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A SURVEY OF A CLASS OF ALGEBRAIC MULTILEVEL ITERATION METHODS FOR POSITIVE DEFINITE SYMMETRIC MATRICES

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The algebraic multilevel iteration method, AMLI, is a recursively defined method to construct spectrally equivalent preconditioners to a sequence of symmetric and positive definite matrices, corresponding to a number of levels with increasing degrees of freedom, such as arises for a sequence of nested finite element meshes. The matrix sequence is connected by the assumption that the Schur complement, for the corresponding two by two partitioning of the matrix on any level, is spectrally equivalent to the matrix on the next lower level with bounds which hold uniformly for any number of levels. It was originally presented for matrices for which there exists a hierarchical basis matrix form with an explicitly given transformation matrix between the standard form and the hierarchical form. This case allowed for arbitrary perturbations of the matrix block, corresponding to the added degrees of freedom, independent of the Schur complement.
For more general matrices, the spectral equivalence still holds if the perturbation of the above block diagonal matrix satisfies a certain spectral relation to the Schur complement. By solving the arising systems for this block with sufficient accuracy one can come arbitrary close to the condition number for the two-level method with exact such blocks.

e1 Introduction

The computational complexity when solving large sparse systems of linear equations can grow rapidly with problem size unless a proper solution method is used. Ideally, we want a solution method whose complexity grows proportionally to the order n of the system, i.e. is of optimal order. To solve a linear system $Ax = b$, where A is symmetric and positive semidefinite, we shall consider the case where A is the final matrix $A^{(J_0)}$ in a sequence of matrices $\{A^{(k)}\}$, $A^{(k)} \in L(\mathrm{R}^{n_k}, \mathrm{R}^{n_k})$, $k = 0, 1, \ldots, J_0$ for a number of J_0 levels and $n_k > n_{k-1}$. On each level, the matrix is partitioned in a two by two block

form

$$A^{(k)} = \begin{bmatrix} A_1^{(k)} & A_{12}^{(k)} \\ A_{21}^{(k)} & A_2^{(k)} \end{bmatrix} \begin{matrix} \} \, n_k - n_{k-1} \\ \} \, n_{k-1} \end{matrix}$$

of dimensions as indicated. Note that $A_2^{(k)}$ has the same dimension as $A^{(k-1)}$. In certain finite element applications the matrices are constructed for a sequence of nested meshes while in other applications the matrices must be constructed algebraically during a preprocessing phase. The matrices in the sequence are connected via the Schur complements

$$S_{A^{(k)}} = A_2^{(k)} - A_{21}^{(k)} A_1^{(k)^{-1}} A_{12}^{(k)}$$

as the basic assumption made is that $S_{A^{(k)}}$ on level k is spectrally equivalent to the matrix $A^{(k-1)}$ on level $k-1$, with spectral equivalence bounds which hold uniformly for all levels. To solve systems with $A^{(k)}$ we shall use the algebraic multilevel iteration (AMLI) method. The AMLI method was first proposed in [10, 11].

The method is based on a preconditioner $M^{(k)}$ which is constructed as an approximation of the block factorization

$$A^{(k)} = \begin{bmatrix} A_1^{(k)} & 0 \\ A_{21}^{(k)} & I_2 \end{bmatrix} \begin{bmatrix} I_1 & A_1^{(k)^{-1}} A_{12}^{(k)} \\ 0 & S_{A^{(k)}} \end{bmatrix}$$

where $A_1^{(k)}$ is replaced with a preconditioner $M_1^{(k)}$ and $S_{A^{(k)}}$ is approximated with a certain matrix polynomial, involving the preconditioner $M^{(k-1)}$ to the previous level matrix $A^{(k-1)}$. In this way, the preconditioner is not defined explicitly but only by recursion via the given levels. The preconditioner is normally used with the conjugate gradient method.

In the original papers the method was presented for finite element matrices and basis functions for which a strengthened form of the Cauchy inequality holds,

$$a(u,v) \leq \gamma \{a(u,u)a(v,v)\}^{\frac{1}{2}}, \quad \text{for all } u \in V_{k-1} \text{ and } v \in V_k \backslash V_{k-1},$$

where V_k denotes the finite element space on level k, and $a(\cdot,\cdot)$ is the symmetric bilinear form corresponding to the given differential operator. Here and in what follows, by $V_k \setminus V_{k-1}$ we will denote a complementary space W_k, i.e., a space such that $V_k = V_{k-1} \oplus W_k$.

It was shown that the method had an optimal order under quite general conditions. In this case there is no restriction on the perturbations $M_1^{(k)}$ of $A_1^{(k)}$. Here a survey of this method is given and it is then shown how it can

be extended to more general positive definite matrices. In the latter case the perturbations $M_1^{(k)}$ must satisfy a certain relation to the Schur complements.

e2 The strengthened Cauchy inequality

In the original version, a sequence of symmetric positive definite sparse matrices $\{A^{(k)}\}_{k=0}^{J_0}$ were given and related variationally, i.e., $A^{(k-1)} = (I_{k-1}^k)^T A^{(k)} I_{k-1}^k$ where $I_{k-1}^k = \begin{bmatrix} J_{12}^{(k)} \\ I_{k-1} \end{bmatrix}$, where I_{k-1} stands for the identity operator at level $k-1$ and $J_{12}^{(k)}$ is typically an interpolation operator from the current coarse to the new components of the solution vector on the next finer level. A common example of such matrices are standard finite element matrices, defined via a bilinear form $a(u,v)$, $u, v \in H^1(\Omega)$ and basis functions $\{\varphi_i\}$, $\varphi_i \in H^1(\Omega)$.

