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DE-FG05-80ET-53088-712 IFSR #712

Line-Broadened Quasilinear Burst Model

H.L. Berk, B.N. Breizman, J. Fitzpatrick,[a] and H.V. Wong
Institute for Fusion Studies
The University of Texas at Austin
Austin, Texas 78712 USA

[a] *permanent address: Dept. of Nuclear Engineering, Univ. of California, Berkeley*

July 1995

THE UNIVERSITY OF TEXAS

AUSTIN

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H.L. Berk, B.N. Breizman, J. Fitzpatrick,[a)b)] and H.V. Wong

Institute for Fusion Studies, The University of Texas at Austin
Austin, Texas 78712 USA

Abstract

A quasilinear model is developed to produce realistic self-consistent saturation levels when modes do not overlap, and give self-consistent diffusion and wave evolution when modes do overlap. Both regimes give steady or pulsating behavior in weakly driven systems with classical relaxation and background dissipation present. An avalanche response is demonstrated: wave momentum release caused by the overlap of closely spaced modes can produce mode overlap of more widely spaced modes (a domino effect) or the growth of modes which would be stable in systems unaffected by the closely-spaced modes' diffusion. Detailed analysis and calculations are performed for the bump-on-tail instability, and extension of the method to more general problems is briefly discussed.

[a)] permanent address: Dept. of Nuclear Engineering, Univ. of California, Berkeley
[b)] Work supported by U.S. DoE Magnetic Fusion Energy Technology Fellowship

I. INTRODUCTION

In an ignited fusion reactor, there is a concern that energetic alpha particles *may excite* Alfvén waves by tapping the universal drift instability mechanism that feeds off the spatial gradient in the distribution function.[1-3] If such instability occurs, the excited waves can either saturate at a low level without appreciably affecting the hot particle stored energy (therefore not affecting the rate of fusion heating of the background plasma), or cause global diffusion of the hot particles with the possible loss of the energetic alpha particles' energy to the first wall. Both cases have been observed experimentally.[4-6] Theoretical analysis has shown how, depending on parameters, both types of responses can arise when the sources and sinks of the energetic particles are accounted for and when dissipation from the background plasma is present.[7-9] Several simulation codes are now being developed to observe the saturation and diffusion mechanisms associated with this behavior and these investigations give results that are consistent with the theoretical description.[10-14] It has also been noted that the nonlinear behavior can be understood from general considerations of weak turbulence theory. Consequently, the bump-on-tail problem can serve as a paradigm for the more complicated problem of describing the effect of the Alfvén-alpha particle instability in a tokamak.

In all these problems there is a natural saturation level for an initially linearly unstable system, where the mode amplitude grows until the bounce frequency of resonant particles in the excited wave becomes comparable to the linear growth rate. Further evolution depends on whether each mode saturates as a single mode or if adjacent modes overlap. It is only when overlap occurs that global diffusion of particles can occur.

In this work we develop a model based on quasilinear theory, to describe the system response independent of whether the modes overlap or not. To construct such a model

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we modify the quasilinear equations in an intuitive manner, by incorporating analytic and numerical scalings for the effect of line broadening due to finite growth rate and resonant particle trapping. The text gives a specific example of the bump-on-tail problem, and indicates how the method could be generalized to the alpha particle problem in realistic tokamak geometry.

Another result of the present work is to show how mode overlap can lead to an avalanche due to the diffusive evolution of the distribution function. In the example discussed in detail in this paper, this phenomena appears as a "domino" effect where, when closely spaced resonances overlap, there is an enhanced release of wave energy, that in turn can cause nearby but more widely spaced resonances to overlap, thereby rapidly expanding the phase space region over which particles can diffuse. This nonlinear effect can also cause the destabilization of a spectrum of modes where most of the spectrum would otherwise be stable on the basis of linear stability.

In Sec. II we describe the physical self-consistent line-broadened quasilinear model. In Sec. III we discuss the theory for the nonlinear domino effect. In Sec. IV we present preliminary numerical results for the bump-on-tail instability. In Sec. V we discuss how the method can in principle be extended to more complicated geometry. In Sec. VI a brief conclusion is presented.

II. LINE BROADENING MODEL

It has been previously observed that the Alfvén wave problem is mathematically similar to the one-dimensional bump-on-tail problem. Here we consider the case where in both problems the wave spectrum is discrete. According to quasilinear theory, diffusion only occurs for the particles that exactly fulfill the resonance condition. In the bump-on-tail problem the resonance condition is $\Omega_n \equiv \omega_n - k_n v = 0$, with ω_n the eigenfrequency for the $n^{\rm th}$ mode (for the bump-on-tail problem we take $\omega_n = \omega_{pe} \equiv$ electron plasma frequency).

The quasilinear equation for the evolution of the distribution function, $f(v)$, takes the form

$$\frac{\partial f}{\partial t} = \hat{Q} f \equiv \frac{\partial}{\partial v} D(v) \frac{\partial f}{\partial v} \tag{1}$$

$$D(v) = \frac{2\pi e^2}{m^2} \sum_n |k_n \phi_{n0}|^2 \delta(\Omega_n) \equiv \sum_n D_n(v). \tag{2}$$

Here $\hat{Q}$ is a shorthand notation for the quasilinear operator, t is time, e and m are the energetic particle charge and mass, v is the energetic particle speed, and ϕ_{n0} the amplitude of the perturbed electrostatic potential. That potential, as a function of time and position x, is given by

$$\phi(x,t) = \sum_n \left[\phi_{n0} \exp(-i\omega_n t + ik_n x) + \text{c.c.}\right],$$

Associated with Eq. (1) is the wave evolution equation, which written as the evolution of wave momentum W_n, is of the form

$$\frac{\partial}{\partial t} W_n = 2\gamma_n W_n \tag{3}$$

where

$$W_n = \frac{|k_n \phi_{n0}|^2}{2\pi v_n}$$

$$\gamma_n = 2\pi^2 \frac{e^2}{m} \frac{\omega_n}{|\omega_n|} \frac{v_n}{k_n} \frac{\partial f(v_n)}{\partial v}, \tag{4}$$

$$v_n = \frac{\omega_n}{k_n}.$$

Note that Eqs. (1)–(4) imply conservation of momentum,

$$m \int_{-\infty}^{\infty} dv\, v\, f + \sum_n W_n = C \tag{5}$$

with C a time-independent constant.

