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On the Time Interval Distribution Between Neutron Counts in a ^{3}He Proportional Counter with Detector Dead Time

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For materials where a spontaneously fissioning isotope (e.g. ^{238}U or ^{240}Pu) initiates fission chains which propagate within a quantity of fissile material (e.g. ^{235}U or ^{239}Pu), the neutrons are not generated according to a Poisson process because a single random event — the spontaneous fission of a radioactive nucleus — can generate many neutrons. We propose a perturbative theory to correct the time interval distribution between neutron counts for materials that spontaneously generate fission chains to accommodate the detector dead time inherent in ^{3}He proportional counters. We also propose a perturbative correction for multi-element detectors where the dead time applies only to neutrons counted within a single element.

I. INTRODUCTION

For a process that generates events as a function of time according to a Poisson distribution, the probability distribution of waiting times from some particular event to the next event is

$$I_0(T)\,dT = e^{-R_1 T} R_1\,dT \tag{1}$$

which is recognized as the *exponential* distribution. An interesting extension of the Poisson process is the problem of neutron detection where, after a count has been recorded, the detector has an inherent dead time δ. When a neutron enters a ^{3}He proportional counter, for example, a series of processes must occur inside the detector and its associated electronics for the count to be registered. These processes happen over a non-zero amount of time during which no other count can be registered, even if another neutron happens to enter the detector. Under certain circumstances, this dead time cannot reasonably be approximated as zero. For the case of a Poisson source with dead time, the time interval distribution from some particular neutron to the next neutron is then given by an exponential distribution shifted by δ for $T > \delta$, and zero for $T \leq \delta$. Thus,

$$J_0(T)\,dT = \frac{\theta(T-\delta) I_0(T)\,dT}{\int_\delta^\infty I_0(T)\,dT} \tag{2}$$

$$= \theta(T-\delta) e^{-R_1(T-\delta)} R_1\ dT \tag{3}$$

where $\theta(x)$ is the Heaviside or unit step function [1] and ensures that $J_0(T) = 0$ for $T \leq \delta$.

Fissile materials, however, do not generate neutrons according to a Poisson distribution [2]; because they support fission chains, a single event — the spontaneous fission of a radioactive nucleus — can generate many neutrons. A theory for the time interval distribution between counts for material spontaneously generating fission chains has been given by Prasad et al. [3]. In this paper, we propose a perturbative correction to this theory to take dead time into account.

II. NEUTRON TIME INTERVAL DISTRIBUTIONS

Consider a multiplying system where a spontaneously fissioning isotope (e.g. ^{238}U or ^{240}Pu) initiates fission chains which propagate within a quantity of fissile material (e.g. ^{235}U or ^{239}Pu). The system experiences spontaneous fissions at a rate of F_{S}. Each spontaneous fission has a probability $C_{\mathrm{S}\nu}$ to create ν neutrons and creates on average $\overline{\nu_{\mathrm{S}}}$ neutrons. These neutrons, each with probability p, go on to induce fissions in the fissile material. An induced fission similarly has a probability C_ν to create ν neutrons and creates on average $\overline{\nu}$ neutrons. The system thus multiplies the initial neutrons from each spontaneous fission by an average factor

$$M = \frac{1}{1-p\overline{\nu}} \tag{4}$$

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The escape multiplication M_e is this number less the average number of neutrons required to sustain the fission chain (equal to the average number of induced fissions in the chain) and is readily seen to be

$$M_\mathrm{e} = (1-p)M \tag{5}$$

The detection efficiency ϵ is the probability to count a neutron which has left the system either through non-fission absorption or leakage. The count rate for a multiplying system is then

$$R_1 = \epsilon\, F_\mathrm{S}\, \overline{\nu_\mathrm{S}}\, M_\mathrm{e} \tag{6}$$

A fission chain initiated by an induced fission from a single neutron has a probability distribution $\mathcal{P}_n$ to create n neutrons. The generating function $\not{\mathcal{P}}(y)$ for this probability distribution can be constructed in the usual way by multiplying $\mathcal{P}_n$ by y^n and summing over n,

$$\not{\mathcal{P}}(y) = \sum_{n=0}^{\infty} \mathcal{P}_n y^n \tag{7}$$

It has been shown that this generating function satisfies the equation [4]

$$\not{\mathcal{P}}(y) = (1-p)y + p\not{\mathcal{C}}\left[\not{\mathcal{P}}(y)\right] \tag{8}$$

where

$$\not{\mathcal{C}}(x) = \sum_{\nu=0}^{\infty} C_\nu x^\nu \tag{9}$$

is the generating function for the neutron multiplicity distribution for induced fission. The solution to Eq. 8 was shown to be [5]