For the analysis of the method we will need the transformed matrices $\widehat{A}^{(k)} = J^T A^{(k)} J$, where $J = \begin{bmatrix} I_1^{(k)} & J_{12}^{(k)} \\ 0 & I_{k-1} \end{bmatrix} = \left[\begin{bmatrix} I_1^{(k)} \\ 0 \end{bmatrix}, I_{k-1}^k \right]$ and $I_1^{(k)}$ is identity operator on the added vector spaces. It follows that $\widehat{A}^{(k)}$ has the following two-level block form,

$$\widehat{A}^{(k)} = \begin{bmatrix} A_1^{(k)} & \widehat{A}_{12}^{(k)} \\ \widehat{A}_{21}^{(k)} & A^{(k-1)} \end{bmatrix}, \tag{1}$$

where $\widehat{A}_{12}^{(k)} = A_{12} + A_{11} J_{12}^{(k)}$. Since the lower left block of $\widehat{A}^{(k)}$ equals $A^{(k-1)}$, the transformed matrices are called two-level hierarchical basis (HB) matrices. An elementary computation shows that $S_{\widehat{A}^{(k)}} = S_{A^{(k)}}$, where $S_{\widehat{A}^{(k)}}$ is the Schur complement of $\widehat{A}^{(k)}$, $S_{\widehat{A}^{(k)}} \equiv A^{(k-1)} - \widehat{A}_{21}^{(k)} A_1^{(k)^{-1}} \widehat{A}_{12}^{(k)}$. Therefore, the following relations hold,

$$\begin{aligned} \widehat{A}_1^{(k)} &= A_1^{(k)} \\ \widehat{A}_2^{(k)} &= A^{(k-1)} \\ S_{\widehat{A}^{(k)}} &= S_{A^{(k)}} \end{aligned} \tag{2}$$

The hierarchical basis matrices admit the following strengthened Cauchy-inequality,

$$\mathbf{v}^T \widehat{A}^{(k)} \mathbf{w} \leq \gamma \left(\mathbf{v}^T \widehat{A}^{(k)} \mathbf{v} \right)^{\frac{1}{2}} \left(\mathbf{w}^T \widehat{A}^{(k)} \mathbf{w} \right)^{\frac{1}{2}}, \quad \text{for all } \mathbf{v} = \begin{bmatrix} \mathbf{v}_1 \\ 0 \end{bmatrix}, \ \mathbf{w} = \begin{bmatrix} 0 \\ \mathbf{v}_2 \end{bmatrix}, \tag{3}$$

for some γ, $0 \leq \gamma < 1$.
For finite element stiffness matrices $A^{(k)}$ it turns out that the constant γ can be determined locally, elementwise, see [14, 1, 7, 17, 5]. For such matrices, corresponding to triangulations obtained by successive refinement generating geometrically similar elements, one can prove that γ remains strictly less than one, independently of the refinement levels $J_0 \geq 1$. Actually, in the analysis one needs the following readily proven relation,

$$(1-\gamma^2)\mathbf{v}_2^T A^{(k-1)}\mathbf{v}_2 \leq \mathbf{v}_2^T S_{A^{(k)}}\mathbf{v}_2 \leq \mathbf{v}_2^T A^{(k-1)}\mathbf{v}_2, \quad \text{for all } \mathbf{v}_2. \tag{4}$$

To illustrate values taken by γ consider piecewise linear ($p = 1$) and quadratic ($p = 2$) basis functions and the bilinear form corresponding to a diffusion problem, on a triangular reference element $\widetilde{e}$,

$$\widetilde{a}_e(\widetilde{u},\widetilde{v}) = \int\int_{\widetilde{e}} \sum_{i,j} a_{ij} \frac{\partial \widetilde{u}}{\partial \widetilde{x}_i} \frac{\partial \widetilde{v}}{\partial \widetilde{x}_j} d\widetilde{x}\, d\widetilde{y},$$

where $\widetilde{x}_1 = \widetilde{x}$, $\widetilde{x}_2 = \widetilde{y}$, $0 < \widetilde{x} < 1$, $0 < \widetilde{y} < 1$ and the coefficients a_{ij} (where $a_{ij} = a_{ji}$) depend on the coordinates, i.e., equivalently on the angles in the triangulation and on the given diffusion matrix $\begin{bmatrix} \widehat{a}_{11} & \widehat{a}_{12} \\ \widehat{a}_{21} & \widehat{a}_{22} \end{bmatrix}$.

Theorem 2.1 [3]. *Consider a sequence of piecewise linear and piecewise quadratic finite element methods for nested meshes. Then the constant γ in the strengthened CBS inequality satisfies*

$$\gamma^2 = \gamma_1^2 = \frac{3}{8}\left(1 + \sqrt{1 - \frac{8}{9} \cdot \frac{1-\widetilde{a}_{12}^2}{1+\widetilde{a}_{12}/\overline{\overline{a}}}}\right) \quad \textit{for } p = 1$$

and $\gamma_2^2 = \frac{4}{3}\gamma_1^2$ for $p = 2$, where $\widetilde{a}_{ij} = a_{ij}/(a_{11}a_{22})^{\frac{1}{2}}$ and $\overline{\overline{a}} = \widetilde{a}_{11} + \widetilde{a}_{12} + \widetilde{a}_{22}$. In particular, $\frac{3}{8} \leq \gamma_1^2 < \frac{3}{4}$ for any triangulation.