There is however an intrinsic difficulty in solving Eqs. (1) and (3) if one takes the expression for $D(v)$ in Eq. (2) literally. This is because the domain of the diffusion coefficient is "over a point." Consequently, as written the distribution function can only relax in infinitesimal interval. In reality the diffusion domain should have a width in v. In fact when a finite growth rate, $\gamma_n > 0$, is taken into account, the diffusion coefficient is broadened as one finds

$$\pi\delta(\omega_n - k_n v) \longrightarrow \frac{\gamma_n}{(\omega_n - k_n v)^2 + \gamma_n^2}. \tag{6}$$

In fact the quasilinear coefficient is really best applicable when there may be waves that cause orbit stochasticity due to mode overlap. Only then is the diffusion coefficient independent of γ_n. Other cases cannot be treated as rigorously. When we do not have orbit stochasticity, we seek a method that realistically models the conversion of particle momentum to wave momentum. The results of the model system we use can be benchmarked with rigorously derived simulation results to ascertain the system's accuracy.

When we have steady waves, without orbit overlap, it is well known that the mean distribution flattens around the resonant particle region over a width that is comparable to the separatrix width of the wave particle interaction.[15,16] A rigorous solution requires accounting for the wave particle phase in calculating the wave-particle interaction. However, one can hope to model the wave-particle interaction by assuming that roughly particles within the separatrix width can stochastically mix in phase space, but particles outside the separatrix width move adiabatically with the wave and do not mix in phase space. For the bump-on-tail problem we take the n^{th} electrostatic wave to be of the form

$$\phi_n(x,t) = 2\phi_{n_0} \sin(k_n x - \omega_n t). \tag{7}$$

A conserved quantity is the energy in the wave frame, which is given by

$$E_n = \tfrac{1}{2} m(v - v_n)^2 + 2\phi_{n_0} \sin(k_n x - \omega_n t). \tag{8}$$

Particles for which $-2e\phi_{n_0} < E_n < 2e\phi_{n_0}$, lie inside the phase space separatrix. Particles on the separatrix satisfy

$$\tfrac{1}{2}(v - v_n)^2 = \frac{2e\phi_0}{m}\left[1 - \sin(k_n x - \omega_n t)\right].$$

The largest and smallest values that $v - v_n$ can take are

$$\frac{\delta v}{2} \equiv \max(v - v_n) = 2\left(\frac{2e\phi_{n0}}{m}\right)^{1/2}, \quad -\frac{\delta v}{2} \equiv \min(v - v_n) = -2\left(\frac{2e\phi_{n0}}{m}\right)^{1/2}.$$

We will assume that for a steady wave the diffusion coefficient is constant over the region $-\Delta v/2 < v - v_n < \Delta v/2$ ($\Delta v = \eta_n \delta v$ and η_n is an adjustable constant ≈ 1), and zero otherwise. In addition, to take into account diffusion during the growing and decaying phases of the wave, we add to Δv a factor proportional to the growth. Thus we choose for the nonzero region of diffusion

$$-\frac{\Delta v}{2} < v - v_n < \frac{\Delta v}{2} \tag{9}$$

with

$$\frac{\Delta v}{2} = \lambda_n \frac{|\gamma_n|}{k_n} + \beta_n \frac{\omega_{b,n}}{k_n} \tag{10}$$

with $\lambda_n \approx 1$ being another adjustable constant and $\omega_{b,n} = (2e\,k_n^2\,\phi_{n0}/m)^{1/2}$ (note $\omega_{b,n}$ is the trapping frequency of a particle deeply trapped in a steady finite amplitude wave) and $\beta_n = 2\eta_n$.

Instead of Eqs. (1) and (3) we choose for $D_n(v)$ and γ_n

$$D_n(v) = \frac{2\pi e^2}{m^2} \frac{\theta\left(v - v_n + \frac{\Delta v}{2}\right)\theta\left(v_n + \frac{\Delta v}{2} - v\right)|k_n\,\phi_{n0}|^2}{|k_n|\,\Delta v} \tag{11}$$

$$\gamma_n = \frac{2\pi^2\,e^2\omega_n}{mk_n} \frac{\left[f\left(v_n + \frac{\Delta v}{2}\right) - f\left(v_n - \frac{\Delta v}{2}\right)\right]}{|k_n|\,\Delta v} \tag{12}$$

This line-broadened model for $D_n(v)$ together with Δv as defined in Eqs. (9) and (10) allows for a self-consistent quasilinear description.

The model described above can be used when modes are separated from each other and when they overlap. When they are separated from each other, the distribution can diffuse and flatten only in a limited phase region that is centered about the resonance velocity, while the distribution maintains its shape in the region between the resonances where there is no diffusion. When there is overlap of two modes, (i.e. if say $v_{n+1} > v_n$, overlap occurs when $v_n + \Delta v_n/2 > v_{n+1} - \Delta v_{n+1}/2$) then particle diffusion can occur in a larger phase space region, as will be discussed in the next section.

