NewtonPlus: Approximating Relativistic Effects in Supernova Simulations

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Abstract

We propose an approximation to full relativity that captures the main gravitational effects of dynamical importance in supernovae. The conceptual link between this formalism and the Newtonian limit is such that it could likely be implemented relatively easily in existing multidimensional Newtonian gravitational hydrodynamics codes employing a Poisson solver. As a test of the formalism's utility, we display results for rapidly rotating (and therefore highly deformed) neutron stars.

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I. INTRODUCTION

The collapsed cores of massive stars are relativistic bodies. Order-of-magnitude estimates of the gravitational potential, infall and outflow velocities, equatorial linear velocity, and microscopic nucleon velocity—computed using typical values of mass, radius, and rotation periods of pulsars—suggest that relativity cannot be neglected. More importantly, detailed comparisons of Newtonian and relativistic supernova simulations in spherical symmetry show more compact cores and higher neutrino luminosities and average energies in relativistic treatments [1,2]. Because of indications that small (several percent) variations in, for example, neutrino luminosities and shock stagnation radius can make the difference in a successful explosion [3], it is clear that even modest relativistic effects comprise an indispensible component of realism in supernova studies.

The multidimensional nature of supernovae must also be recognized as an important aspect of realism. Various observations, especially data from SN 1987A, point to the asphericity of supernova explosions (see e.g. [3] for an overview, and [4] for a recent polarimetry analysis of several supernovae). Convection may play an important role in the explosion mechanism [5–8], and the differing results in various simulations in up to two dimensions (2D) [9–12,3,13,14] show that further study is needed. Ultimately consideration of the third dimension will be necessary. Based on an initial exploration of 3D effects, it has been reported [3] that the sizes of convective cells in 3D simulations are about half as large as in 2D simulations. Moreover, rapid rotation can significantly affect the strength and spatial distribution of convection [15]. Detailed studies of magnetic field generation, jet formation, and neutron star kicks also invite 3D treatments.

Since neutrinos—which carry away about 99% of the gravitational binding energy released in the collapse—are believed to drive the explosions, accurate neutrino transport is also essential to realistic simulations. Simulations with Boltzmann transport have been recently performed in spherical symmetry [16,17,1].

Including all the physics necessary for realism in a supernova simulation is a daunting task. The multidimensional simulations mentioned above, which had simplified neutrino transport, taxed the computational resources of their time; the same is true of the recent simulations involving Boltzmann transport in spherical symmetry. Adding general relativity to the list of desired physics makes things all the more challenging. While numerical relativity has been successful in spherical and axisymmetric cases, "...in the general three-dimensional (3D) case which is needed for the simulation of realistic astrophysical systems, it has not been possible to obtain stable and accurate evolutions...," and it is argued that the difficulties are more fundamental than insufficient resolution [18].

In order to overcome the difficulties associated with 3D general relativistic simulations and to save resources for 3D hydrodynamics and accurate neutrino transport, an approximate multidimensional treatment of gravity that captures the phenomena of dynamical importance in supernovae would be desirable. The list of new gravitational phenomena introduced by relativity includes the nonlinearity of the gravitational field, the inclusion of all forms of energy and stress as sources, and gravitational waves. The first two of these effects are of dynamical importance in supernovae, while gravitational waves will probably not exert a strong back-reaction (unless bar or breakup instabilities of some sort become operative in the core). The post-Newtonian expansion is systematic and useful in perturbative applications, but in the present context, an approach that probes the nonlinear nature of gravity more deeply would be desirable. A method that could be incorporated into existing Newtonian hydrodynamics codes would be even more useful. We here describe such an approximation—which we call "NewtonPlus"—and present results for rapidly rotating (and therefore highly deformed) neutron stars, which show that this simple "NewtonPlus" approach to gravity is indeed a significant improvement over the Newtonian limit. A more detailed exposition is given in Ref. [19].