Theorem 2.1 can be used in particular to show how γ^2 depends on the angles in the triangulation. Using an algebraic derivation such a result was derived already in [17], for the differential operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ and a general triangular mesh. For a triangle τ with angles $\theta_i(\tau)$, $a_{11} = 1$, $a_{22} = 1$, $a_{12} = 0$, using a different method of derivation, it was shown that

$$\gamma_1^2 = \frac{3}{4} - \frac{3-g}{2[(4g-3)^{\frac{1}{2}} + 3]},$$

where $g = \max \sum_1^2 \cos^2\theta_i$. Hence $\frac{3}{4} \leq g < 3$, so $\frac{3}{8} \leq \gamma_1^2 < \frac{3}{4}$. The upper bound $\gamma_1^2 = \frac{3}{4}$ is taken only for a degenerate triangle where the maximum angle

equals π, so $g = 3$. The value $\gamma^2 = \frac{1}{2}$ is taken for any right angled triangle, including a degenerate with a zero angle. The smallest value $\gamma^2 = \frac{3}{8}$ is taken for equilateral triangles where $g = \frac{3}{4}$. Our result shows more generally that these bounds hold for an arbitrary second order elliptic operator. The upper bound follows from $\widetilde{a}_{12}^2 < 1$ and the lower bound is taken when $\widetilde{a}_{11} = \widetilde{a}_{22} = 1$ and $\widetilde{a}_{12} = -\frac{1}{2}$.

e3 The two-level preconditioning methods

There are two major preconditioning schemes to solve the block partitioned algebraic system.

(i) *Multiplicative scheme* [7]
This is based on the exact block matrix factorization

$$A = \begin{bmatrix} A_1 & A_{12} \\ A_{21} & A_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ A_{21} & S \end{bmatrix} \begin{bmatrix} I_1 & A_1^{-1}A_{12} \\ 0 & I_2 \end{bmatrix}, \; S = A_2 - A_{21}A_1^{-1}A_{12}.$$

Here A_1 is replaced with some approximation M_1, by (2), S equals

$$S = A^{(k-1)} - \widehat{A}_{21}A_1^{-1}\widehat{A}_{12},$$

and is replaced with some sparse approximation M_2, such as $A^{(k-1)}$ and the preconditioning takes then the form

$$M_{mlt} = \begin{bmatrix} M_1 & 0 \\ A_{21} & M_2 \end{bmatrix} \begin{bmatrix} I_1 & M_1^{-1}A_{12} \\ 0 & I_2 \end{bmatrix}.$$

The action of M_{mlt}^{-1} requires two inversions of M_1, an inversion of M_2 and a multiplication with A_{12} and A_{21}.

(ii) *Additive scheme* [14], [7]
To be efficient this requires the use of the transformation matrix J and the iterations are performed via the hierarchical basis function matrix (which, however, need not be formed explicitly)

$$J^T A J = \widehat{A} = \begin{bmatrix} \widehat{A}_1 & \widehat{A}_{12} \\ \widehat{A}_{21} & \widehat{A}_2 \end{bmatrix} \sim \begin{bmatrix} \widehat{A}_1 & 0 \\ 0 & \widehat{A}_2 \end{bmatrix},$$

where $\widehat{A}_1 = A_1$, $\widehat{A}_2 = A^{(k-1)}$ and diag $(\widehat{A}_1, \widehat{A}_2)$ is spectrally equivalent to $\widehat{A}$. Here A_1 may be replaced by M_1 and $A^{(k-1)}$ by some approximation M_2 to form the preconditioner

$$M_{add} = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}.$$

Clearly, the action of M_{add}^{-1} requires a single inversion of the blocks M_1 and M_2. The following spectral relations hold.

Theorem 3.1 *Assume, that the following conditions are satisfied:*

$$\mathbf{v}_1^T A_1 \mathbf{v}_1 \leq \mathbf{v}_1^T M_1 \mathbf{v}_1 \leq (1+\delta_1)\mathbf{v}_1^T A_1 \mathbf{v}_1 \quad \textit{for all } \mathbf{v}_1$$
$$\mathbf{v}_2^T A_2 \mathbf{v}_2 \leq \mathbf{v}_2^T M_2 \mathbf{v}_2 \leq (1+\delta_2)\mathbf{v}_2^T A_2 \mathbf{v}_2 \quad \textit{for all } \mathbf{v}_2,$$

where δ_1 and δ_2 are some positive constants. Then the following inequalities hold:

$$c_{mlt}\mathbf{v}^T A\mathbf{v} \leq \mathbf{v}^T M_{mlt}\mathbf{v} \leq C_{mlt}\mathbf{v}^T A\mathbf{v} \quad \textit{for all } \mathbf{v},$$
$$c_{add}\mathbf{v}^T A\mathbf{v} \leq \mathbf{v}^T M_{add}\mathbf{v} \leq C_{add}\mathbf{v}^T A\mathbf{v} \quad \textit{for all } \mathbf{v},$$

where

$$c_{mlt} = 1,$$
$$C_{mlt} = \frac{1}{1-\gamma^2}\left\{1 + \frac{1}{2}\left[\delta_1 + \delta_2 + \sqrt{(\delta_1-\delta_2)^2 + 4\delta_1\delta_2\gamma^2}\right]\right\},$$
$$c_{add} = \frac{1}{1+\gamma},$$
$$C_{add} = \frac{1}{1-\gamma^2}\left\{1 + \frac{1}{2}\left[\delta_1 + \delta_2 + \sqrt{(\delta_1-\delta_2)^2 + 4(1+\delta_1)(1+\delta_2)\gamma^2}\right]\right\}.$$

Frequently, A_1 is similar to a mass matrix and M_1 may then correspond to a simple smoother, possibly even $M_1 = \mathrm{diag}(A_1)$. By performing a sufficient number of such smoothing steps, δ_1 can be made arbitrarily small. M_2 corresponds normally to an approximation of $A^{(k-1)}$, possibly defined recursively via a multilevel approach. By making M_2 sufficiently close to $A^{(k-1)}$, δ_2 becomes arbitrarily small. When $\delta_1 = \delta_2 = 0$ Theorem 3.1 shows that the condition numbers become

$$\kappa_1 = \kappa\left(M_{mlt}^{-1}A\right) \leq \frac{1}{1-\gamma^2},$$
$$\kappa_2 = \kappa\left(M_{add}^{-1}A\right) \leq \frac{1+\gamma}{1-\gamma}.$$