III. STEADY STATE AND BURSTING

We now introduce extra processes in our model. These include a particle source, a classical particle relaxation mechanism, wave damping from the plasma background and an intrinsic wave fluctuation level. As discussed elsewhere, these mechanisms are needed to exhibit a pulsating response of the system. However, under the appropriate conditions, even with such mechanisms present, we can have a steady-state response. In this paper we will consider a particle source $S(v)$ and particle annihilation at an annihilation rate ν. Our complete system of equations is then

$$\frac{\partial f}{\partial t} = \hat{Q} f - \nu(v) f + S(v)$$
$$\frac{\partial W_n}{\partial t} = 2(\gamma_n - \gamma_{dn}) W_n + \epsilon_n. \tag{13}$$

Thus, without waves, $D(v) = 0$, and the steady state particle distribution function is

$$f_0(v) \equiv \frac{S(v)}{\nu(v)}.$$

We of course treat the case where $\frac{v}{|v|}\frac{\partial f_0(v_n)}{\partial v} > 0$, so that in absence of background dissipation the system is unstable with a predicted linear growth $\gamma_{L_n} = \frac{2\pi^2 e^2 v_n \omega_n}{m|\omega_n| k_n}\frac{\partial f_0(v_n)}{\partial v}$. The damping rate for mode n in the absence of hot particles is γ_{dn}, and the wave thermal fluctuation level in the absence of hot particles is given by

$$W_n = \frac{\epsilon_n}{2\gamma_{dn}}.$$

Our model has been created to exhibit the same pulsating behaviors for driven systems that have been analyzed elsewhere. Previously Berk, *et al.*[8,9] have discussed the cases where the modes are discrete and non-overlapping where one has a benign steady or benign pulsating response, or when there is mode overlap giving rise to an explosive pulsation or a steady quasilinear response. Let us first analyze the non-overlapping case, where each mode acts independent of the others.

To find the maximum saturation amplitude, we neglect the background damping. This serves as an estimate for saturation if $\gamma_d \ll \gamma_L$. Then the mode will grow until the distribution flattens over the region in an interval I, defined as $-\Delta v/2 < v - v_n < \Delta v/2$, with Δv determined by the condition that the total momentum (wave momentum plus particle momentum) and particle number in the final state equals the initial particle momentum and particle number within the interval I. Thus, the saturation level is determined by

$$f_F \Delta v = \int_{v_n - \Delta v/2}^{v_n + \Delta v/2} dv\, f_L \tag{14}$$

$$W_n = m \int_{v_n - \Delta v/2}^{v_n + \Delta v/2} f_L v\, dv - m \int_{v_n - \Delta v/2}^{v_n + \Delta v/2} dv\, f_F v \tag{15}$$

with f_F the final distribution function that is flattened over the interval I. With $f_L(v) \doteq f_L(v_n) + (v - v_n) f_L'(v_L)$, we find that

$$\Delta v_n^3 = \frac{12 W_n}{m f_L'(v_n)} = \frac{3\pi |\omega_{bn}|^4}{k^3 \gamma_{L_n}}.$$

Now, at saturation, we have from Eq. (10) $\dfrac{k_n \Delta v_n}{2} = \beta_n |\omega_{bn}|$. Thus we find that the bound for the saturation level for a pulsation is given by

$$|\omega_{bn}| \equiv \left| \frac{2e\,\phi_n\, k_n^2}{m} \right|^{1/2} = \frac{8\beta_n^3}{3\pi} \gamma_{L_n}. \tag{16}$$

This result shows that the saturation level for a single mode scales so that the trapping frequency of a particle in a wave is proportional to the linear growth rate. When background damping has to be taken into account ω_{bn} will be less than Eq. (16) as the wave momentum will be transferred to the background plasma through the linear dissipation mechanism, and the equilibrium distribution does not have to completely flatten before stabilization occurs.

Now let us consider what happens when waves overlap. Suppose DV is the interval in velocity over which the linear modes overlap (we assume $DV \gg \gamma_L/k$). For simplicity let us take $\partial f_L/\partial v$ a constant in this region, and again we neglect damping. Then if the distribution is flattened over the phase space region that is active with linear modes, then a repeat of the above logic shows that the wave momentum that is released is

$$\sum_n W_n = \frac{m(DV)^3}{12} f_L'(v). \tag{17}$$

Let us assume that the phase velocity spacing, $v_{n+1} - v_n \simeq \gamma_{L_n}/k_n \simeq DV/N$, with N the number of unstable modes. In this case the total wave momentum released is

$$\sum_n W_n \simeq \frac{mN^3}{12} \left(\frac{\gamma_n}{k}\right)^3 f_L'(v) \tag{18}$$

which is N^3 the wave momentum released by a single mode and N^2 the wave momentum released by N modes that do not quite overlap. As noted in previous works, this large release of wave momentum causes rapid global diffusion of particles over the active velocity space region as a large fraction of the momentum stored in the equilibrium distribution can be converted to wave momentum. If the particles can reach the boundary, and be lost at say $v = 0$, then in principle all of the stored particle momentum can be converted to wave momentum. If there is particle loss, then the bound on the level of wave momentum conversion is even four times larger than if the particles were not lost at the $v = 0$ boundary.

We now note that overlap can occur even when the average spacing between modes, $\overline{v_{n+1} - v_n}$, is larger than the average $\overline{\gamma_n/k_n}$. This arises because of the possibility of a "nonlinear domino" effect. As a set of modes in an interval in velocity, δv_F, flattens, the distribution takes on a sharp velocity gradient at the interface between the region of velocity where waves have been excited, and the region of velocity where waves have not been excited.

