

A 3-D model of superfluid helium suitable for numerical analysis

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The two-fluid description is a very successful phenomenological representation of the properties of Helium II. A 3-D model suitable for numerical analysis based on the *Landau-Khalatnikov* description of Helium II is proposed. In this paper we introduce a system of partial differential equations that is both complete and consistent as well as practical, to be used for a 3-D solution of the flow of Helium II.

The development of a 3-D numerical model for Helium II is motivated by the need to validate experimental results obtained by observing the normal component velocity distribution in a Helium II thermal counter-flow using the Particle Image Velocimetry (PIV) technique.

INTRODUCTION

The phenomenological representation of superfluid helium (Helium II) in the two-fluid dynamics model permits us to derive a system of hyperbolic and parabolic partial differential equations to be used for a 3-D solution of the flow of Helium II. A set of convenient variables for the partial differential equations system is chosen to describe the Helium II hydrodynamic approximations. A macroscopic approach to the conservation equations is taken to model thermal counter-flow.

MATHEMATICAL FORMULATION- SYSTEM OF PDE FOR THE TWO-FLUID MODEL

The mathematical formulation of liquid helium has been established for many years [1].

In the two-fluid model the mass, momentum and heat transfer of He II are interpreted in terms of the motion of two independent fluids, a normal one, with velocity, v_n , and a superfluid one, with velocity, v_s . The superfluid component is inviscid and carries no entropy, while the normal component is viscous, with dynamic viscosity, η , and carries the entropy, s .

Using notations and conventions of Khalatnikov [2] and Roberts & Donnelly [3-4], we here take a macroscopic approach to the conservation equations of the two-fluid model, assuming local thermodynamic equilibrium, so that the state of Helium II as well as that of each of the two-fluid components can be described by two independent state variables (i.e. pressure p , and temperature T).

Mass balance conservation

The total mass density ρ of the fluid is given by the sum of the densities of the normal components and superfluid components, ρ_n , and ρ_s , respectively. The overall conservation equation of mass becomes:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \quad (1)$$

Momentum balance conservation

To derive the equations of motion of the superfluid and normal components we start by postulating conservation of the total momentum carried by the two species, i.e.:

$$\frac{\partial(\rho_n \mathbf{v}_n + \rho_s \mathbf{v}_s)}{\partial t} + \nabla \cdot (\rho_n \mathbf{v}_n \mathbf{v}_n + \rho_s \mathbf{v}_s \mathbf{v}_s) + \nabla p = -\nabla \cdot \bar{\bar{\tau}} + \rho \mathbf{g} \quad (2)$$

where \mathbf{g} is the acceleration of the gravity field and $\bar{\bar{\tau}}$ is the stress tensor, which only receives contributions from the normal fluid and can be written as follows:

$$\nabla \cdot \bar{\bar{\tau}} = -\eta \left\{ \nabla^2 \mathbf{v}_n + \frac{1}{3} \nabla (\nabla \cdot \mathbf{v}_n) \right\} \quad (3)$$

Let us introduce \mathbf{w} , the difference of normal and superfluid velocities and the thermodynamic potential, Φ as follows:

$$\mathbf{w} = \mathbf{v}_n - \mathbf{v}_s \quad (4)$$

The momentum equation for the superfluid can be written as:

$$\rho_s \frac{\partial \mathbf{v}_s}{\partial t} + \rho_s \mathbf{v}_s \nabla \cdot \mathbf{v}_s + \nabla \Phi = \mathbf{F}_t + \rho_s \mathbf{g} \quad (5)$$

where \mathbf{F}_t is the friction force associated with turbulence is given by the empirical expressions, proposed by Gorter and Mellink for counterflow situations [5-6]:

$$\mathbf{F}_t = A_{GM} \rho_s \rho_n w^2 \mathbf{w} \quad (6)$$

where A_{GM} is a function of T and, possibly, of w ; and where Φ , the potential function is:

$$\Phi = i + \frac{p}{\rho} - sT - \left(\frac{\rho_n}{2\rho} \right) w^2 \quad (7)$$

with i is the internal energy, and s , the entropy.