For a triangular mesh and a general second order diffusion operator with constant coefficients on each element of the coarsest mesh, as follows from Theorem 2.1 these condition numbers satisfy

$$1.6 \leq \kappa_1 < 4, \quad 4.16 \,.\, \kappa_2 < 14.$$

Remark 3.1 *In many practical problems A_1 is not well-conditioned with respect to some problem parameters such as anisotropic diffusion coefficients or nearly-degenerate triangles. In such cases one must construct more sophisticated approximations M_1 of A_1, see* [9] *for a discussion of this topic.*

e4 Extension to the Multilevel Case. Polynomial Stabilization Procedures

The two-level methods could be extended to multilevel methods by simple recursion, letting $M_2^{(k)} = M^{(k-1)}$ or some polynomial function of $M^{(k-1)}$. However, as will be seen in Section 5, this does not necessarily preserve the advantageous condition numbers given in Theorem 3.1, which permitted arbitrary perturbations of the matrix block $A_1^{(k)}$. One way to handle this is to base the method on hierarchical basis function matrices or, at least, let the transformation matrix J be involved in the method.

Let then $\{V_k\}$ be a sequence of finite element spaces corresponding to a nested sequence of finite element meshes.

Since the hierarchical matrices are less sparse than the standard basis function matrices, it is desirable to still use the latter in the actual implementation of the method. This can be achieved as follows.

Multiplicative multilevel preconditioner

Definition 4.1

- *Let* $M^{(0)} = A^{(0)}$.
- *For* $k = 1, 2, \ldots, J_0$, *assuming that* $M^{(k-1)}$ *has been defined, define first* $\widetilde{M}^{(k-1)}$,

$$\widetilde{M}^{(k-1)} = [I - P_{\nu_k}(M^{(k-1)^{-1}} A^{(k-1)})]A^{(k-1)^{-1}}.$$

- *Then*

$$M^{(k)} = \begin{bmatrix} M_1^{(k)} & 0 \\ \widetilde{A}_{21}^{(k)} & \left(\widetilde{M}^{(k-1)}\right)^{-1} \end{bmatrix} \begin{bmatrix} I_1^{(k)} & M_1^{(k)^{-1}} \widetilde{A}_{12}^{(k)} \\ 0 & I_{k-1} \end{bmatrix} \begin{matrix} \} V_k \backslash V_{k-1} \\ \} V_{k-1} \end{matrix}$$

where

$$\begin{aligned} \widetilde{A}_{12}^{(k)} &= A_{12}^{(k)} + (A_1^{(k)} - M_1^{(k)}) J_{12}^{(k)}, \\ \widetilde{A}_{21}^{(k)} &= A_{21}^{(k)} + J_{12}^{(k)^T} (A_1^{(k)} - M_1^{(k)}) \end{aligned} \tag{5}$$

Here $\widetilde{M}^{(k-1)}$ is an approximation of $A^{(k-1)^{-1}}$. It follows that the preconditioner $M^{(k)}$ is only implicitly (recursively) defined.

Further, P_{ν_k} are given polynomials of degree ν_k which are normalized, $P_{\nu_k}(0) = 1$ and should be chosen such that $\max|P_{\nu_k}|$ is as small as possible on the spectrum of $M^{(k-1)^{-1}} A^{(k-1)}$. Clearly, the smaller the norm of P_{ν_k}

on the above spectrum, as closer $\widetilde{M}^{(k-1)}$ is to $A^{(k-1)}$ and, hence, to the two-level method. As it turns out, there is no loss of efficiency by choosing $0 \leq P_{\nu_k}(t) \leq 1$ with $\max P_{\nu_k}(t)$ as small as possible on the spectrum. With this choice and assuming $\mathbf{v}_1^T M_1^{(k)} \mathbf{v}_1 \geq \mathbf{v}_1^T A^{(k)} \mathbf{v}_1$, for all $\mathbf{v}_1 \in \mathrm{R}^{n_k - n_{k-1}}$, it follows

$$\mathbf{v}^T M^{(k)} \mathbf{v} \geq \mathbf{v}^T A^{(k)} \mathbf{v}, \quad \text{for all } \mathbf{v} \in R^{n_k}. \tag{6}$$

If $\nu_k = 1$, we let then $P_{\nu_k}(t) = 1 - t$.

The reason for perturbing the off-diagonal block matrices as done in (5) is that in this way

$$\widehat{M}^{(k)} \equiv J^{(k)^{\mathrm{T}}} M^{(k)} J^{(k)}$$

takes the form

$$\widehat{M}^{(k)} = \begin{bmatrix} M_1^{(k)} & \widehat{A}_{12}^{(k)} \\ \widehat{A}_{21}^{(k)} & \widetilde{M}^{(k-1)^{-1}} + \widehat{A}_{21}^{(k)} M_1^{(k)^{-1}} \widehat{A}_{12}^{(k)} \end{bmatrix}, \tag{7}$$

which follows from an elementary computation. Hence $\widehat{M}^{(k)}$ can be considered as a preconditioner to $\widehat{A}^{(k)}$ and the extreme eigenvalues of $M^{(k)^{-1}} A^{(k)}$ equal those of $\widehat{M}^{(k)^{-1}} \widehat{A}^{(k)}$. Since the off-diagonal blocks in $\widehat{M}^{(k)}$ equal those in $\widehat{A}^{(k)}$ the estimate of the extreme eigenvalues of $\widehat{M}^{(k)^{-1}} \widehat{A}^{(k)}$ can be readily done and although the preconditioner $M^{(k)}$ does not involve matrices in the hierarchical basis, the condition number of $M^{(k)^{-1}} A^{(k)}$ can now be estimated via the hierarchical matrix, see Theorem 4.1.