II. EINSTEIN EQUATIONS IN THE NEWTONPLUS APPROXIMATION

Use of the metric

$$ds^2 = -(1+2\Phi)dt^2 + (1-2\Phi)d\mathbf{x}^2 \tag{1}$$

in the Einstein equations yields the Newtonian limit, provided the gravitational potential $\Phi \ll 1$ and velocities (including microscopic velocities, large values of which lead to significant stresses) are much less than the speed of light. In order to capture the nonlinearity of gravity, the significance of stresses as gravitational sources, and relativistic fluid velocities, we propose the use of the following metric:

$$ds^2 = -e^{2\Phi+2\delta}dt^2 + e^{-2\Phi}(dr^2 + r^2 d\Omega^2), \tag{2}$$

where $d\Omega^2 \equiv d\theta^2 + r^2 \sin^2\theta\, d\phi^2$. In comparison with Eq. (1), the "linearized" metric functions have been promoted here to full exponentials, and a second metric function, δ, has been added. Eq. (2) will then reduce to the Newtonian case if $\delta \rightarrow 0$; we shall see that this is in fact the case under conditions in which the Newtonian limit is valid. This provides a tight conceptual link with the Newtonian limit. Since it has two independent metric functions, this "NewtonPlus" metric should also provide an exact solution in spherical symmetry.

The metric, a symmetric 4×4 matrix, has ten independent components, but the invariance of relativity under coordinate transformations implies that in reality there are only six degrees of freedom. It is apparent that the NewtonPlus metric of Eq. (2) contains only two of the six degrees of freedom that should be present. This means that examination of the complete set of Einstein equations should reveal inconsistencies, but we have argued that these are not too serious in the supernova environment [19]. Here we present highlights of that more detailed discussion, which involves the (3+1) formulation of the Einstein equations [20–23], in which spacetime is foliated into spacelike slices labeled by a time coordinate.

We begin with the Hamiltonian constraint. This yields

$$\nabla^2\Phi = 4\pi e^{-2\Phi}E + \frac{1}{2}(\partial\Phi)^2 - \frac{3}{2}e^{-4\Phi-2\delta}(\partial_t\Phi)^2. \tag{3}$$

In this expression ∇^2 is the usual 3D flat-space Laplacian. The energy density as viewed by an "Eulerian" observer (i.e., one whose 4-velocity is orthogonal to the spacelike slices, having covariant components $n_\mu = (-\alpha, 0, 0, 0)$, where α is the lapse function) is denoted by $E \equiv T^{\mu\nu}n_\mu n_\nu$; where $T^{\mu\nu}$ is the stress-energy tensor. For a perfect fluid, $E = \Gamma^2(\rho+p) - p$, where $\Gamma = (1-v^2)^{-1/2}$, v is the magnitude of the fluid velocity as measured by an Eulerian observer, and ρ and p are respectively the total energy density and pressure in the fluid rest frame. We have employed the notation

$$\partial X\,\partial Y \equiv \partial_r X\,\partial_r Y + \frac{1}{r^2}\,\partial_\theta X\,\partial_\theta Y + \frac{1}{r^2\sin^2\theta}\,\partial_\phi X\,\partial_\phi Y. \tag{4}$$

As expected from the conceptual link between the Newtonian and NewtonPlus metrics, Eq. (3) identifies Φ as a glorified gravitational potential. In addition to the rest energy, the source includes internal and kinetic energies and pressure, all boosted by nonlinear contributions from Φ itself.

The momentum constraints relate $\partial_t\Phi$ to the fluid momentum $\mathbf{s} = \Gamma^2(\rho+p)\mathbf{v}$, where $\mathbf{v}$ is the physical velocity measured by an Eulerian observer. Because the test calculations reported here are of stationary configurations, $\partial_t\Phi = 0$, and the momentum constraints don't come into play. The condition $\partial_t\Phi = 0$ can be taken as a first approximation in the supernova environment as well, but it will fail if high density matter moves at relativistic velocities. Relativistic velocities may be achieved by infalling matter outside the shock at late times, or in outflowing jets or winds; but these situations involve matter at low density in comparison with the core. Convection may occur deep in the core, where densities are high, but this likely involves nonrelativistic velocities. This means that $\mathbf{s} = \Gamma^2(\rho+p)\mathbf{v}$ will be arguably small enough everywhere for $\partial_t\Phi$ to be neglected.