$$\not{\mathcal{P}}(y) = (1-p)y + \sum_{k=1}^{\infty} \frac{P_p(k|n+k)}{k+n}\left[\not{\mathcal{C}}(y)\right]^k \tag{10}$$

where the quantity k has the interpretation of the total number of fissions in the fission chain and where

$$P_p(k|n+k) = \frac{(k+n)!}{k!n!} p^k (1-p)^n \tag{11}$$

is the binomial distribution. The probability distribution $\mathcal{P}_n$ is then the coefficients on y^n in the expansion of the function $\not{\mathcal{P}}(y)$ as a power series about the origin [6], denoted

$$\mathcal{P}_n = [y^n]\,\not{\mathcal{P}}(y) \tag{12}$$

Given the generating function for the fission chain neutron multiplicity distribution $\not{\mathcal{P}}(y)$ and the neutron multiplicity distribution for spontaneous fission $C_{\mathrm{S}\nu}$, it is then possible to calculate the fission chain neutron multiplicity distribution for fission chains initiated by spontaneous fission, $\mathcal{P}_{\mathrm{S}n}$. Each neutron created by the spontaneous fission is capable — with probability p — of causing a subsequent fission, and by extension, a fission chain with a neutron multiplicity distributed according to $\mathcal{P}_n$. The generating function for the distribution $\mathcal{P}_{\mathrm{S}n}$ was shown to be [5]

$$\not{\mathcal{P}}_\mathrm{S}(y) = \not{\mathcal{C}}_\mathrm{S}[\not{\mathcal{P}}(y)] \tag{13}$$

and, again, the probability distribution $\mathcal{P}_{\mathrm{S}n} = [y^n]\,\not{\mathcal{P}}_\mathrm{S}(y)$.

The probability to count m neutrons from a fission chain is then [7]

$$\mathrm{e}_m(\epsilon) = \sum_{n=m}^{\infty} \mathcal{P}_{\mathrm{S}n} \binom{n}{m} \epsilon^m (1-\epsilon)^{n-m} \tag{14}$$

The probability that no additional neutrons from the *same* fission chain are counted within a time T after some particular neutron gets counted is [8]

$$r_0(T) = \frac{F_\mathrm{S}}{R_1} \sum_{m=1}^{\infty} \mathrm{e}_m(\epsilon) \left(\sum_{k=0}^{n-1} e^{-k\lambda T} \right) \tag{15}$$

where neutrons from the same fission chain diffuse on a time scale of λ^{-1} before being detected. During a time interval T, the average number of instances of counting k neutrons coming from the *same* fission chain is [7]

$$\begin{aligned}\Lambda_k(T) = \sum_{n=k}^{\infty} \mathcal{P}_{\mathrm{S}n} \sum_{m=k}^{n} \binom{n}{m} \epsilon^m (1-\epsilon)^{n-m} \binom{m}{k} \\ \times \Bigg\{ \int_{-\infty}^{0} \left[\int_0^T e^{-\lambda(t-s)} \lambda\, dt \right]^k \left[1 - \int_0^T e^{-\lambda(t-s)} \lambda\, dt \right]^{m-k} F_\mathrm{S}\, ds \\ + \int_0^T \left[\int_s^T e^{-\lambda(t-s)} \lambda\, dt \right]^k \left[1 - \int_s^T e^{-\lambda(t-s)} \lambda\, dt \right]^{m-k} F_\mathrm{S}\, ds \Bigg\}\end{aligned} \tag{16}$$

under the (usually reliable) approximation that a fission chain happens instantaneously. Evaluating the integrals yields

$$\Lambda_k(T) = F_{\rm S} \sum_{n=k}^{\infty} \mathcal{P}_{{\rm S}n} \sum_{m=k}^{n} \binom{n}{m} \epsilon^m (1-\epsilon)^{n-m} \binom{m}{k} \Bigg\{ \frac{1}{\lambda} B[(1-e^{-\lambda T});\, k,\, m-k+1] - \frac{1}{\lambda} \sum_{j=0}^{k-1} B[(1-e^{-\lambda T});\, k-j,\, m-k+1] + (1-\delta_{km}) \frac{1-e^{-\lambda T(m-k)}}{\lambda(m-k)} + \delta_{km} T \Bigg\} \tag{17}$$

where $B(z;\, a,\, b)$ is the incomplete beta function [1].