Additive multilevel scheme

Definition 4.2

$$M_{add}^{(k)} = \begin{bmatrix} M_1^{(k)} & 0 \\ 0 & \widetilde{M}^{(k-1)^{-1}} \end{bmatrix} \begin{matrix} \} \, V_k \backslash V_{k-1} \\ \} \, V_{k-1} \end{matrix}$$

where

$$\widetilde{M}^{(k-1)} = \left[I - P_{\nu_k} \left(M^{(k-1)^{-1}} A^{(k-1)} \right) \right] A^{(k-1)^{-1}}.$$

With the same assumption $\mathbf{v}_1^T M_1^{(k)} \mathbf{v}_1 \geq \mathbf{v}_1^T A_1 \mathbf{v}_1$ *we have as before* $\mathbf{v}^T M^{(k)} \mathbf{v} \geq \mathbf{v}^T A^{(k)} \mathbf{v}$ *for all* $\mathbf{v} \in \mathrm{R}^{n_k}$.

For P_ν we take a shifted and scaled Chebyshev polynomial,

$$P_\nu(t) = \frac{T_\nu \left(\frac{1+\alpha-2t}{1-\alpha} \right) + 1}{T_\nu \left(\frac{1+\alpha}{1-\alpha} \right) + 1}$$

where

$$T_\nu(x) = \frac{1}{2}\left[\left(x+\sqrt{x^2-1}\right)^\nu + \left(x-\sqrt{x^2-1}\right)^\nu\right]$$

i.e., T_ν is the Chebyshev polynomial of the first kind. Further α, $\alpha > 0$ is a lower bound of the eigenvalues of $M^{(k-1)^{-1}}A^{(k-1)}$. By (6) the upper bound of these eigenvalues is bounded by the unit number. We have

$$\alpha = \alpha_{k-1} = \min_{\mathbf{v}_2} \frac{\mathbf{v}_2^T A^{(k-1)}\mathbf{v}_2}{\mathbf{v}_2^T M^{(k-1)}\mathbf{v}_2} = \lambda_{k-1}^{-1}.$$

Assume now first that $\nu_k = \nu$, i.e. is fixed on each level. Then the following holds for the multiplicative version.

Theorem 4.1 *Let $M_1^{(k)}$ be spectrally equivalent approximations to $A_1^{(k)}$ such that, uniformly in k, one has*

$$\mathbf{v}_1^T A_1^{(k)}\mathbf{v}_1 \le \mathbf{v}_1^T M_1^{(k)}\mathbf{v}_1 \le (1+b)\mathbf{v}_1^T A_1^{(k)}\mathbf{v}_1, \quad \textit{for all } \mathbf{v}_1.$$

Let $\nu > (1-\gamma^2)^{-\frac{1}{2}}$ and $\rho \in (0,1)$ be a solution of the inequality,

$$\left(\frac{1+\rho}{1-\rho}\right)^2 \alpha \le 1-\gamma^2-\alpha b, \quad \alpha = \left(\frac{1-\rho^{\frac{1}{\nu}}}{1+\rho^{\frac{1}{\nu}}}\right)^2. \tag{8}$$

Then, the corresponding AMLI preconditioning matrix $M^{(k)}$ is spectrally equivalent to $A^{(k)}$ with the following bounds:

$$\mathbf{v}^T A^{(k)}\mathbf{v} \le \mathbf{v}^T M^{(k)}\mathbf{v} \le \frac{1}{1-\gamma^2}\left(b+\left(\frac{1+\rho}{1-\rho}\right)^2\right)\mathbf{v}^T A^{(k)}\mathbf{v}$$

$$\le \frac{1}{\alpha}\mathbf{v}^T A^{(k)}\mathbf{v}, \quad \textit{for all } \mathbf{v}.$$

For $\nu = 2$ and $4\gamma^2 < 3$, one has $\alpha = \frac{3-4\gamma^2}{1+2b+\sqrt{3-4\gamma^2+(1+2b)^2}}$ and $\lambda \le \left(\frac{1}{\alpha}\left(\frac{1+\alpha}{2}\right)^2 + b\right)/(1-\gamma^2)$. Λ

Due to the factor α in αb, the same bound for ν holds for any b as for $b = 0$, i.e., the inequality (8) has a solution $\alpha \in (0,1)$ if $\nu > (1-\gamma^2)^{-\frac{1}{2}}$.

It can be shown that for the additive version the same bounds hold as in Theorem 4.1 when we replace $\frac{1}{1-\gamma^2}$ with $\frac{1+\gamma}{1-\gamma}$.

Remark 4.1 (Growth of the computational complexity)
Since $P_\nu(0) = 1$, it is readily seen that each action of $M^{(k-1)^{-1}}$ requires $\nu-1$ actions of $A^{(k-1)}$ and ν actions of $M^{(k-1)^{-1}}$. Therefore, to get an optimal order, $O(n)$, $n = n_{J_0}$ of computational complexity of each action of $M^{(J_0)}$ the polynomial degree chosen must be bounded above and related to the ratio n_k/n_{k-1} of the degrees of freedom. For a uniform recursive triangulation of a plane domain where on each new level each triangle is divided in four parts, it holds $\nu < 4$. On the other hand, as we have seen, to get an optimal order condition number, cond$(M^{(k)^{-1}} A^{(k)}) = O(1)$ which is bounded as $k \to \infty$, we must choose ν sufficiently large.

It follows from the above that the following bounds must hold to get a method of optimal order of computational complexity for a uniform partitioning of a mesh in a d-dimensional space, $d = 2$ or $d = 3$,

$$\frac{1}{\sqrt{1-\gamma^2}} < \nu < 2^d, \quad \sqrt{\frac{1+\gamma}{1-\gamma}} < \nu < 2^d$$

for the multiplicative and additive methods, respectively. Since $\gamma^2 < \frac{3}{4}$ for any triangulation and any diffusion operator it suffices to take $\nu = 2$ when $d = 2$ for the first method. For the second method we must choose a triangulation and restrict the diffusion operator so that $\gamma < \frac{4}{5}$, in which case we can take $\nu = 3$. For problems in $3D$, the restrictions are more severe.
The value $\nu = 2$ corresponds to what in the context of multigrid methods are called a W-cycle and $\nu = 1$ corresponds to a V-cycle.