Finally we consider the evolution equations of γ_{ij} and K_{ij}. For the NewtonPlus metric, explicit calculation shows that the former turn out to be identities leading to no new information. The latter give three different equations for the subdominant metric function δ that are probably inconsistent, though it is not obvious (to us) how to prove this rigorously. It turns out that stresses (e.g., pressure) constitute the primary source terms in the equations determining δ, confirming the expectation expressed previously, that δ should vanish as the Newtonian limit is approached. The observation that δ will only be appreciable at the highest densities, where pressure begins to make a nontrivial contribution in comparison with energy density, suggests a reasonable path forward. In typical cases it is expected that the deepest portion of the core will be roughly spherical, even if rapid rotation causes an equatorial bulge of lower density material (e.g., Fig. 2 of Ref. [24]). If this is the case, neglecting the angular derivatives of δ is justified, removing many of the apparent inconsistencies in the three equations for δ. The remaining discrepancies have to do with angular derivatives of Φ and particular components of the stress tensor that appear in each equation. As it happens, if one adds all three equations, these remaining discrepancies disappear. Hence the equation we shall use to determine δ is

$$\partial_r\partial_r\delta + \frac{1}{r}\partial_r\delta = 4\pi e^{-2\Phi}({S^\theta}_\theta + {S^\phi}_\phi) - [\partial_r(\Phi+\delta)]^2, \tag{5}$$

where $S_{ij} \equiv T^{km}h_{ik}h_{jm}$, and the spacelike projection tensor is defined by $h_{ij} \equiv g_{ij} + n_i n_j$. When it is recalled that ${S^\theta}_\theta = {S^\phi}_\phi = p$ in spherical symmetry, this is precisely the equation obtained in the spherical case. The source on the right-hand side is to be angle-averaged in solving for δ.

It should be straightforward to include the solution of Φ and, if desired, δ, in existing multidimensional gravitational hydrodynamics codes. The solution of Φ would make use of the Poisson solver normally used to solve for the Newtonian gravitational potential, the only difference being that one would have to iterate on Eq. (3) (with the $\partial_t \Phi$ term dropped) to get a self-consistent Φ. The simplest approximation would be to simply solve for Φ in this manner, and ignore δ altogether. The next level of approximation would involve solving Eq. (5) for δ, but ignoring δ on the right hand side. Since δ is already something of a correction, δ appearing on the right hand side is essentially a "correction to the correction." If desired, however, Eq. (5) could be solved as it stands, with iteration being required.

The equations of hydrodynamics are obtained from the vanishing divergences of the baryon flux vector and stress-energy tensor, employing the NewtonPlus metric. Our study of these equations [19] shows that they can be cast in a form similar to that used by at least one Newtonian multidimensional PPM hydrodynamics code, VH-1. We plan to adapt this code to the NewtonPlus approach in the immediate future.

III. TESTING NEWTONPLUS GRAVITY WITH RAPIDLY ROTATING STARS

In this section we present calculations of neutron stars undergoing rapid uniform rotation in order to assess the strengths and weaknesses of the NewtonPlus approximation to relativistic gravity. Our models were computed with a code described in Ref. [25], which was written to compute the structure of relativistic axisymmetric stars. We have modified the code to include the ability to perform computations in the Newtonian and various NewtonPlus limits: with vanishing metric function δ, with "linearized" δ (i.e. ignoring δ on the right hand side of Eq. (5)), and "full" δ (solving Eq. (5) as written). All of the NewtonPlus limits solve a two-dimensional (and stationary) version of the nonlinear Poisson-type Eq. (3) for the enhanced "gravitational potential" Φ. For the results presented here, the high-density portion of the equation of state (EOS) is taken from Ref. [26], and is based on a field-theoretic description of cold dense matter. We also performed calculations with a polytropic EOS of adiabatic index 2, and found qualitatively similar results.

Panel (a) of Fig. 1 exhibits mass vs. radius curves for spherical stars. It shows that while the Newtonian limit exhibits no maximum mass with this EOS,[1] the NewtonPlus approximation does yield a maximum mass. Even with vanishing δ, the approximation captures this consequence of nonlinear gravity. The "linear δ" approximation follows the exact relativistic curve until the most dense configurations are reached. Since pressure is the main source for δ (see equation (5)), the large pressures associated with such high densities

[1] No turnover in the mass vs. radius curve appears in the Newtonian limit, up to the high-density boundary of the tabulated EOS. A configuration with central baryon mass density (total energy density) of 3.07×10^{15} g cm^{-3} (4.65×10^{15} g cm^{-3}) has a gravitational mass of 15.6 $M_\odot$ and radius 18.9 km in the Newtonian limit, while the relativistic configuration with this central density has a gravitational mass of 1.68 $M_\odot$. and a radius of 9.40 km. We remind the reader that the Chandrasekhar mass phenomenon is a property of stars built on a polytropic equation of state with adiabatic index equal to 4/3 (suitable for white dwarfs), but that stars constructed on "realistic" nuclear equations of state do not necessarily exhibit this behavior. Instead, the upper mass limit of neutron stars derives from the the general relativistic instability indicated by the turning point in the mass vs. radius curve.

raise δ to large enough values that it cannot be neglected on the right-hand side of equation (5). As expected, the "full δ" approximation is indistinguishable from the relativistic results in spherical symmetry, where only two metric functions are needed to describe the spacetime exactly.