Hence, the equation of motion for the normal component is obtained by subtracting equation (5) from the conservation of the total momentum equation (2).

$$\rho_n \frac{\partial \mathbf{v}_n}{\partial t} + \rho_n \mathbf{v}_n \nabla \cdot \mathbf{v}_n + \frac{\rho_n}{\rho} \nabla p + \rho_s s \nabla T + \frac{\rho_s \rho_n}{2\rho} \nabla w^2 = -\nabla \cdot \bar{\bar{\tau}} - \mathbf{F}_t + \rho_n \mathbf{g} - m \mathbf{w} \quad (8)$$

$$\rho_s \frac{\partial \mathbf{v}_s}{\partial t} + \rho_s \mathbf{v}_s \nabla \cdot \mathbf{v}_s + \frac{\rho_s}{\rho} \nabla p - \rho_s s \nabla T - \frac{\rho_s \rho_n}{2\rho} \nabla w^2 = \mathbf{F}_t + \rho_s \mathbf{g} \quad (9)$$

In these two sets of equations the first and second term on the left hand side are the acceleration terms. The third term on the left hand side is the force due to the pressure gradient. The fourth term is the pressure originated by the thermomechanical effect, and demonstrates how a temperature gradient can generate a counterflow in the mixture. The fifth term originates from the mass exchange among the two fluids in He II. It is important to note that the orders of magnitude of the terms above can be very different. For small accelerations, and modest Mach numbers, the thermomechanical and turbulent force terms tend to dominate the balance in equations (8) and (9).

At this point it is also interesting to note that the effect of the transformation of superfluid into normal fluid, and vice-versa, produces a term mw in the normal fluid equation (8), but not in the superfluid equation (9). This is a consequence of the assumption of conservation of total momentum (equation (2)) and of irrotational superfluid flow, equation (5).

Energy balance conservation

We turn now to the equation of energy conservation, using the conservation of the irreversible motion of the entropy [5]. A convenient form, derived from the conservation of the total energy density, is given by [4]:

$$\begin{aligned} & \frac{\partial}{\partial t} \left(\rho i + \frac{\rho_n \mathbf{v}_n^2}{2} + \frac{\rho_s \mathbf{v}_s^2}{2} \right) + \nabla \cdot \left(\rho i \mathbf{v} + \frac{\rho_n \mathbf{v}_n^2}{2} \mathbf{v}_n + \frac{\rho_s \mathbf{v}_s^2}{2} \mathbf{v}_s \right) \\ & + \nabla p \mathbf{v} + \nabla \rho_s s T \mathbf{w} + \nabla \frac{\rho_s \rho_n}{2\rho} w^2 \mathbf{w} - \nabla (k \nabla T) = -\nabla \cdot (\bar{\tau} \mathbf{v}_n) + \rho \mathbf{g} \mathbf{v} + q \end{aligned} \quad (10)$$

The first two terms on the left hand side of equation (10) represent the change in total energy density, i.e. the sum of the internal energy density and the specific kinetic energy of the two flows. We simplify the energy balance by subtracting the kinetic term from equation (10) and by subtracting the total continuity equation (Equation (1)) multiplied by i , leading to the final form of the internal energy conservation:

$$\rho \frac{\partial}{\partial t} + \rho \mathbf{v} \nabla \cdot i + p \nabla \mathbf{v} + T \nabla \rho_s s \mathbf{w} + w^2 \left(\nabla \frac{\rho_s \rho_n}{2\rho} \mathbf{w} - \frac{m}{2} \right) - \mathbf{F}_t \mathbf{w} - \nabla \cdot (k \nabla T) = -\bar{\tau} \cdot \nabla \cdot \mathbf{v}_n + q \quad (11)$$

Notice that several terms in equation (11) are non-standard. The term $T \nabla \rho_s s \mathbf{w}$, represents an internal heat convection through entropy transport. The terms $w^2 \left(\nabla \frac{\rho_s \rho_n}{2\rho} \mathbf{w} - \frac{m}{2} \right)$ originate from the transformation of superfluid into normal fluid and vice versa. Finally, the term $-\mathbf{F}_t \mathbf{w}$ represents the internal energy dissipation associated with turbulence.

Closure of the PDE system

The two-fluid model is completely described by equations (1) (total mass balance), (8) (normal fluid momentum balance), (9) (superfluid momentum balance), and (11) (internal energy balance). In addition we require a suitable equation of state, providing thermodynamic quantities as a function of two state variables. We choose pressure and temperature as state variables: ρ, ρ_n, i, s, k and η are function of (p, T) .