As was noted in [19], [6] and [15], to relax the above restrictions on ν or γ one can better let $\nu = 1$ on most levels and stabilize the condition number on certain levels using polynomials of sufficiently high degree there. The use of such polynomials correspond to using inner iterations. As was shown in [3], [4], [9], it is then most efficient to let the levels (k_s) where stabilization, or inner iterations, will take place be determined as follows,

$$k_{s+1} = (1 + \alpha_s)k_s,$$

where $\alpha_{s+1} = \frac{\alpha_s}{1+\alpha_s}\eta$, $s = k_0, k_0 + 1, \ldots, J - 1$. Here $\eta = \frac{1}{2}\log\sigma/\log\rho_0$, where ρ_0 is a lower bound on the average increase of degrees of freedom, i.e. $\left(n_{k_{s+1}}/n_{k_s}\right)^{1/(k_{s+1}-k_s)} \geq \rho_0$, and

$$\sigma = \left\{\kappa\left(M^{(k_{s+1},k_s)^{-1}} A^{(k_{s+1})}\right)\right\}^{1/(k_{s+1}-k_s)}$$

is the average condition number between the two levels. Here $M^{(k_{s+1},k_s)}$ denotes the block diagonal preconditioner between levels k_{s+1} and k_s.

From the definition of $M^{(k)}$ it follows that the computational complexity of one action of the corresponding preconditioner from level k_{s+1} to k_s is

$$W_{k_{s+1}} \leq c\left(n_{k_{s+1}} - n_{k_s}\right) + C_0 n_{k_s}^{1+\alpha_{k_s}}.$$

Here c is a constant which depends on the sparsity of the matrices $A^{(k)}$, i.e. c is an upper bound of the average number of nonzeros per row in $A_1^{(k)}$ for all k. As shown in [3], [4], [9], if $\eta < 1$ it follows that $\alpha_s \to 0$ and the computational complexity becomes asymptotically of optimal order.

In general, σ is not known but can be approximated using values for a regular mesh refinement, where both ρ_0 and γ are known. Furthermore, it turns out that the method is quite insensitive to the choice of stabilization levels, see [9]. Alternatively, one may use the parameter–free AMLI method of additive type as proposed in [12].

e5 The AMLI method for more general positive definite matrices

In this section we consider the construction of an AMLI method for general positive definite matrices, i.e., without assuming any underlying hierarchy of meshes and thus avoiding any (implicit or explicit) transformation to a corresponding HB block structure of the matrices. It will be shown that in order to construct an optimal order preconditioner the approximations $B_1^{(k)}$ to $A_1^{(k)}$ must be related to the Schur complements $S_{A^{(k)}}$ in a certain way. It suffices to consider the two level form of the method as the multilevel extension can be done as shown in Section 4 . For convenience, we delete then the superscripts (k).

Consider the preconditioner in the block matrix factored form

$$B = \begin{bmatrix} B_1 & 0 \\ A_{21} & S_B \end{bmatrix} \begin{bmatrix} I_1 & B_1^{-1}A_{12} \\ 0 & I_2 \end{bmatrix}, \tag{9}$$

where B_1 is an approximation of A_1 and S_B is an approximation of S_A. Note that

$$B = \begin{bmatrix} B_1 & A_{12} \\ A_{21} & S_B + A_{21}B_1^{-1}A_{12} \end{bmatrix},$$

so S_B is the Schur complement of B.
We assume that B_1 is spectrally equivalent to A_1 and S_B to S_A and that the

following inequalities hold for some $\beta \geq 1$, $\eta \geq 1$.

$$\beta \mathbf{v}_1^T A_1 \mathbf{v}_1 \geq \mathbf{v}_1^T B_1 \mathbf{v}_1 \geq \mathbf{v}_1^T A_1 \mathbf{v}_1, \text{ for all } \mathbf{v}_1 \in \mathrm{R}^{n_k - n_{k-1}} \tag{10,i}$$

$$\eta \mathbf{v}_2^T S_A \mathbf{v}_2 \geq \mathbf{v}_2^T S_B \mathbf{v}_2 \geq \mathbf{v}_2^T S_{\widetilde{A}} \mathbf{v}_2, \text{ for all } \mathbf{v}_2 \in \mathrm{R}^{n_{k-1}}, \tag{10,ii}$$

where $\widetilde{A} = \begin{bmatrix} B_1 & A_{12} \\ A_{21} & A_2 \end{bmatrix}$ and $S_A = A_2 - A_{21} A_1^{-1} A_{12}$, $S_{\widetilde{A}} = A_2 - A_{21} B_1^{-1} A_{12}$.

We shall also assume that

$$\alpha \mathbf{v}_2^T S_A \mathbf{v}_2 \geq \mathbf{v}_2^T S_{\widetilde{A}} \mathbf{v}_2 \geq \mathbf{v}_2^T S_A \mathbf{v}_2, \text{ for all } \mathbf{v}_2, \tag{10,iii}$$

where $\alpha \geq 1$ and the left inequality is sharp, i.e., there exists a vector $\widehat{\mathbf{v}}_2$ such that

$$\alpha \widehat{\mathbf{v}}_2^T S_A \widehat{\mathbf{v}}_2 = \widehat{\mathbf{v}}_2^T S_{\widetilde{A}} \widehat{\mathbf{v}}_2.$$

The right inequality in (iii) follows from the right inequality in (i), because (i) implies $\mathbf{v}^T \widetilde{A} \mathbf{v} \geq \mathbf{v}^T A \mathbf{v}$ for all $\mathbf{v} = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix}$, and hence $\mathbf{v}_2^T S_{\widetilde{A}} \mathbf{v}_2 \geq \mathbf{v}_2^T S_A \mathbf{v}_2$.

