpha \right)^{-\frac{1}{2}} \sin\alpha \end{aligned}$$

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The reason for introducing the angle ϕ is seen when one considers in the rest system

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \bar{\gamma} & i\beta\bar{\gamma} \\ 0 & -i\beta\bar{\gamma} & \bar{\gamma} \end{pmatrix} \overset{\text{lab.}}{\begin{pmatrix} \sin\phi \; (\hat{\underline{m}}) \\ \bar{\gamma}\cos\phi \; (\hat{\underline{\ell}}) \\ i\beta\bar{\gamma}\cos\phi \end{pmatrix}} = \overset{\text{rest.}}{\begin{pmatrix} \sin\phi \; (\hat{\underline{m}}) \\ \cos\phi \; (\hat{\underline{\ell}}) \\ 0 \end{pmatrix}}$$

Thus, ϕ gives the orientation of $\hat{\underline{T}}^{\text{rest}}$, $[T_4^{\text{rest}} = 0]$, along directions parallel and perpendicular to the coordinate grid specified by $\hat{\underline{\ell}}$.

Substituting Eq. (16.35) into Eq. (16.33), using Eqs. (16.15), (16.16), and (16.17), and defining:

(16.37)
$$\Omega = d/d\tau \; \phi$$

one arrives at the relations:

(16.38)
$$\Omega = -e\,(\bar{\gamma}^2\beta)^{-1}(\underline{E}\cdot\hat{\underline{m}}) \qquad , \quad \text{and}$$

(16.39)
$$d/d\tau \; \hat{\underline{m}} = e\,\bar{\gamma}^{-1}\left\{ -\hat{\underline{\ell}}/\beta \left[\hat{\underline{m}}\cdot(\underline{E}+\underline{\beta}\times\underline{B})\right] \right.$$
$$\left. + \hat{\underline{m}}\times\hat{\underline{\ell}}\left[-(\bar{\gamma}\beta)^{-1}\cot\phi\,(\hat{\underline{m}}\times\hat{\underline{\ell}}\cdot\underline{E}) + \hat{\underline{\ell}}\cdot\underline{B}\right]\right\}.$$

Now, from Eqs. (16.16) and (16.17), one finds:

(16.40)
$$d\hat{\underline{\ell}}/d\tau = e\,(\bar{\gamma}\beta)^{-1}\hat{\underline{\ell}} \times \left[\beta\underline{B} + (\underline{E}\times\hat{\underline{\ell}})\right]$$

Since $\hat{\underline{m}}$ is perpendicular to $\hat{\underline{\ell}}$, it is always true that:

$$d/d\tau \; (\hat{\underline{\ell}}\cdot\hat{\underline{m}}) = 0 = \hat{\underline{\ell}}\cdot d/d\tau\,(\hat{\underline{m}}) + \hat{\underline{m}}\cdot d/d\tau\,(\hat{\underline{\ell}})$$

hence,

$$(d\hat{\underline{m}}/d\tau)\cdot\hat{\underline{\ell}} = -e(\bar{\tau}\ell)^{-1}\left\{\hat{\underline{m}}\cdot[\underline{E} + (\underline{\ell}\times\underline{B})]\right\}$$

Thus, the first term in Eq. (16.39) represents the change in the perpendicular component of polarization due to the changing direction of motion of the particle, and merely signifies that the direction of polarization transverse to the motion remains transverse. The second term of Eq. (16.39), in the $(\hat{\underline{m}} \times \hat{\underline{\ell}})$ direction, represents a rotation of the transverse direction of polarization about the direction of motion, in a left-handed sense. Consequently, relative to a coordinate system moving with the particle, the equations,

$$\Omega = -e(\bar{\tau}^2\ell)^{-1}(\underline{E}\cdot\hat{\underline{m}})$$, and

(16.41)

$$d\hat{\underline{m}}/d\tau = (\hat{\underline{m}}\times\hat{\underline{\ell}})e\bar{\tau}^{-1}\left[(\bar{\tau}\ell)^{-1}\cot\varphi(\underline{\ell}\times\underline{m}\cdot\underline{E}) + \hat{\underline{\ell}}\cdot\underline{B}\right],$$

give the precession of polarization in the particle's rest system in terms of quantities computed in the laboratory system. These equations were originally derived by Bargmann, Michel, and Telegdi,[13] and specialize to give agreement with earlier precession equations obtained by Tolhoek and deGroot.[3] Four cases are of special interest:

(i) $\underline{E}$ longitudinal ($\underline{E} = \pm E\hat{\underline{\ell}}$), $\underline{B} = 0$.