The probability to count no neutrons during a random time interval T is [7]

$$b_0(T) = e^{-\left(\sum_{k=1}^{\infty} \Lambda_k\right)} \tag{18}$$

and the probability that no additional neutrons are counted within a time T after some particular neutron gets counted is [8]

$$\mathrm{n}_0(T) = r_0(T) b_0(T) \tag{19}$$

It has been shown that the probability distribution of waiting times from some particular count to the next count for a multiplying source is [3]

$$I_0(T)\, dT = \frac{F_{\rm S}}{R_1} \sum_{m=2}^{\infty} \mathrm{e}_m(\epsilon) \left(\sum_{k=1}^{n-1} k e^{-k\lambda T} \right) b_0(T) \lambda\, dT + r_0(T) \mathrm{n}_0(T) R_1\, dT \tag{20}$$

In Eq. 20, the first term is associated with adjacent counts coming from the same fission chain and the second term is associated with adjacent counts coming from different chains.

III. NEUTRON TIME INTERVAL DISTRIBUTIONS WITH DEAD TIME

We propose a perturbative correction to Eq. 20 to account for dead time. Up to a normalization factor, the time interval distribution from some particular neutron to the next neutron can be calculated from Eq. 20 as

$$I_0'(T)\, dT = dT\, \theta(T-\delta) \Bigg\{ I_0(T) + \int_0^\delta I_0(t) \left[I_0(T-t) - I_0(T) \right] dt + \int_0^\delta I_0(t) \int_0^{\delta - t} I_0(u) \left[I_0(T-t-u) - I_0(T) \right] du\, dt + \cdots \Bigg\} \tag{21}$$

where the zeroth-order, first-order, and second-order corrections have been explicitly written out. This can be normalized in the usual way as

$$J_0(T)\, dT = \frac{I_0'(T)\, dT}{\int_\delta^\infty I_0'(T)\, dT} \tag{22}$$

To zeroth-order, the distribution is merely $I_0(t)\, dT$ for $T > \delta$ and with the normalization correspondingly adjusted. The first-order correction, however, takes into account the case where the next neutron enters the detector a time t after the original neutron but within the dead time such that $t \leq \delta$, and the next-to-next neutron enters a time T after the original neutron — or $(T-t)$ after the next neutron — outside the dead time such that $T > \delta$. Because the next neutron entered the detector within the dead time, the observed time-to-next is no longer t and $(T-t)$, but rather becomes only the actual time-to-next-to-next T. Thus, the probability that the next neutron is detected between T and $T + dT$ gets *increased* by an amount proportional to $I_0(t) \times I_0(T-t)$. But it's not that simple because, in exactly the same way, the probability that the next neutron is detected between $(T+t)$ and $(T+t) + dT$ also gets increased by an amount proportional to $I_0(t) \times I_0(T)$. However, the probability that the next neutron is detected between T and $T + dT$ gets corresponding *decreased* by exactly this amount. This is shown conceptually in Figure 1.

Higher order corrections follow the same pattern: The second-order correction takes into account the case where both the next and the next-to-next neutrons enter the detector within the dead time, but the next-to-next-to-next neutron enters the detector outside of the dead time. A similar reshuffling of probability naturally results. The third-order correction has three neutrons entering the de-

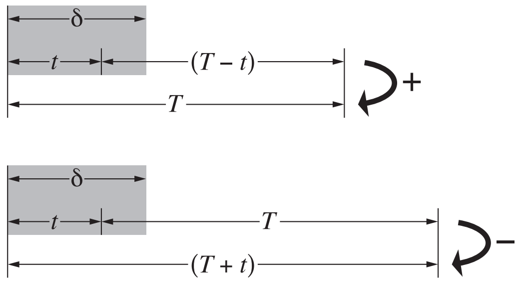

FIG. 1. In the first order correction, dead time causes the probability $I_0(T)\,dT$ to gain probability from $I_0(T-t)\,dT$ but lose probability to $I_0(T+t)\,dT$ as the *observed* time-to-next is replaced with the *actual* time-to-next-to-next.

tector within the dead time, but the fourth entering outside the dead time, and so on.

IV. MULTIPLE-ELEMENT DETECTORS

Using similar reasoning as above, we can extend the theory to correct for dead time in neutron detector systems with multiple elements where the dead time only applies to an individual element within the detector. That is to say, only successive neutrons counted by the *same* detector element experience the effects of dead time. As an example, the Ortec Fission Meter neutron multiplicity counter employs 30 ^{3}He tubes, and the dead time per tube is known to be $\sim 1\ \mu$s [9].