As we shall see, η will be a lower bound for the estimate of the condition number of $B^{-1}A$. The value of η taken will depend on both the accuracy of the approximation of S_B to S_A and of B_1 to A_1. More precisely, the latter dependence follows from

$$\eta \geq 1 + \sup_{\mathbf{v}_2} \frac{\mathbf{v}_2^T (S_{\widetilde{A}} - S_A) \mathbf{v}_2}{\mathbf{v}_2^T S_A \mathbf{v}_2} \geq 1 + \frac{\mathbf{v'}_2^T A_{21} (A_1^{-1} - B_1^{-1}) A_{12} \mathbf{v'}_2}{\mathbf{v'}_2^T S_A \mathbf{v'}_2} \tag{11}$$

where $\mathbf{v'}_2$ is the eigenvector for the smallest eigenvalue of S_A.
In general, unless $S_B = S_{\widetilde{A}}$, a strict inequality $\alpha < \eta$ holds. Further $\alpha = 1$ if $\beta = 1$ and α is related to β in the following way. By (10,iii),

$$\begin{aligned} (\alpha - 1) \widehat{\mathbf{v}}_2^T S_A \widehat{\mathbf{v}}_2 &= \widehat{\mathbf{v}}_2^T (S_{\widetilde{A}} - S_A) \widehat{\mathbf{v}}_2 = \widehat{\mathbf{v}}_2^T A_{21} (A_1^{-1} - B_1^{-1}) A_{12} \widehat{\mathbf{v}}_2 \\ &= \widehat{\mathbf{v}}_2^T \widetilde{A}_{21} (I - \widetilde{B}_1^{-1}) \widetilde{A}_{12} \widehat{\mathbf{v}}_2 \\ &\leq (1 - \beta^{-1}) \widehat{\mathbf{v}}_2^T A_{21} A_1^{-1} A_{12} \widehat{\mathbf{v}}_2 \end{aligned}$$

where $\widetilde{A}_{21} = A_{21} A_1^{-1}$, $\widetilde{B}_1 = A_1^{-\frac{1}{2}} B_1 A_1^{-\frac{1}{2}}$. Hence

$$\alpha - 1 \leq q(\widehat{\mathbf{v}}_2)(1 - \beta^{-1}) \tag{12}$$

where

$$q(\widehat{\mathbf{v}}_2) = \frac{\widehat{\mathbf{v}}_2^T A_{21} A_1^{-1} A_{12} \widehat{\mathbf{v}}_2}{\widehat{\mathbf{v}}_2^T S_A \widehat{\mathbf{v}}_2}.$$

Here q can take large values for some vector, typically for a "smooth" vector close to the lowest harmonic vector. However, for the particular vector $\widehat{\mathbf{v}}_2$ it can be expected that q take moderate values.
It is further seen from (11) that for (ii) to hold for a not too large constant η, B_1 must be related to A_1 so that

$$A_1^{-1}A_{12}\mathbf{v}_2 = B_1^{-1}A_{12}\mathbf{v}_2$$

or

$$A_1 B_1^{-1}A_{12}\mathbf{v}_2 = A_{12}\mathbf{v}_2$$

for some "smooth" vector $\mathbf{v}_2$ such as the eigenvector to the smallest eigenvalue of S_A. Otherwise

$$\frac{\mathbf{v}_2^T(S_{\widehat{A}} - S_A)\mathbf{v}_2}{\mathbf{v}_2^T S_A \mathbf{v}_2} \simeq q(\mathbf{v}_2)(1-\beta^{-1})$$

where $q(\mathbf{v}_2)$ is large, because $\mathbf{v}_2^T S_A \mathbf{v}_2 / \mathbf{v}_2^T \mathbf{v}_2$ is small. In this case η would be much larger than $1-\beta^{-1}$. The conclusion is that in order for B in (9) to be an efficient preconditioner for a general spd matrix A, the approximation B_1 of A_1 must be related to the Schur complement S_A, and in particular, the action of B_1^{-1} must be close to the action of A_1^{-1} for "smooth" vectors.

The inequalities (10,i-iii) imply the following spectral relation between A and B, see [13].

Theorem 5.1 *Let* (10,i-iii) *hold. Then*

$$\mathbf{v}^T A\mathbf{v} \le \mathbf{v}^T B\mathbf{v} \le \kappa \mathbf{v}^T A\mathbf{v}, \quad \textit{for all } \mathbf{v}$$

where

$$\kappa \le \beta + \tfrac{1}{2}[\eta - 1 + (\alpha-2)(\beta-1)] + \tfrac{1}{2}\sqrt{[\eta-1+(\alpha-2)(\beta-1)]^2 + 4\beta(\alpha-1)(\beta-1)}.$$

In particular, if $\beta = 1$ *then* $\kappa \le \eta$, *if* $\alpha = \eta$ *then* $\kappa \le \beta\eta$ *and when* $\alpha < \eta$ *then* $\kappa < \beta\eta$.