Panels (b)-(f) of Fig. 1 show various physical parameters of rapidly rotating configurations. In order to test the NewtonPlus approximation in a nonspherical setting, we ask the question: Given a definite number of baryons rotating at a given uniform angular velocity Ω, what do the various treatments of gravity do with those baryons? (The fact that baryon number is a conserved quantity makes this an obvious way to compare different treatments of gravity.) To answer this question we have computed constant baryon mass sequences beginning at zero rotation (marked by squares) and ending at the mass shedding limit (marked by stars). The value of baryon mass chosen, 1.8 $M_\odot$, is close to the maximum baryon mass of 1.95 $M_\odot$ for the equation of state we employed. The quantities plotted, as a function of the (uniform) stellar angular velocity, are gravitational mass; equatorial radius; total angular momentum; eccentricity, defined as $1 - r_p/r_e$, where r_p and r_e are respectively the polar and equatorial coordinate radii; and the linear equatorial velocity.

In panels (b)-(f), the efficacy of the NewtonPlus approximations can be judged by choosing a value of angular velocity and seeing how close the approximate quantities come to the fully relativistic value. While the "full δ" approximation is indistinguishable from full relativity in the spherical case, the two curves representing these treatments deviate from one another with increasing angular velocity. As expected, the angular velocities at mass shedding of the NewtonPlus approximations are closer to the relativistic values than the Newtonian case. The NewtonPlus treatments are quite successful at approximating the gravitational mass, radius, and eccentricity, while the success of the results for angular momentum and equatorial velocity is more modest. (It is expected that the mass and radius of the collapsed core are more important to the supernova explosion mechanism than the angular momentum.)

IV. CONCLUSION

Accurate neutrino transport, 3D hydrodynamics, and relativity are all essential for realistic supernova simulations. Given the constraints of current hardware, these cannot all be treated simultaneously with the detail they deserve. We have therefore presented an approximation to full relativity (or set of related approximations) that captures the most relevant relativistic effects in the quasispherical supernova environment: nonlinearity creating a deeper potential well and pressure being a nontrivial source of gravitation.

This "NewtonPlus" approach to gravity has a tight conceptual link with the Newtonian limit that yields certain advantageous features. The gravitational portion of multidimensional Newtonian calculations involves only the solution of the Poisson equation for the gravitational potential Φ, and taking its gradient to find the gravitational force. The basic idea of our NewtonPlus approach is to promote the Newtonian metric functions—which are linear in Φ—to full exponentials. We also add a second metric function, δ, whose main source is pressure; hence this metric function vanishes in the Newtonian limit. The Einstein equations yield a nonlinear Poisson-type equation for a (now enhanced) "gravitational potential," whose solution in 3D can be obtained in a manner similar to what is currently done in the Newtonian limit. The inconsistencies in the Einstein equations arising from the reduced number of degrees of freedom turn out to be relegated to the subdominant metric function δ; they can be removed by ignoring angular variations in δ. This is expected to be successful in the supernova context because the region where δ makes the greatest difference is where pressure is significant in comparison with rest mass density. Normally, this is the deepest portion of the collapsed core, which is roughly spherical even when the outer layers bulge at the equator due to rapid rotation.[2] This strategy—allowing the main contribution to the gravitational field to be multidimensional and nonlinear, while allowing a spherical correction for the contribution of stresses—reproduces fairly well many of the physical characteristics of rapidly rotating relativistic stars. Importantly, the hydrodynamics equations obtained from the NewtonPlus metric are of the same form as those used in a popular Newtonian hydrodynamics algorithm, providing the expectation that existing Newtonian codes might be adapted fairly easily to the NewtonPlus approach.

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[2]Ultra-strong magnetic fields [27] or differential rotation (e.g., Ref. [28]) can give rise to off-center density maxima, making the spherical correction for stresses via the metric function δ less useful. The NewtonPlus approximation with $\delta = 0$ could still be employed in such (probably exceptional) cases, however.