Equations in $(p, \mathbf{v}_n, \mathbf{v}_s, T)$ Form

The two-fluid flow phenomena can be analyzed as a set of hyperbolic and parabolic PDEs. The PDE hyperbolic class component is due to the normal component of the superfluid thermo- and hydro-dynamics while the parabolic one is due to the superfluid component behavior. The PDE under study is strongly non-linear due to the complexity of the interactions between the two types of fluids and the unique properties of the superfluid. The equations are complex, involve terms that are non-standard in nature, and contain terms that largely dominate the balances, e.g. the thermomechanical force or the mutual friction in the momentum balances. For this reason it is convenient to rearrange them and put them in a simplified form where pressure, velocities and temperature appear explicitly as variables in the derivatives. We refer to this choice as the $(p, \mathbf{v}_n, \mathbf{v}_s, T)$ form of the PDE system. The main advantage is that these variables are the leading orders in all dominating terms of the balances, and treating them implicitly in the solution algorithm will

largely improve the stability of the integration. To modify the equations as desired, however, we make the assumption that the thermodynamic state is independent on the composition and relative motion of the mixture of the two fluids. As shown by Roberts and Donnelly [4], this is not exact. Indeed, for the two-fluid system the internal energy i and thermodynamic potential Φ depend on the relative motion of the two fluids. The advantage is that standard thermodynamic relations can be used, and in particular the following hold:

$$h = i + \frac{P}{\rho} \quad (12)$$

$$di = \left(\frac{P}{\rho} - \phi C_v T \right) \frac{d\rho}{\rho} + C_v dT \quad (13)$$

$$d\rho = \frac{1+\phi}{c^2} dp - \frac{\phi\rho}{c^2} dh \quad (14)$$

where ϕ is the Gruneisen parameter, c , is the speed of the (first) sound, C_v is the specific heat at constant density, and h is the specific enthalpy.

Using the conservation laws and the thermodynamic relations, the two-fluid flow can be modeled with associated PDEs. To simplify the 3-D problem, we will first study the case where all quadratic terms and small terms are zero.

To modify the equations, we make the assumption that the thermodynamic state is independent of the composition and relative motion of the mixture of the two fluids.

- The contributions related explicitly to the mass exchange, m , are small when compared to other terms, and we can drop them from the balances;
- The energy dissipated by viscous dissipation is small compared to other sources of heat transport e.g. mutual friction can be treated as a source perturbation.
- All terms containing differentials of quantities other than the set of variables $(p, \mathbf{v}_n, \mathbf{v}_s, T)$ are small compared to the terms containing the differentials of these variables. In other terms we can regard them as perturbations with respect to the leading terms of the equations;
- We assume that the variations of the Gruneisen parameter, ϕ , are small, so that we can write:

$$\phi \nabla(k \nabla T) \approx \nabla(\phi k \nabla T) \quad (15)$$

With the assumptions above it is finally possible to come to the set of approximate equations that we seek, given in Appendix I, where we show the time derivative term, the convective flux terms (gradient of the system variables), the diffusive terms (Laplacian of the system variables), the non-linear source terms (proportional to the system variables), the linear source terms, and the perturbations. Equation (16) governs the evolution of pressure. Equations (17) and (18) give the flow field in 3-D and Equation (19) governs the evolution of the temperature.

APPLICATION TO PIV TECHNIQUE USING A PARTICLE FLOW SOLVER

One motivation for developing a 3-D numerical model for superfluid Helium is driven by the need to validate experimental results obtained at National High Magnetic Field Laboratory (NHMFL). The velocity distribution in a Helium II thermal counter-flow using the Particle Image Velocimetry (PIV) technique was observed [7-8]. Numerical analyses permit to decouple the normal and superfluid component velocities to examine experimental phenomena. Hence, NHMFL PIV results provide an ideal benchmark, which can be used to test the numerical 3-D code for the calculation of the velocity distribution in the two-fluid model.

Particle tracks the normal component velocity (which has viscosity) in a thermal counter-flow channel.

The normal and superfluid components of the physical thermal counter-flow channel problem can be modeled in terms of pressure, velocities and temperature, for a simple rectangular geometry simulating the

thermal counter-flow channel. The numerical implementation in a 3-D solver of the resulting system of partial differential equations is possible using equations in $(p, \mathbf{v}_n, \mathbf{v}_s, T)$ form. Modeling of the normal and superfluid components can be done taking these eight degrees of freedom in the 3-D space. The formalism of the 3-D numerical solution is introduced in [9].

The boundary conditions of the physical problem assume non-slip conditions at the wall of the channel, hence the normal component velocity only is zero. Adiabatic and symmetrical conditions are used to simulate the experimental channel. The normal fluid velocity, v_n , of the heated surface is assumed by equation (20):

$$v_n = \frac{q}{\rho \cdot s \cdot T} \quad (20)$$

where q is the heat generating the normal fluid velocity.

The set of PDE derived from the conservation balances can therefore be used to simulate the experimental results. The main difficulty is that the PDE is strongly non-linear due to the complexity of the interactions between the two types of fluids.