In this case, we find from Eqs. (16.41) that:

$$\Omega = 0 \quad ; \quad d\hat{\underline{m}}/d\tau = 0$$.

Thus, referred to axes continuously moving with the particle, the direction of polarization does not change with time. Since, from Eq. (16.41)

$d\hat{\underline{\ell}}/d\tau = 0$, the above statement is also valid with reference to axes instantaneously fixed in the particle.

(ii) $\underline{B}$ transverse ($\underline{B} \cdot \hat{\underline{\ell}} = 0$), $\underline{E} = 0$.

One finds, from Eqs. (16.41), that

$$\Omega = 0 \quad ; \quad d\hat{\underline{m}}/d\tau = 0 \quad .$$

Accordingly, referred to axes continuously moving with the particle, the polarization direction remains fixed. However, from Eq. (16.40),

$$d\hat{\underline{\ell}}/d\tau = e\,\bar{\gamma}^{-1}(\hat{\underline{\ell}} \times \underline{B})$$

One sees that the polarization rotates about the $-\hat{\underline{B}}$ axis with the relativistic cyclotron frequency,

(16.42) $$\omega_L = e\,\bar{\gamma}^{-1} B$$

Thus, referred to axes instantaneously fixed in the particle, the normal component of polarization appears to rotate with the relativistic cyclotron frequency about the $-\hat{\underline{B}}$ direction.

(iii) $\underline{B}$ longitudinal ($\underline{B} = \pm B\hat{\underline{\ell}}$), $\underline{E} = 0$.

From Eqs. (16.41) one finds

$$\Omega = 0 \quad ; \quad d\hat{\underline{m}}/d\tau = \omega_L (\hat{\underline{m}} \times \hat{\underline{\ell}}) \quad .$$

Accordingly, referred to axes continuously moving with the particle, the longitudinal component of polarization is constant, and the transverse component of polarization rotates about the direction of motion, in a left-handed

sense, with the relativistic cyclotron frequency ω_L . Since, in this case,

$d\hat{\underset{\sim}{m}}/d\tau = 0$, the continuously moving axes are equivalent to a set

of axes instantaneously fixed in the particle. One may, with Tolhoek and

deGroot, write

$$\Delta\hat{\underset{\sim}{m}} = e B \bar{\gamma}^{-1}\Delta t = e B \underset{\sim}{\ell} \Delta\tau (\bar{\gamma}\underset{\sim}{\ell})^{-1}$$

$$= \bar{\underset{\sim}{p}}^{-1} e B \Delta (\text{distance particle travels})$$

where $\bar{\underset{\sim}{p}}$ is the instantaneous momentum of the particle.

(iv) $\underset{\sim}{E}$ transverse ($\underset{\sim}{E}\cdot\hat{\underset{\sim}{\ell}} = 0$), $\underset{\sim}{B} = 0$, $(\hat{\underset{\sim}{\ell}}\times\hat{\underset{\sim}{m}}\cdot\underset{\sim}{E}) = 0$.

For this case, Eqs. (16.41) give

(16.43) $\qquad \Omega = -e(\bar{\gamma}^2\underset{\sim}{\ell})^{-1}(\underset{\sim}{E}\cdot\hat{\underset{\sim}{m}})$; $d\hat{\underset{\sim}{m}}/d\tau = 0$.