REFERENCES

[1] M. Liebendörfer, A. Mezzacappa, F.-K. Thielemann, O. E. B. Messer, W. R. Hix, and S. W. Bruenn, Phys. Rev. D **63**, 103004 (2001).

[2] S. W. Bruenn, K. R. De Nisco, and A. Mezzacappa, submitted to Astrophys. J., astro-ph/0101400.

[3] H.-T. Janka and E. Müller, Astron. Astrophys. **306**, 167 (1996).

[4] L. Wang, D. A. Howell, P. Höflich, and J. C. Wheeler, Astrophys. J. **550**, 1030 (2001).

[5] R. I. Epstein, Mon. Not. R. Astron. Soc. **188**, 305 (1979).

[6] L. Smarr, J. R. Wilson, R. T. Barton, and R. L. Bowers, Astrophys. J. **246**, 515 (1981).

[7] W. D. Arnett, in *The Origin and Evolution of Neutron Stars,* edited by D. J. Helfland and J.-H. Huang (D. Reidel, Dordrecht, 1987).

[8] H. A. Bethe, G. E. Brown, and J. Cooperstein, Astrophys. J. **322**, 201 (1987).

[9] J. R. Wilson and R. W. Mayle, Phys. Rep. **227**, 97 (1993).

[10] D. S. Miller, J. R. Wilson, and R. W. Mayle, Astrophys. J. **415**, 278 (1993).

[11] M. Herant, W. Benz, W. R. Hix, C. L. Fryer, and S. A. Colgate, Astrophys. J. **435**, 339 (1994).

[12] A. Burrows, J. Hayes, and B. A. Fryxell, Astrophys. J. **450**, 830 (1995).

[13] A. Mezzacappa, A. C. Calder, S. W. Bruenn, J. M. Blondin, M. W. Guidry, M. R. Strayer, and A. S. Umar, Astrophys. J. **493**, 848 (1998).

[14] A. Mezzacappa, A. C. Calder, S. W. Bruenn, J. M. Blondin, M. W. Guidry, M. R. Strayer, and A. S. Umar, Astrophys. J. **495**, 911 (1998).

[15] C. L. Fryer and A. Heger, Astrophys. J. **541**, 1033 (2000).

[16] A. Mezzacappa, M. Liebendörfer, O. E. B. Messer, W. R. Hix, F.-K. Thielemann, and S. W. Bruenn, Phys. Rev. Lett. **86**, 1935 (2001).

[17] M. Rampp and H.-T. Janka, Astrophys. J. Lett. **539**, 33 (2000).

[18] M. Alcubierre, G. Allen, B. Brügmann, E. Seidel, and W.-M. Suen, Phys. Rev. D **62**, 124011 (2000)

[19] C. Y. Cardall, A. Mezzacappa, and M. Liebendörfer, astro-ph/0106105.

[20] R. Arnowitt, S. Deser, and C. W. Misner, in *Gravitation: An Introduction to Current Research,* edited by L. Witten (Wiley, New York, 1962).

[21] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (W. H. Freeman, New York, 1973).

[22] L. Smarr and J. W. York, Phys. Rev. D **17**, 2529 (1978).

[23] J. W. York, in *Sources of Gravitational Radiation,* edited by L. Smarr (Cambridge UP, Cambridge, 1979).

[24] S. Bonazzola, E. Gourgoulhon, M. Salgado, and J. A. Marck, Astron. Astrophys. **278**, 421 (1993).

[25] C. Y. Cardall, M. Prakash, and J. M. Lattimer, Astrophys. J. (to be published), astro-ph/0011148.

[26] M. Prakash, J. R. Cooke, and J. M. Lattimer, Phys. Rev. D **52**, 661 (1995).

[27] M. Bocquet, S. Bonazzola, E. Gourgoulhon, and J. Novak, Astron. Astrophys. **301**, 757 (1995).

[28] T. W. Baumgarte, S. L. Shapiro, and M. Shibata, Astrophys. J. Lett. **528**, L29 (2000).

FIGURES

FIG. 1. Panel (a): Mass vs. radius curves for spherical configurations computed with various treatments of gravity. Panels (b)-(f): Various physical quantities characterizing uniformly rotating configurations, plotted as a function of angular velocity. Each curve represents a constant baryon mass sequence computed with the treatments of gravity labeled in panel (a).