CONCLUSION

A new Helium II hydrodynamic approximation was established based on the two-fluid model and the theory of Gorter-Mellink mutual friction. The equations are expressed in terms of variables $(p, \mathbf{v}_n, \mathbf{v}_s, T)$. The validation of the numerical code by means of a comparison with the NHMFL experimental results is possible and on-going. A simple thermal counter-flow channel was modeled and results will identify the contribution of the two fluids.

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APPENDIX

Table I. Final system of Partial Differential Equations (PDE) for the thermodynamic state and 3-D flow of the normal- and superfluid components in Helium II

$$\begin{aligned} & \frac{\partial p}{\partial t} + \frac{\rho_n \mathbf{v}_n + \rho_s \mathbf{v}_s}{\rho} \nabla p + \rho_n c^2 \nabla \cdot \mathbf{v}_n + \phi w^2 \frac{\rho_s \rho_n}{2\rho} \nabla \cdot \mathbf{v}_n + \phi T \rho_s s \nabla \cdot \mathbf{v}_n + \rho_s c^2 \nabla \cdot \mathbf{v}_s - \phi w^2 \frac{\rho_s \rho_n}{2\rho} \nabla \cdot \mathbf{v}_s - \phi T \rho_s s \nabla \cdot \mathbf{v}_s - \\ & - \nabla \cdot (\phi k \nabla T) - \phi A \rho_s \rho_n w^2 \mathbf{w} \mathbf{v}_n + \phi A \rho_s \rho_n w^2 \mathbf{w} \mathbf{v}_s = \end{aligned} \quad (16)$$

$$\phi q - \phi \bar{\tau} \cdot \nabla \cdot \mathbf{v}_n - \rho c^2 \mathbf{v}_n \nabla \frac{\rho_n}{\rho} - \rho c^2 \mathbf{v}_s \nabla \frac{\rho_s}{\rho} - \phi T \mathbf{w} \nabla \rho_s s - \phi w^2 \mathbf{w} \nabla \frac{\rho_s \rho_n}{2\rho}$$

$$\rho_n \frac{\partial \mathbf{v}_n}{\partial t} + \frac{\rho_n}{\rho} \nabla p + \rho_n \mathbf{v}_n \nabla \cdot \mathbf{v}_n + \frac{\rho_s \rho_n}{\rho} \mathbf{w} \nabla \cdot \mathbf{v}_n - \frac{\rho_s \rho_n}{\rho} \mathbf{w} \nabla \cdot \mathbf{v}_s + \rho_s s \nabla T + A \rho_s \rho_n w^2 \mathbf{v}_n - A \rho_s \rho_n w^2 \mathbf{v}_s = -\nabla \cdot \bar{\tau} + \rho_n \mathbf{g} \quad (17)$$

$$\rho_s \frac{\partial \mathbf{v}_s}{\partial t} + \frac{\rho_s}{\rho} \nabla p - \frac{\rho_s \rho_n}{\rho} \mathbf{w} \nabla \cdot \mathbf{v}_n + \rho_s \mathbf{v}_s \nabla \cdot \mathbf{v}_s + \frac{\rho_s \rho_n}{\rho} \mathbf{w} \nabla \cdot \mathbf{v}_s - \rho_s s \nabla T - A \rho_s \rho_n w^2 \mathbf{v}_n + A \rho_s \rho_n w^2 \mathbf{v}_s = \rho_s \mathbf{g} \quad (18)$$

$$\begin{aligned} & \rho C_v \frac{\partial T}{\partial t} + \rho_n \phi C_v T \nabla \cdot \mathbf{v}_n + w^2 \frac{\rho_s \rho_n}{2\rho} \nabla \cdot \mathbf{v}_n + T \rho_s s \nabla \cdot \mathbf{v}_n + \rho_s \phi C_v T \nabla \cdot \mathbf{v}_s - w^2 \frac{\rho_s \rho_n}{2\rho} \nabla \cdot \mathbf{v}_s - T \rho_s s \nabla \cdot \mathbf{v}_s + \rho C_v \frac{\rho_n \mathbf{v}_n + \rho_s \mathbf{v}_s}{\rho} \nabla T - \\ & - \nabla \cdot (k \nabla T) - A \rho_s \rho_n w^2 \mathbf{w} \mathbf{v}_n + A \rho_s \rho_n w^2 \mathbf{w} \mathbf{v}_s = \end{aligned} \quad (19)$$

$$q - \bar{\tau} \cdot \nabla \cdot \mathbf{v} - \rho \phi C_v T \mathbf{v}_n \nabla \frac{\rho_n}{\rho} - \rho \phi C_v T \mathbf{v}_s \nabla \frac{\rho_s}{\rho} - T \mathbf{w} \nabla \rho_s s - w^2 \mathbf{w} \nabla \frac{\rho_s \rho_n}{2\rho}$$