Consider now the case where the opposite inequalities hold, i.e.

$$\beta^{-1}\mathbf{v}_1^T A_1 \mathbf{v}_1 \le \mathbf{v}_1^T B_1 \mathbf{v}_1 \le \mathbf{v}_1^T A_1 \mathbf{v}_1, \text{ for all } \mathbf{v}_1 \in \mathrm{R}^{n_k - n_{k-1}} \qquad (13,\text{i})$$

$$\eta^{-1}\mathbf{v}_2^T S_A \mathbf{v}_2 \le \mathbf{v}_2^T S_B \mathbf{v}_2 \le \mathbf{v}_2^T S_{\widehat{A}} \mathbf{v}_2, \text{ for all } \mathbf{v}_2 \in \mathrm{R}^{n_{k-1}} \qquad (13,\text{ii})$$

$$\alpha^{-1}\mathbf{v}_2^T S_A \mathbf{v}_2 \le \mathbf{v}_2^T S_{\widehat{A}} \mathbf{v}_2 \le \mathbf{v}_2^T S_A \mathbf{v}_2, \text{ for all } \mathbf{v}_2 \in \mathrm{R}^{n_{k-1}}, \qquad (13,\text{iii})$$

where $\beta \geq 1$, $\eta \geq 1$ and $\alpha \geq 1$. Here the latter inequality is sharp for a vector $\widehat{\mathbf{v}}_2$, i.e.

$$\alpha^{-1}\widehat{\mathbf{v}}_2^T S_A \widehat{\mathbf{v}}_2 = \widehat{\mathbf{v}}_2^T S_B \widehat{\mathbf{v}}_2.$$

The following spectral relation between A and B holds (see [13]).

Theorem 5.2 *Assume that* (13,i-iii) *hold. Then*

$$\kappa^{-1}\mathbf{v}^T A\mathbf{v} \leq \mathbf{v}^T B\mathbf{v} \leq \mathbf{v}^T A\mathbf{v}, \text{ for all } \mathbf{v}$$

where

$$\kappa \leq \beta + \tfrac{1}{2}[\eta - 1 + ((1-\alpha^{-1})\eta - 1)(\beta - 1)] + \tfrac{1}{2}\sqrt{[\eta - 1 + ((1-\alpha^{-1})\eta - 1)(\beta - 1)]^2 + 4\beta(1-\alpha^{-1})\eta(\beta - 1)}.$$

Further $\kappa \leq \eta$ *if* $\beta = 1$ *and* $\kappa \leq \eta\beta$ *if* $\alpha = \eta$. *If* $\alpha < \eta$ *then* $\kappa < \eta\beta$.

Remark 5.1 *The bound* $\eta\beta$ *has been derived earlier in* [18].

eReferences

1. O. Axelsson, On multigrid methods for the two-level type. In W. Hackbusch and U. Trottenberg (editors), Multigrid Methods, *Lecture Notes in Mathematics*, 960, pages 352-367, Springer Verlag, Berlin, Heidelberg, New York, 1982.
2. O. Axelsson, Iterative Solution Methods, Cambridge University Press, New York, 1994.
3. O. Axelsson, Stabilization of algebraic multilevel iteration methods; additive methods, Numerical Algorithms, to appear.
4. O. Axelsson, The stabilized V-cycle method, *J. Comp. Appl. Math.* 74 (1996), 33-50.
5. O. Axelsson, A. Barker, Finite element solution of boundary value problems, Theory and Computation, Academic Press, Orlando, Fl. 1984.
6. O. Axelsson, V. Eijkhout, The nested recursive two-level factorization method for nine-point difference matrices, *SIAM J. Sci. Stat. Comp.* 12 (1991), 1371-1400.
7. O. Axelsson and I. Gustafsson, Preconditioning and two-level multigrid methods of arbitrary degree of approximations, *Math. Comp.* 40 (1983), 219-242.
8. O. Axelsson, M. Neytcheva, Algebraic multilevel iteration method for Stieltjes matrices, *Numer. Linear Algebra with Applications*, 1 (1994), 213-236.
9. O. Axelsson, A. Padiy, On the additive version of the algebraic multilevel iteration method for anisotropic elliptic problems, SIAM J. Sci. Comp., 20 (1999), 1807-1830.
10. O. Axelsson and P.S. Vassilevski, Algebraic multilevel preconditioning methods, I, *Numer. Math.* 56 (1989), 157-177.

11. O. Axelsson and P.S. Vassilevski, Algebraic multilevel preconditioning methods, II, *SIAM J. Numer. Anal.* 27 (1990), 1569-1590.
12. O. Axelsson and P.S. Vassilevski, Variable-step multilevel preconditioning methods, Numer. Linear Algebra with Applications 1(1994), 75–101.
13. O. Axelsson, P.S. Vassilevski, The AMLI method; A class of algebraic multilevel iteration methods for positive definite symmetric matrices, in preparation.
14. R.E. Bank and T. Dupont, Analysis of a two-level scheme for solving finite element equations, *Report CNA-159,* Center for Numerical Analysis, The University of Texas at Austin, 1980.
15. Zhong-Zhi Bai, O. Axelsson, A unified framework for the construction of various algebraic multilevel preconditioning methods, *Acta Mathematica Sinica,* 15 (1999), 132-143.
16. Y.A. Kuznetsov, Algebraic multigrid domain decomposition methods, *Soviet J. Numer. Meth. and Math. Modelling* 4 (1989), 351-380.
17. J.F. Maitre and F. Musy, The contraction number of a class of two-level methods; an exact evaluation for some finite element subspaces and model problems. *In* W. Hackbusch and U. Trottenberg (editors), Multigrid Methods, *Lecture Notes in Mathematics,* 960, pages 535-544, Springer Verlag, Berlin, Heidelberg, New York, 1982.
18. Y. Notay, Optimal order preconditioning of finite difference matrices, Technical Report GANMN 97-02, ULB Brussels, Belgium, 1997.
19. P. S. Vassilevski, Hybrid V-cycle algebraic multilevel preconditioners, Math. Comp. 58(1992), 489-512.
20. P.S. Vassilevski, On two ways of stabilizing the hierarchical basis multilevel methods, *SIAM Review* 39 (1997), 18-53.
21. H. Yserentant, On the multilevel splitting of finite element spaces, *Numer. Math.*, 49 (1986), 379-412.

Auspices statement will go here.