For a detector system with N_{tubes} elements, and making the approximation that all elements within the detector system have the same detection efficiency, we can define

$$\varepsilon = \frac{1}{N_{\text{tubes}}} \tag{23}$$

Up to the normalization factor, the time interval distribution from some particular neutron to the next for a multiple element detector system can be calculated from Eq. 20 as

$$\begin{aligned} I_0'(T)\,dT = {} & (1-\varepsilon)I_0(T)\,dT \\ & + dT\ \theta(T-\delta)\left\{\varepsilon I_0(T) + \varepsilon^2 \int_0^\delta I_0(t)\left[I_0(T-t) - I_0(T)\right]\,dt \right. \\ & \left. \quad + \varepsilon^3 \int_0^\delta I_0(t) \int_0^{\delta-t} I_0(u)\left[I_0(T-t-u) - I_0(T)\right]\,du\,dt + \cdots \right\} \\ & + dT(1-\varepsilon)\left\{\varepsilon \int_0^\delta I_0(t)\left[I_0(T-t) - I_0(T)\right]\,dt \right. \\ & \left. \quad + \varepsilon^2 \int_0^\delta I_0(t) \int_0^{\delta-t} I_0(u)\left[I_0(T-t-u) - I_0(T)\right]\,du\,dt + \cdots \right\} \end{aligned} \tag{24}$$

where the zeroth-order, first-order, and second-order corrections have been explicitly written out. The term in the first set of curly brackets is the case where all the counts occur within the *same* detector element. This term is similar to the above case for a single-element detector system except that each term is adjusted by the probability that, given a count within a particular element, the following counts all occur, each with probability ε, within the same element. The term in the second set of curly brackets covers the case where all counts but the last occur, each with probability ε, within the same detector element while the last count occurs, with probability $(1-\varepsilon)$, within a *different* element. Note that in this term, T is unrestricted, and indeed the region $T \leq \delta$ is important. This distribution can again be normalized in the usual way with Eq. 22.

V. MONTE CARLO SIMULATIONS

If we want to generate waiting times δt between random events, we simply apply the inverse transform method [10] to the exponential distribution, Eq. 1. Starting with a random number u which is uniformly distributed on $[0,1]$, the waiting time is then

$$\delta t = -\frac{\ln u}{r} \tag{25}$$

Source	mass (kg)	F_S (s^{-1})	M	M_e	ϵ (%)	λ^{-1} (μs)	Observation Time	Number of Neutrons	Count Rate (s^{-1})
50 kg DU/HEU	^{238}U: 50	338.9	2	1.6	3	50	2 hours	241,502	33.5
46 kg HEU	^{238}U: 3.22	21.8	20	12.5			2 hours	119,432	16.6
6 kg WGPu	^{240}Pu: 0.36	1.7×10^5	6.67	5.0			3 min	9,508,143	5.9×10^4
10 kg RGPu	^{240}Pu: 2.5	9.1×10^5	10	7.0			1 min	24,768,231	4.1×10^5

TABLE I. Monte Carlo input parameters for each source.

Consider a fission chain which produces n neutrons. The probability of detecting exactly m neutrons out of the possible n, if the probability of detection is ϵ, is again just the binomial distribution $P_\epsilon(m|n)$, Eq. 11. The fission chain neutron multiplicity $\mathcal{P}_{Sn}$ is the probability that a fission chain initiated by spontaneous fission generates n neutrons. The probability of detecting m neutrons from such a fission chain, when the detection efficiency is ϵ, is $\mathrm{e}_m(\epsilon)$, Eq. 14. The rate of detecting m neutrons from a single fission chain is then the spontaneous fission rate F_S times this probability,

$$r_m = F_S \, \mathrm{e}_m(\epsilon) \tag{26}$$

Fission chains in which $m = 1, 2, 3, \cdots$ neutrons are detected can be considered as different kinds of events, and each can be considered individually. The rates r_m can be used to generate waiting times between events where an "event" is a fission chain in which m neutrons were detected. From Eq. 25, a list of times of fission chain initiations is generated as

$$t_{m,i} = \delta t_{m,i} + t_{m,i-1} \tag{27}$$

$$= -\frac{\ln u_i}{r_m} + t_{m,i-1} \tag{28}$$

with $t_{m,0} = 0$. Iteration is stopped once $t_{m,i}$ exceeds the observation time.

The neutron does not get detected the instant it is created in the fission, however. The neutron lifetime against detection λ^{-1} represents the time scale for the neutron to move from the site of the fission chain out to the detector. Thus, for each $t_{m,i}$, a second list of waiting times $\delta\tau_j$, $j = 1, 2, \cdots, m$ must be generated according to

$$\delta\tau_j = -\frac{\ln u_j}{\lambda} \tag{29}$$

The neutron detection times $t_{m,k}$ are thus

$$t_{m,k} = t_{m,i} + \delta\tau_j \tag{30}$$

$$k = m(i-1) + j \tag{31}$$

To produce the final list of detection times, all the $t_{m,k}$ must simply be combined and then sorted. In this way, time-tagged neutron data for multiplying sources can be produced quickly.