Referred to axes continuously moving with the particle, the polarization

stays in the plane specified by $\hat{\underset{\sim}{\ell}}$ and $\underset{\sim}{E}$ and rotates about $\hat{\underset{\sim}{\ell}}\times\hat{\underset{\sim}{E}}$ axis

with frequency,

$$\omega = e E (\bar{\gamma}^2\underset{\sim}{\ell})^{-1}$$

It is seen that this type of electromagnetic field converts, in a harmonic

fashion, longitudinal to transverse polarization. Also, from Eqs. (16.40),

the particle rotates about the $\hat{\underset{\sim}{\ell}}\times\hat{\underset{\sim}{E}}$ axis with the frequency $e E (\bar{\gamma}\underset{\sim}{\ell})^{-1}$,

so, referred to axes instantaneously fixed in the particle, the polarization ro-

tates about the $\hat{\underset{\sim}{\ell}}\times\hat{\underset{\sim}{E}}$ direction with frequency,

$$\omega_{Fixed} = e E (\bar{\gamma}^2\underset{\sim}{\ell})^{-1}(\bar{\gamma}-1)$$

One may, with Tolhoek, and deGroot write,

$$(\omega_{Fixed})/(\omega_{particle}) = \mathcal{E}_{kin.} / \mathcal{E}$$

where the total energy $\mathcal{E}$ is $\bar{\gamma}$, and the kinetic energy is $(\bar{\gamma} - 1)$.

17. Anomalous Magnetic Moment Considerations

Under the assumption that the anomalous magnetic moment of a Dirac particle can be described, in single particle theory, by the addition of a co-variant Pauli[22] term to the Hamiltonian, one has

$$(17.1) \qquad H = i \gamma_4 \underline{\gamma} \cdot \underline{\Pi} + \gamma_4 - ie A_4 + (1/8) ie \hbar \mu F_{\alpha\rho} \gamma_4 \gamma_\alpha \gamma_\beta$$

where μ is a dimensionless scalar (the anomalous contribution), and the g-factor is given by

$$(17.2) \qquad g = 2 + \mu$$

With T_μ still defined by Eq. (15.1), where the definition $T_4 = \gamma_5 (i\gamma_4 - iH + ie\phi)$ is taken, one now finds that:

$$(17.3) \qquad d\langle T_\lambda \rangle / d\tau = \tfrac{1}{2} g \langle ie \gamma_4 \gamma_5 \gamma_\mu F_{\lambda\nu} \rangle + (1/8)\mu e \langle \gamma_4 \gamma_5 \gamma_\alpha \gamma_\beta [F_{\alpha\beta}, \Pi_\lambda]_+ \rangle$$

The packet considerations follow as in Sec. 16. Using Eq. (16.9), it is found that

$$(17.4) \qquad \langle i \gamma_4 \gamma_5 \gamma_\mu \rangle = \bar{\gamma}^{-1} \langle T_\mu \rangle + order (\hbar)$$

$$(17.5) \qquad \langle \gamma_4 \gamma_5 \gamma_\alpha \gamma_\beta F_{\alpha\beta} \Pi_\lambda \rangle = -2 \bar{\gamma}^{-1} G_\lambda + order (\hbar)$$

where

$$(17.6) \quad G_\lambda = i\, \varepsilon_{jk\tau} \langle \gamma_5 (\gamma_j B_\tau - E_\tau \beta \sigma_j) \pi_\lambda \pi_k \rangle$$

$$+ \langle (\underline{\sigma} \cdot \underline{E}) \pi_\lambda \rangle$$

Also, since

$$(17.7) \quad \langle F_{\alpha\beta} \pi_\beta T_\alpha \pi_\lambda \rangle = G_\lambda + order(\hbar)$$

one finds

$$(17.8) \quad \langle \gamma_4 \gamma_5 \gamma_\alpha \gamma_\beta F_{\alpha\beta} \pi_\lambda \rangle = -2\, \bar{\gamma}^{-1} \langle F_{\alpha\beta} \pi_\beta T_\alpha \pi_\lambda \rangle + order(\hbar)$$

Substitution of Eqs. (17.4) and (17.8) into Eq. (17.3) yields

$$(17.9) \quad d\langle T_\lambda \rangle / d\tau = \tfrac{1}{2} e\, \bar{\gamma}^{-1} \left(g F_{\lambda\nu} \langle T_\nu \rangle - \mu \langle F_{\alpha\beta} \pi_\beta T_\alpha \pi_\lambda \rangle \right) + order(\hbar) .$$