Thus, at this interface, $\frac{\partial f}{\partial v} \doteq \Delta f\, \delta(v - v_I)$ where v_I is the interface velocity. By assuming f is flat between $v_I > v > v_I - \delta v_1$, and f has its value before excitation of instability for $v > v_I$, we find that $\Delta f = \frac{\partial f_L}{\partial v}\frac{\delta v_1}{2}$. For such a step function distribution, the dispersion relation for a mode that is excited by a phase velocity near v_I, $\gamma \ll \omega_{pe}$, can be written as

$$2(\omega - \omega_{pe}) = \frac{-4\pi e^2}{m} \int \frac{dv \frac{\partial f}{\partial v}}{\omega - k_n v} \doteq \frac{-\gamma_{L_n} k_n \delta v_1}{\pi(\omega - k_n v_1)}. \tag{19}$$

For $k_n v_I \doteq \omega_{pe}$, we then find that the growth rate, γ_i, for a mode at the interface is

$$\gamma_I = \gamma_{L_n} \left(\frac{\delta v_1}{2\pi \delta v_{L_n}}\right)^{1/2} \tag{20}$$

where $\delta v_{L_n} \equiv \gamma_n / k_n$ is the "velocity width" of linear mode. Thus we see that the linear growth rate is enhanced by a factor $(\delta v_1 / 2\pi \delta v_{L_n})^{1/2}$ at the interface between the flattened and unperturbed distribution. In order for this enhanced growing mode to saturate, the wave momentum has to grow until the trapping frequency of this mode, ω_{bI}, is comparable to γ_I. Here saturation occurs when $\omega_{bI} \sim \gamma_I$ (this result can also be obtained from momentum and particle conservation arguments). This implies that the wave momentum of the enhanced growing mode, $W_I \propto \omega_{bI}^4$, saturates at a level

$$\frac{W_I}{W_n} \approx \left(\frac{\delta v_1}{\delta v_{L_n}}\right)^2. \tag{21}$$

This result gives a comparable saturation level as our previous estimate for the saturation level with mode overlap found in Eq. (18) which assumed $\delta v_1 / \delta v_{L_n} \approx N$. Thus, the release of wave momentum due to mode overlap is consistent with the linear stability that is implied by the evolution of a nonlinearly distorted distribution function. It should also be noted that when there is strong mode overlap, the quasilinear diffusion is insensitive to the mode widths of the individual waves.

Because the growth rate enhances as the distribution function flattens, one can achieve global overlap even if the mean phase velocity spacing between unstable modes exceeds

$\overline{\gamma_n/k_n}$. We do require some modes to have a separation $v_{n+1} - v_n \lesssim \gamma_n/k_n$. However, as these modes flatten, the saturation levels of the edge modes increase. For example, suppose we have a set of modes denoted by the index p, with a mode separation

$$v_{p+1} - v_p \approx \frac{\gamma_n}{k_n} p.$$

Thus, the saturation of the modes with $p = 1$ and 2 that are separated just enough to cause mode overlap of two neighboring modes, can trigger a reaction where mode overlap arises for the more widely spaced modes at larger p. The result gives rise to an "avalanche" that produces global particle diffusion, and can even give particle loss at the edge of the velocity space boundary. Another consequence of this enhanced growth rate is that regions that are stabilized in the linear theory due to the background damping can be destabilized by the nonlinear enhancement of the growth rate that occurs as the "diffusion front" propagates into the linearly stable region.

Another way the domino effect can arise is if there is a spectrum of relatively close modes which are predominantly stable due to the presence of background damping. If a few of these modes are unstable with a growth large enough to cause local mode overlap, the resulting steepening in the distribution function will enhance the drive so that the spectrum can be excited and cause mode overlap in the otherwise stable region. This mechanism is a strong candidate to explain energetic particle losses that have been observed in experiment.

IV. SIMULATION OF TWO-STREAM INSTABILITY

In this section we report on numerical results that follow the evolution of Eqs. (13). We first of all need to choose the parameters λ_n and β_n. Our results were found to be insensitive to λ_n, and we have set $\lambda_n = 1$ in the simulations. The parameter $\beta_n = 2\eta_n$ was established by synchronizing the steady state wave energy for a single mode with sources and sinks found in Ref. 14. The system is defined by the parameters $\gamma_L/\omega = 0.176$, $\nu/\gamma_L = 0.035$,

$\gamma_d/\gamma_L = .031$ and $\frac{v_{\max}}{\nu}\frac{\partial S}{\partial x} = 1$ with $x = 1 - v/v_{\max}$, with $v_{\max}$ the maximum velocity of the simulation. We assume particles are lost at $v = 0$ and are reflected at $v = v_{\max}$. With a choice $\eta_n = 0.73$ we achieve the same steady state wave momentum as found by the more basic simulation of Ref. 14. This result is shown in Fig. 1. However, unlike the result in Ref. 14, the quasilinear simulation has an overshoot not seen in the more basic simulation. If we change parameters so that $\gamma_L/\omega = .0039$, $\nu/\gamma_L = .0082$, $\gamma_d/\gamma_L = 0.082$, $\frac{v_{\max}}{\nu}\frac{\partial S}{\partial x} = 1$, pulsations of a single mode result. Figure 2 shows the pulsating results of the quasilinear theory which is of similar character to the corresponding results in Ref. 14 with the same value of $\eta_n = 0.73$. The quasilinear model gives an average pulse width $(\overline{\omega_{bn}(\max)/\gamma_L})$ [with $\omega_{bn}(\max)$ the maximum trapping frequency of a single pulse] that is a factor 1.14 larger than that seen in the more basic simulation, while the first pulse [that is due to the initialized unstable distribution function where $\frac{\partial f}{\partial v} = \frac{\partial}{\partial v}\left(\frac{S(v)}{\nu}\right)$] saturates when the normalized bounce frequency, ω_{bn}/γ_L, is .88 of the value found in the more basic simulation. If we run a case without a loss mechanism, without a source and without damping, our choice of β_n gives a value of ω_{bn}/γ_L that is .81 of the more basic simulation result. Thus, though there is some discrepancy, one finds that the essential scalings of the more basic simulations can be reproduced.

We now exhibit what happens when mode overlap occurs due to the domino effect. We first choose parameters so that a domino effect does not occur. In Fig. 3, we show the mode width as a function of time when overlap does not occur. We see the modes pulsate in time, but saturation is at a relatively low level. In Fig. 4 we see the resulting distribution. It contains local flattening, but the overall stored particle momentum (or particle energy) is essentially the same as if the pulsation were absent. In this figure $x = 1 - v/v_{\max}$. By increasing γ_{L_n}/ω by 5%, we can achieve mode overlap.