Dead time was then added by recursively removing count times from the list of time-tagged neutron data: starting with the first neutron, if the waiting time from the last surviving count time to the next count time was less than the detector dead time, this count time was removed from the list of time-tagged data.

For multiple element detectors, each count time was assigned a random tube number chosen from a uniform distribution. Dead time was then added in the same way except that for a count time to be removed from the list of time-tagged data, both the count in question and the last surviving count had to share the *same* tube number.

Monte Carlo simulations were done for four sources: 50 kg ^{238}U (DU) driving ^{235}U with multiplication $M = 2$, 46 kg highly enriched uranium (HEU) (93% ^{235}U, 7% ^{238}U) with multiplication $M = 20$, 6 kg weapons-grade plutonium (94% ^{239}Pu, 6% ^{240}Pu) with multiplication $M = 6.67$, and 10 kg reactor-grade plutonium (94% ^{239}Pu, 6% ^{240}Pu) with multiplication $M = 10$. The input parameters for the Monte Carlo simulations are show in Table I. Values for C_n and C_{Sn} used in both the Monte Carlo simulations and in Eqs. 9 and 13 are shown in Table II.

For a single-element detector with dead time, plots comparing the distributions computed from Eqs. 20, and 21 and 22 to the distributions resulting from the Monte Carlo simulations are shown for the four sources in Figures 2, 3, 4, and 5. For a two-element detector with dead time, plots comparing the distributions computed from Eqs. 20, and 24 and 22 to the distributions resulting from the Monte Carlo simulations are shown for the four sources in Figures 6, 7, 8, and 9.

VI. ACKNOWLEDGEMENTS

The author wishes to thank Manoj Prasad and Neal Snyderman for many thoughtful discussions.

		C_n		C_{Sn}	
		^{235}U	^{239}Pu	^{238}U	^{240}Pu
n	0	0.0238	0.0085	0.0831	0.0632
	1	0.1556	0.0790	0.2370	0.2320
	2	0.3217	0.2536	0.3377	0.3333
	3	0.3150	0.3290	0.2403	0.2528
	4	0.1445	0.2328	0.0854	0.0986
	5	0.0356	0.0800	0.0152	0.0180
	6	0.0034	0.0156	0.0013	0.0020
	7	0.0005	0.0012		
	8		0.0003		

TABLE II. Values for C_n and C_{Sn} for uranium and plutonium.

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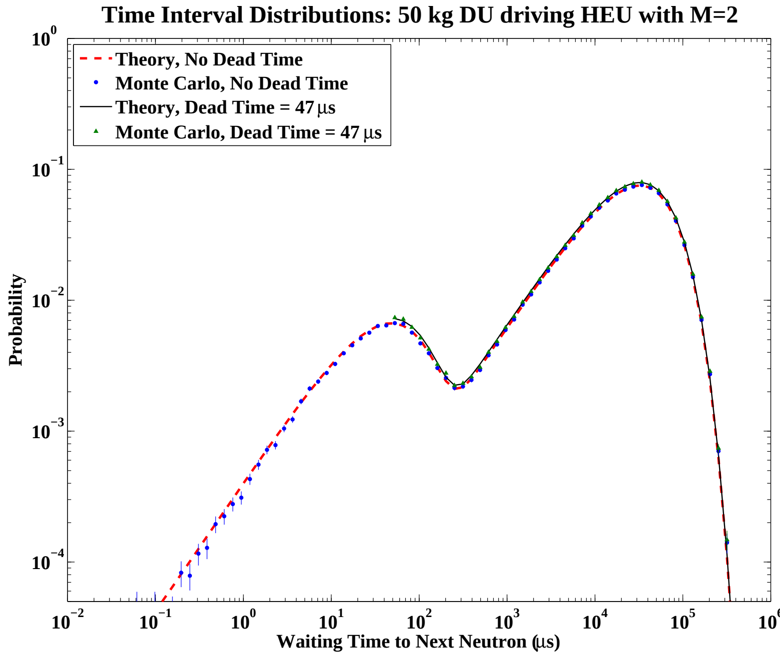

FIG. 2. Comparison of theory and simulated data with and without dead time for a neutron source comprised of 50 kg of DU driving HEU with $M = 2$. The dead time was set to 47 μs in order to maximize its effects on the distribution. Dead time corrections were carried out to third-order.