In the classical particle picture, the order ($\hbar$) may be neglected, and

the expectation values may be regarded as classical values. Identifying

$\overline{\pi}_\mu$ as the four-vector velocity, Eq. (17.9) corresponds to the equation

of motion derived by Bargmann, Michel, and Telegdi.[13] By a procedure

similar to that given in Sec. 16, one obtains, after lengthy algebra, the pre-

cession equations

$$\Omega = \tfrac{1}{2} g\, (\Omega)_{\mu=0} + \mu e (2\,\theta)^{-1} (\hat{\underline{n}} \times \hat{\underline{b}}_\perp \cdot \hat{\underline{b}}) \qquad , \text{ and}$$

$$(17.10)$$

$$d\hat{\underline{m}} / d\tau = \tfrac{1}{2} g (d\hat{\underline{m}} / d\tau)_{\mu=0} + (\hat{\underline{m}} \times \hat{\underline{b}}) \mu e (2\,\theta)^{-1} \cot\varphi (\hat{\underline{m}} \cdot \underline{b}_\perp)$$

where $\underline{b}_\perp$ is a vector which, when non-zero, points in the direction of the

instantaneous normal to the plane of motion,

(17.11)
$$\mathbf{\mathcal{f}_{\perp}} = \hat{\mathbf{v}} \times [\mathbf{E} + (\mathbf{v} \times \mathbf{B})]$$

and the terms for $\mu = 0$ are given by Eqs. (16.41). Eqs. (17.10) are valid relative to a coordinate system rigidly attached to the particle and continuously moving with it.

In the special cases discussed in detail at the end of Sec. 16, the presence of an anomalous moment produces a qualitative change only for the case in which $\mathbf{B}$ is transverse, $\mathbf{E} = 0$. The results for the same four special cases are:

(i) $\mathbf{E}$ longitudinal, $\mathbf{B} = 0$.

Here $\mathbf{\mathcal{f}_{\perp}} = 0$, and the direction of polarization is unchanged, as before.

(ii) $\mathbf{B}$ transverse, $\mathbf{E} = 0$.

Referred to axes moving continuously with the particle, the normal component of polarization rotates until it is in the plane of motion, and thereafter the polarization rotates about the perpendicular to the plane, ($-\hat{\mathbf{B}}$), with the frequency

$$\omega = \tfrac{1}{2} \mu \bar{\gamma} \omega_L$$

Consequently, the anomalous moment creates plane polarization, and converts longitudinal polarization into transverse polarization, in a harmonic fashion. Referred to axes instantaneously fixed, the polarization appears to rotate about the $-\hat{\mathbf{B}}$ axis with frequency,

$$\omega_{Fixed} = \omega_L (1 + \tfrac{1}{2} \mu \bar{\gamma})$$

This last result was obtained by Carrassi,[23] and by Mendlowitz and Case.[24]

(iii) $\underset{\sim}{B}$ longitudinal, $\underset{\sim}{E} = 0$.

Here, $\underset{\sim}{\mathcal{A}}_{\perp} = 0$, and in the coordinate system continuously moving with the particle, the longitudinal polarization is constant, and the transverse polarization rotates about the direction of motion with a frequency $\frac{1}{2}g$ times the relativistic cyclotron frequency.

(iv) $\underset{\sim}{E}$ transverse, $\underset{\sim}{B} = 0$. $(\hat{\underset{\sim}{\beta}} \times \hat{\underset{\sim}{m}} \cdot \underset{\sim}{E}) = 0$.

In this case,

$$\Omega = (1 - \tfrac{1}{2}\mu \bar{\gamma}^2 \beta^2)(\Omega)_{\mu=0} \qquad \text{, and}$$

$$d\hat{\underset{\sim}{m}}/dt = (1 - \tfrac{1}{2}\mu \bar{\gamma}^2 \beta^2)(d\hat{\underset{\sim}{x}}/dt)_{\mu=0} \qquad ,$$

where the quantities at $\mu = 0$ are given by Eqs. (16.43). Consequently, in the coordinate system continuously moving with the particle, the transverse polarization remains in the plane specified by $\hat{\underset{\sim}{\beta}}$ and $\underset{\sim}{E}$, and rotates about the ($\hat{\underset{\sim}{E}} \times \hat{\underset{\sim}{\beta}}$) axis with frequency,

$$\omega = (1 - \tfrac{1}{2}\mu \bar{\gamma}^2 \beta^2) e E (\bar{\gamma}^2 \beta)^{-1}$$

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