Figures 5–7 show, respectively, the "explosion" that occurs in the resonance width, in the wave momentum, and in the distribution function when we attain mode overlap. This explosion leads to global diffusion to the edge of the domain that gives rise to particle loss at the boundary. Nearly all the particles are lost as a result of the explosion. One can also observe in these figures that the flattening starts in the interior, and the diffusion pulse propagates to the outside as time evolves. After the particles are first lost at $v = 0$, the wave amplitude continues to increase and the diffusion pulse then rapidly "cleans" out all the particles in nearly the entire phase space. After this pulse, the wave damps and the particle distribution builds up again, until another pulse is triggered. The retriggering gives a series of bursts similiar to the bursts in Fig. 2, and the domino effect as shown will arise at every other pulsation, the pulse shown in Fig. 5 being the first to display it.

V. GENERALIZATION OF METHOD

In more complex problems, such as the hot particle interaction with Alfvén waves in a toroidally symmetric tokamak the phase space is of higher dimension than the two-stream problem, but the line broadened model we have described can be extended. In the appropriate quasilinear equations[17] for the energetic particle evolution, one has a diffusion coefficient similar to Eq. (2), where the resonance function Ω_n in the delta function $\delta(\Omega_n)$ has the form

$$\Omega_n = \omega_n - n\varpi_\phi(E, \mu, p_\phi) - p\varpi_\theta(E, \mu, p_\phi)$$

with E, μ, p_ϕ respectively, the constants of motion: energy, magnetic moment and canonical angular momentum. (Equivalently one can choose other constants of motion.) The frequencies ϖ_ϕ and ϖ_θ are, respectively, the hot particle's mean toroidal angular frequency, and the mean poloidal angular frequency; n is the toroidal quantum number of the mode and p is an integer. (There are in general multiple p values even for a given n-value due to the poloidal angular structure of the mode and the particle orbit response). The phase space region covered by the condition $\Omega_n = 0$ are surfaces in the three-dimensional E, μ, p_ϕ phase space. Hence, as in the two-stream case the quasilinear diffusion is localized in phase space, and may not be global. The resonant surfaces can be given a finite "thickness" in a manner similar to what is described in Sec. II. First, broadening arises due to a finite growth rate. Broadening also arises due to "wave trapping" of particles in the phase space. The generic trapping frequency is proportional to $A^{1/2}$ where A is the amplitude of a wave. Specific forms for the trapping frequency in a tokamak for mode frequencies less than the cyclotron frequency have been calculated in Refs. 18 and 13. From these solutions the width $\Delta\Omega_n$ can be calculated. Note that, in toroidal geometry, the nth low-frequency wave causes a change ΔE_n and $\Delta p_{\phi n}$ in the particle orbit, while $\Delta\mu_n = 0$. Further $\Delta E_n = \frac{\omega_n}{n}\Delta p_{\phi n}$ and thus the spread of Ω_n is given by

$$\Delta\Omega_n = \left[\frac{\omega_n}{n}\frac{\partial\Omega_n(E,\mu,p_\phi)}{\partial E} + \frac{\partial\Omega_n(E,\mu,p_\phi)}{\partial p_\phi}\right]\Delta p_{\phi n}$$

with $\Delta p_{\phi n} \propto A_n^{1/2}$. Detailed evaluations of the proportionality constant can be extracted from Refs. 7,13, and 18 and will be presented in later work.

The spread in Ω_n given by the above equation can now be incorporated into a more general quasilinear equation. The evolution of waves and the distribution function can then be treated in a manner similar to the two-stream problem to follow wave excitations and particle diffusion in the benign (non-overlap) and explosive (overlap) regimes. The evolution of waves and particle diffusion in the benign and explosive regimes can be followed with an approach similar to our analysis of the two-stream instability.

VI. CONCLUSIONS

In this work we have indicated how to develop a quasilinear model to exhibit benign or global diffusive behavior of particles, established by the balance of sources and sinks, which have a distribution function that causes weak instability. Detailed calculations have been performed for the two-stream instability, and we have indicated how this method can be extended to more general situations, such as Alfvén instabilities caused by energetic particles in a tokamak.

A key to understanding the response is to study the resonance structure and to understand the condition for mode overlap which will cause global diffusion. Mode overlap will arise if the resonance width is comparable to or in excess of separation of the active modes (i.e. the modes in the linearly unstable region). In this work we point out a "domino" effect that allows a small cluster of modes that satisfy this overlap condition, to give rise to a nonlinear enhancement of the wave energy that in turn allows for more widely spaced modes to overlap. In the example given, we show that energetic particles can be rapidly lost with a very small change of plasma parameters once the domino effect is triggered.

Acknowledgments

This work was supported by the U.S. Dept. of Energy contract No. DE-FG05-80ET-53088.

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FIGURE CAPTIONS

FIG. 1. Evolution of wave momentum of a single mode with source, sink, and background dissipation giving rise to a steady saturation level. Parameters γ_L/ω, normalized particle absorption rate ν/γ_L, and normalized background wave dissipation rate γ_d/γ_L are given in text.

FIG. 2. Evolution of wave momentum of a single mode with source, sink, and background dissipation present giving rise to a pulsating response. Parameters given in text.

FIG. 3. Time evolution of resonance widths (shaded areas) for a multi-mode system where mode overlap does not occur. Time range of graph selected to coincide with that of Fig. 5. System evolved from $f(v, t = 0) = 0$ and $W_n(t = 0)$ values at thermal noise levels to give benign pulsations, with period of order of that of bursts in Fig. 2.

FIG. 4. Particle distribution function as a function of time for simulation shown in Fig. 3. Resonance locations are shaded.

FIG. 5. Time evolution of resonance widths for a multi-mode system where mode overlap leads to the domino effect. The curves, in different line styles, show the boundaries of individual resonances. System evolved from same initial conditions as those of Figs. 3-4; γ_L/ω is 5% larger than in those figures.