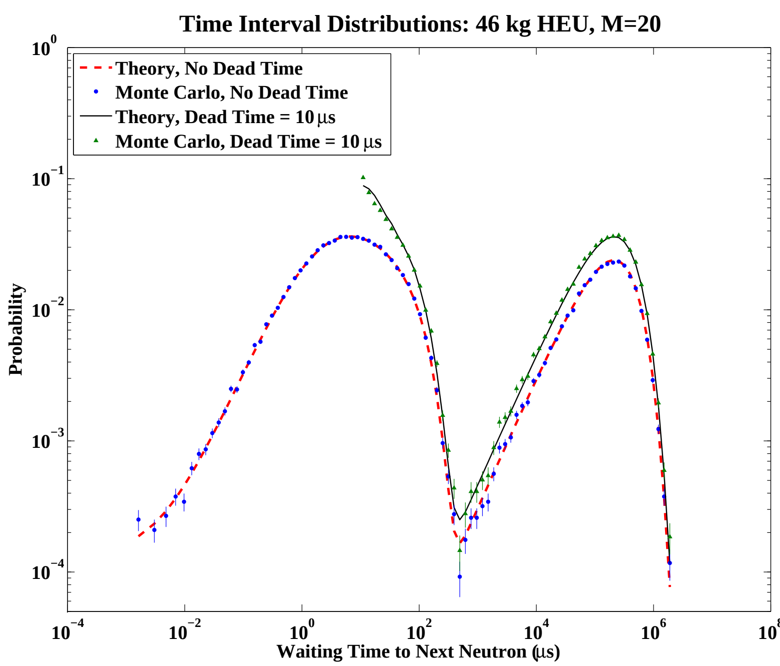

FIG. 3. Comparison of theory and simulated data with and without dead time for a neutron source comprised of 46 kg of HEU with $M = 20$. The dead time was set to 10 μs in order to maximize its effects on the distribution. Dead time corrections were carried out to third-order.

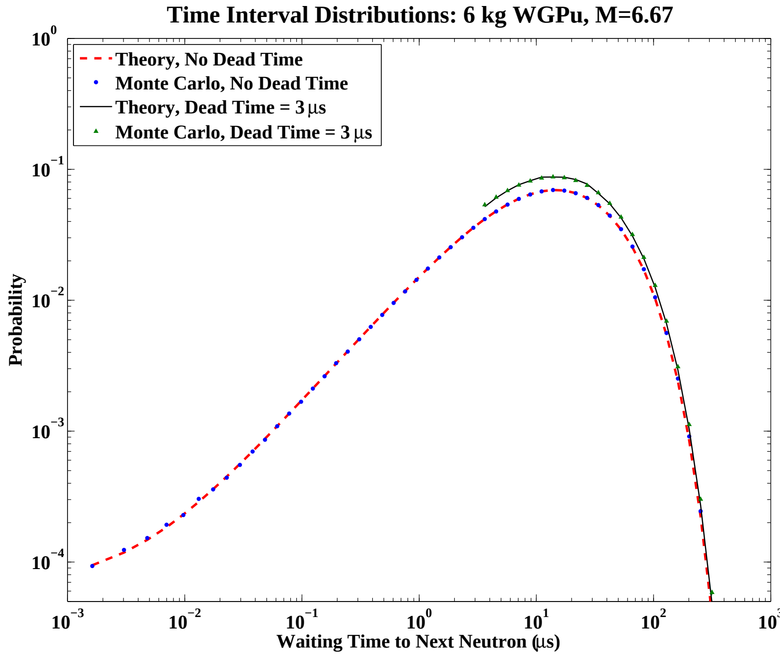

FIG. 4. Comparison of theory and simulated data with and without dead time for a neutron source comprised of 6 kg of WGPu (6% ^{240}Pu) with $M = 6.67$. The dead time was set to 3 μs in order to maximize its effects on the distribution. Dead time corrections were carried out to third-order.

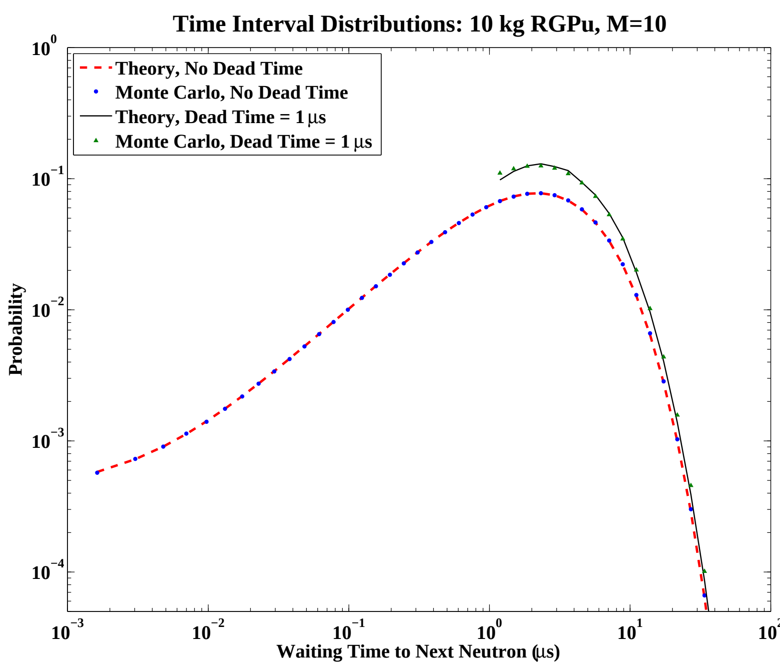

FIG. 5. Comparison of theory and simulated data with and without dead time for a neutron source comprised of 10 kg of RGPu (25% ^{240}Pu) with $M = 10$. The dead time was set to 1 μs in order to maximize its effects on the distribution. Dead time corrections were carried out to third-order.

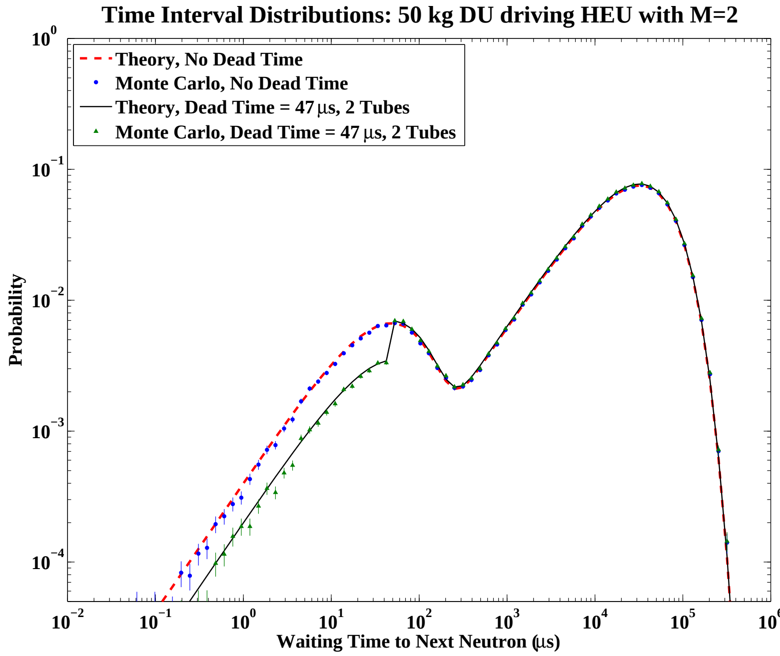

FIG. 6. Comparison of theory and simulated data for a two-element neutron detector with and without dead time for a neutron source comprised of 50 kg of DU driving HEU with $M = 2$. The dead time was set to 47 μs in order to maximize its effects on the distribution. Dead time corrections were carried out to third-order.

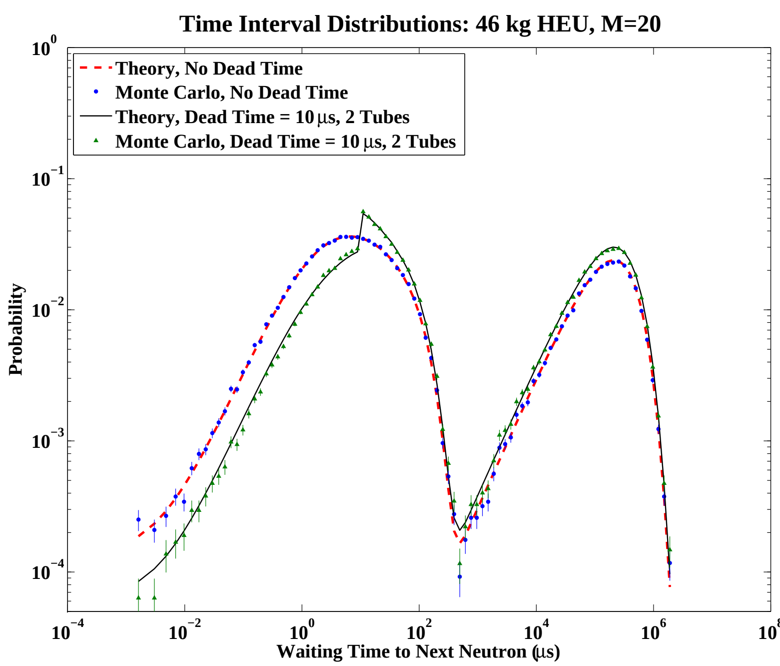

FIG. 7. Comparison of theory and simulated data for a two-element neutron detector with and without dead time for a neutron source comprised of 46 kg of HEU with $M = 20$. The dead time was set to 10 μs in order to maximize its effects on the distribution. Dead time corrections were carried out to third-order.