FIG. 6. Evolution of normalized wave momenta for the modes of Fig. 5. Modes located at lower values of x are located further to the left side of the graph. Modes at higher values of x saturate at higher levels because they can absorb particle momentum from particles transported from lower x, as well as from particles originally within their resonance widths.

FIG. 7. Particle distribution function as a function of time, for the simulation shown in Fig. 5.

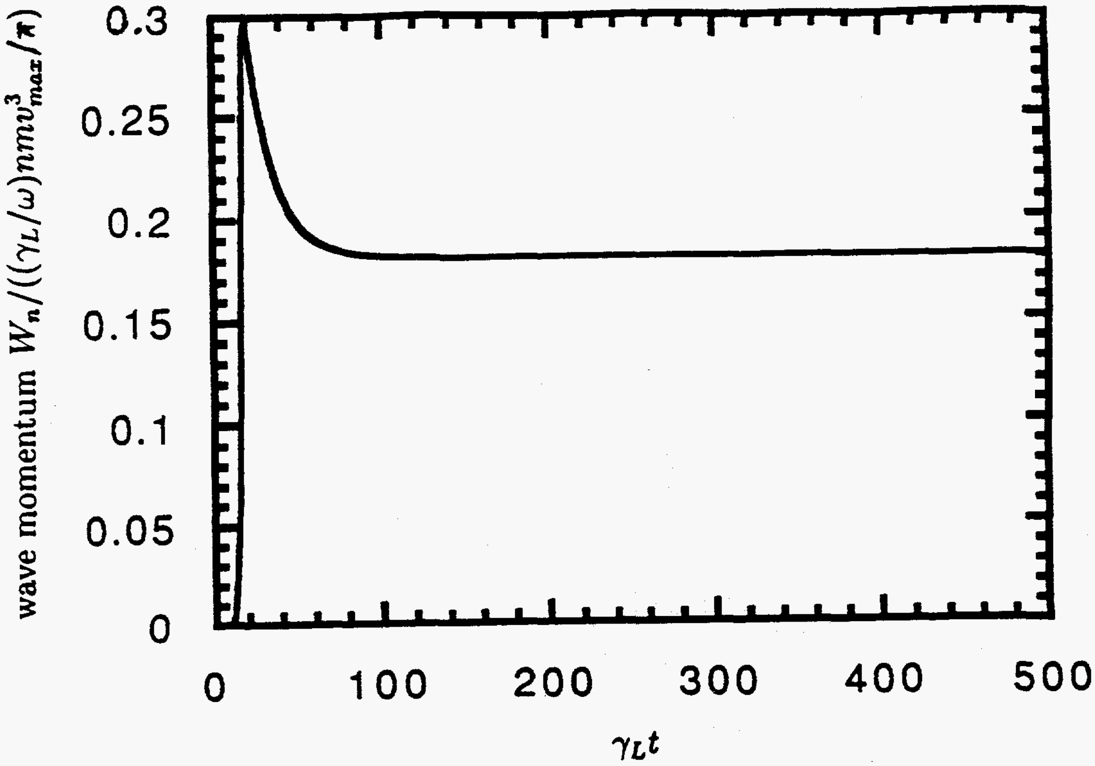

Figure 1

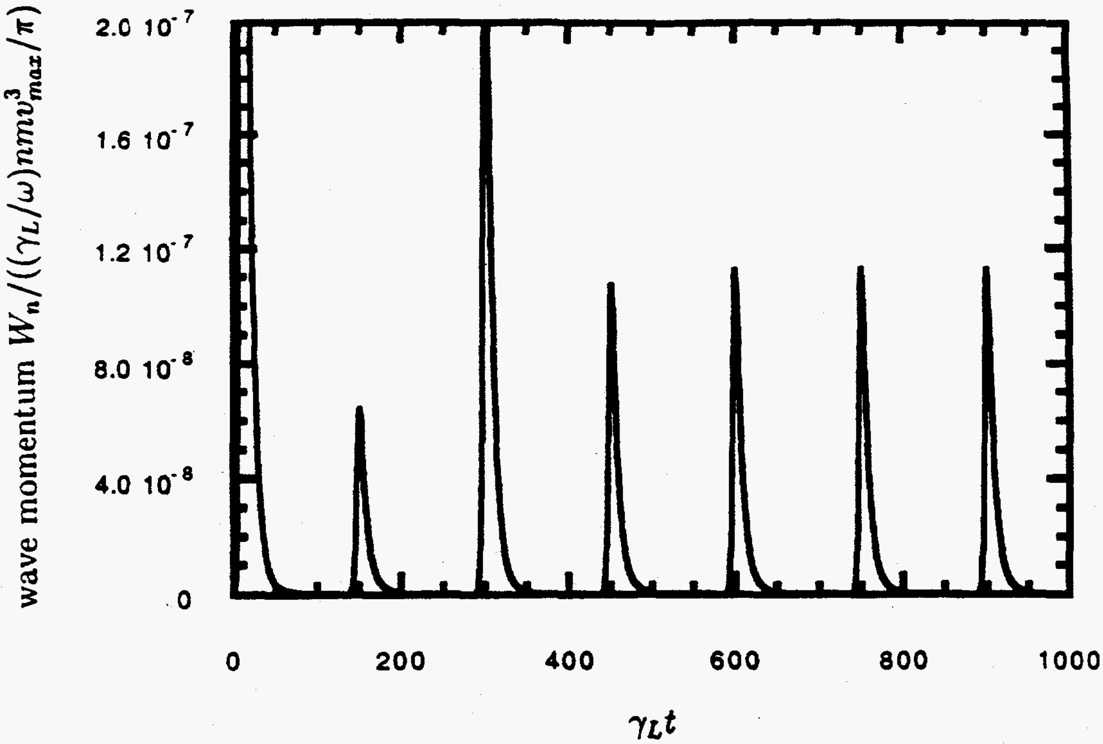

Figure 2

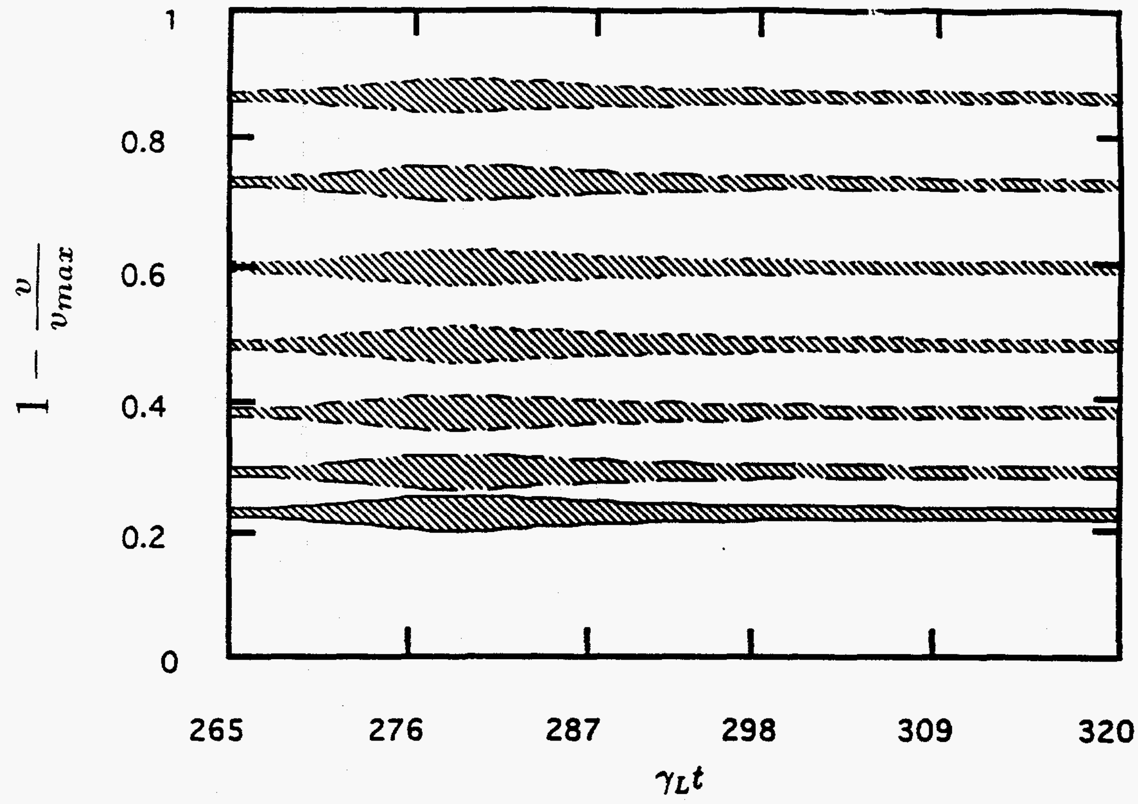

Figure 3

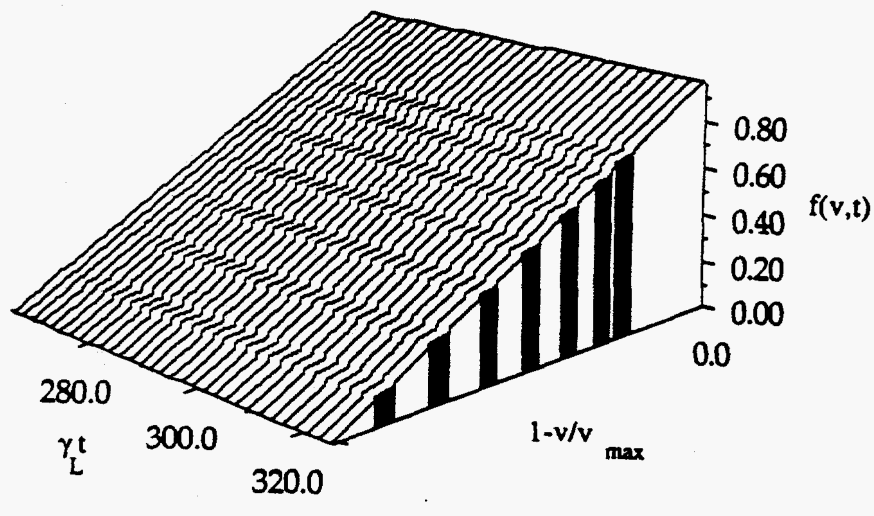

Figure 4

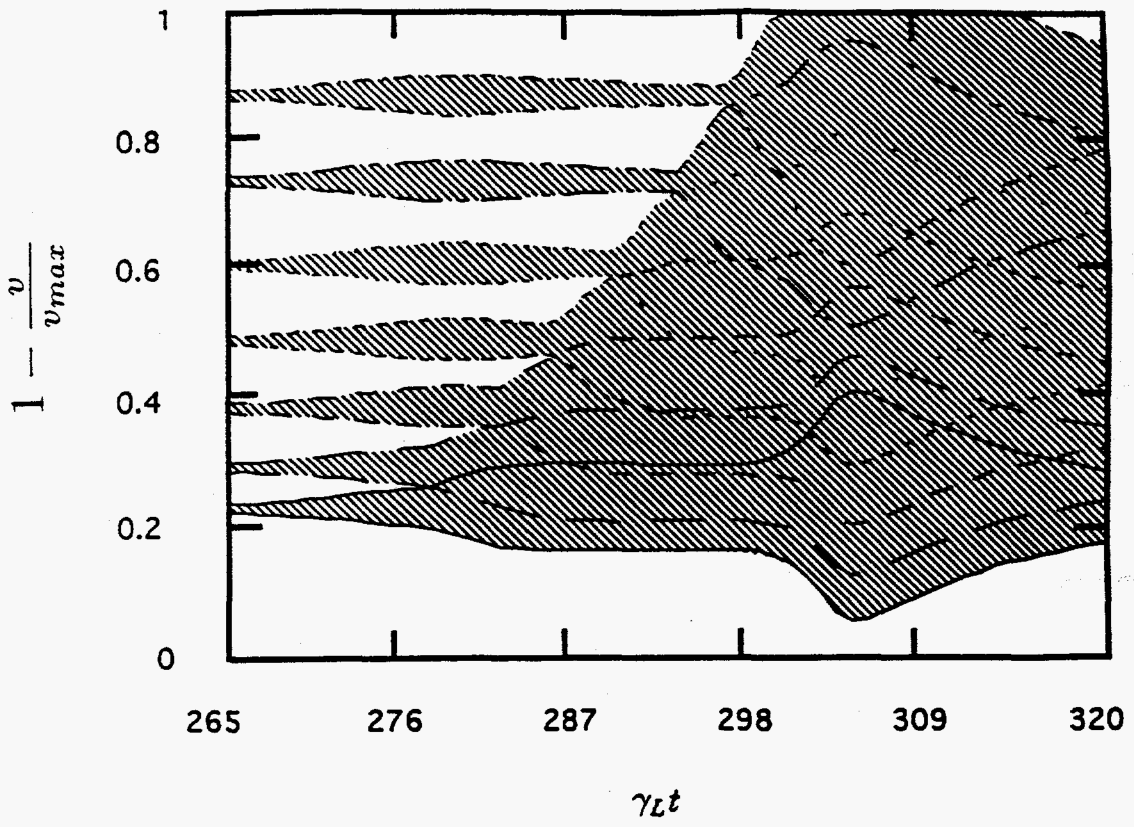

Figure 5

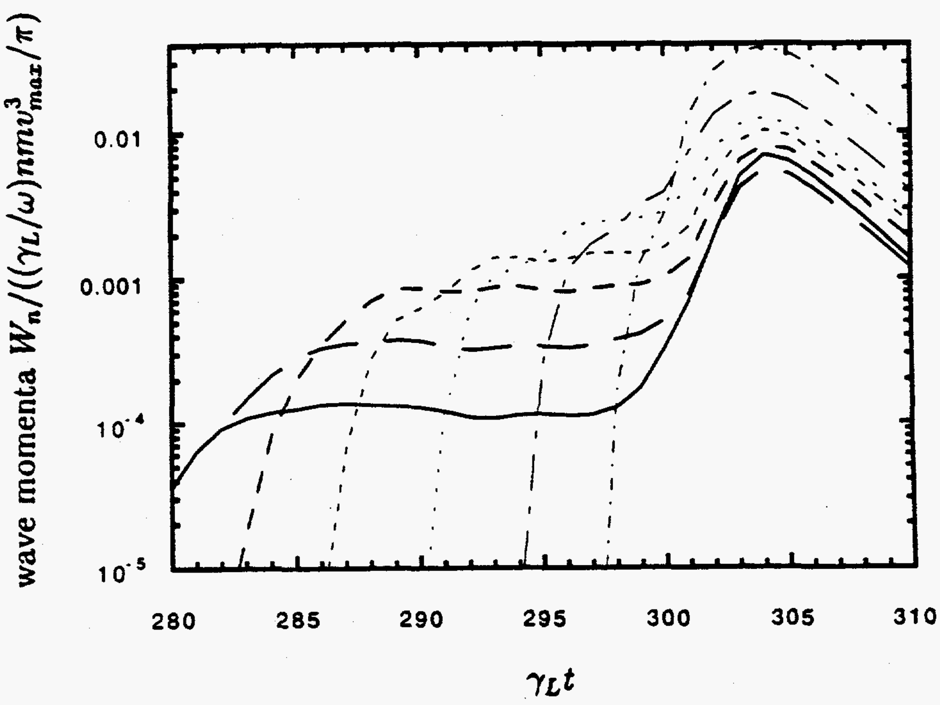

Figure 6

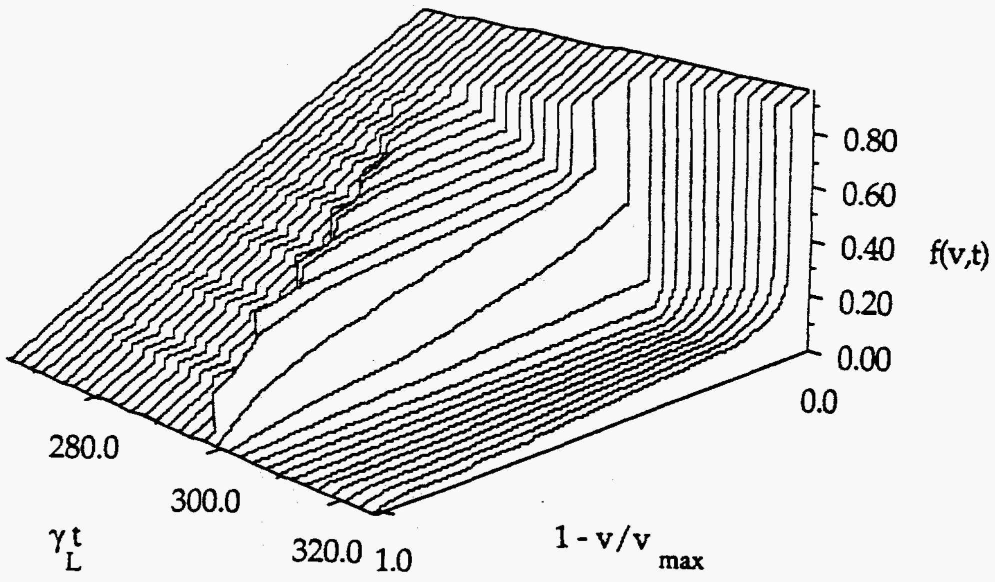

Figure 7