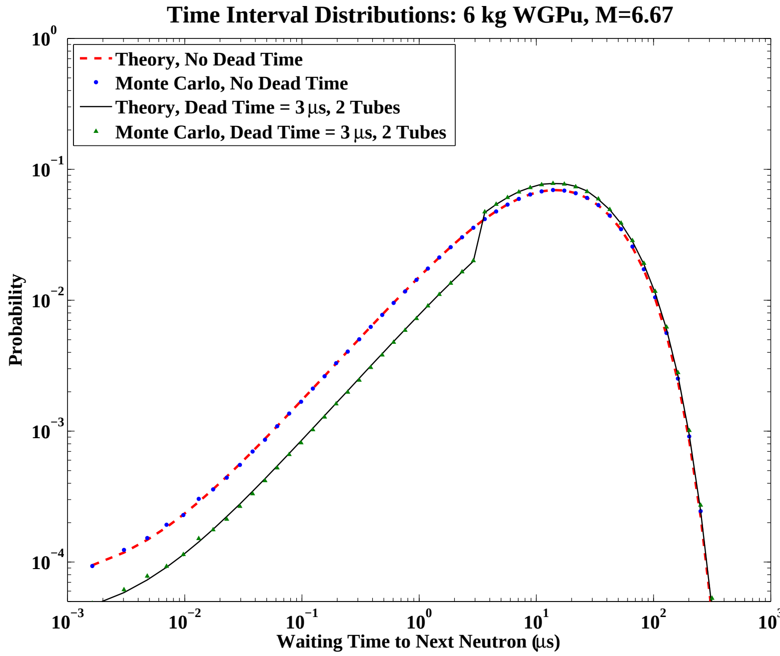

FIG. 8. Comparison of theory and simulated data for a two-element neutron detector with and without dead time for a neutron source comprised of 6 kg of WGPu (6% ^{240}Pu) with $M = 6.67$. The dead time was set to 3 μs in order to maximize its effects on the distribution. Dead time corrections were carried out to third-order.

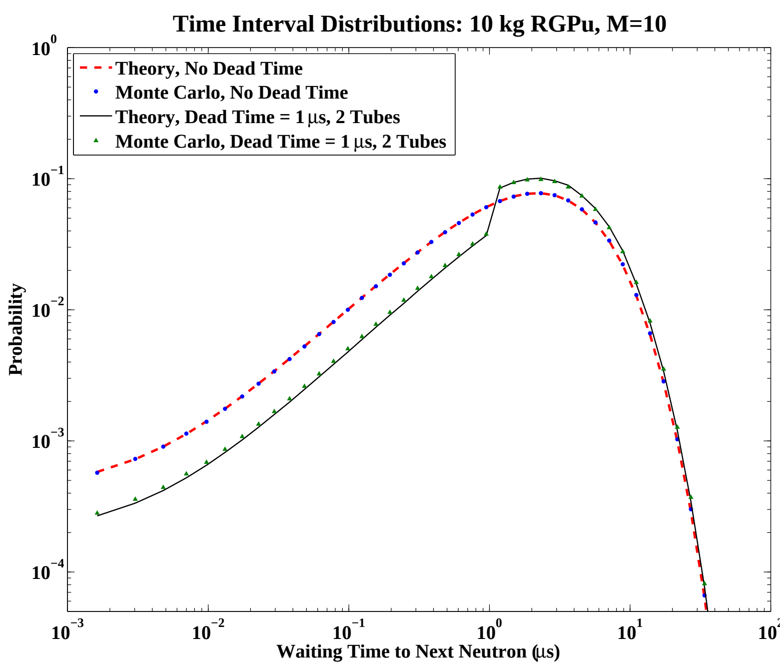

FIG. 9. Comparison of theory and simulated data for a two-element neutron detector with and without dead time for a neutron source comprised of 10 kg of RGPu (25% ^{240}Pu) with $M = 10$. The dead time was set to 1 μs in order to maximize its effects on the distribution. Dead time corrections were carried out to third-order.