SAND99-0700C

Numerical techniques to evaluate moments of dynamic system response

Richard V. Field, Jr., Thomas L. Paez, John R. Red-Horse

Sandia National Laboratories[1], Albuquerque, NM 87185

rvfield@sandia.gov, tlpaez@sandia.gov, jrredho@sandia.gov

Probabilistic uncertainty is a phenomenon that occurs to a certain degree in many engineering applications. The effects that this uncertainty has upon a given system response are a matter of some concern. Techniques which provide insight to these effects will be required as modeling and prediction becomes a more vital tool in the engineering design process. The purpose of this paper is to outline a procedure to evaluate uncertainty in dynamic system response exploiting various numerical methods. Specifically, the goal is to attain the statistics of the response with minimal computational effort. Numerical interpolation and integration techniques are utilized in conjunction with the iterative form of the Advanced Mean Value (AMV+) method to efficiently and accurately estimate statistical moments of the response random process. A numerical example illustrating the use of this analytical tool in a practical framework is presented.

Introduction

Certain response characteristics of structural dynamic systems exhibit behavior that can only be quantified to within some level of uncertainty. These uncertainties are often incorporated into system models as parametric quantities, such as material and geometrical properties. A previous paper [4] developed a technique for the analysis of this class of uncertainty using a probabilistic approach where the system parameters are assumed to be random variables with known probability distributions. The technique, suitable for approximating the response cumulative distribution function (CDF) at given response levels, is based on the AMV method, an approach that was developed specifically for application to system reliability analysis by Wu and Wirsching [5]. AMV is strongly motivated by the fact that the relationship mapping the random parameters to the response quantity of interest is approximated using point analyses, or function evaluations. Thus, the analytical functional relationship is not required.

In this article, the issue of evaluating statistical moments is addressed through the use of a numerical quadrature scheme. The goal is to achieve pertinent statistical information at a cost which is far lower than what is necessary to evaluate the full CDF. This method discussed herein involves three steps: (1) using the iterative form of AMV, AMV+, to estimate the CDF at a discrete number of abscissa values, (2) implementing interpolation tools to approximate the CDF and corresponding PDF at arbitrary abscissa locations, and (3) using numerical integration tools to compute the moments of the response. The work presented here is aimed at refining the method reported in [1], which proved promising but became inaccurate when considering highly non-Gaussian response variables.

[1] *Sandia National Laboratories is a multiprogram laboratory operated by Sandia Corporation, a Lockheed Martin Company, for the United States Department of Energy under Contract DE-AC04-94AL85000.*

DISCLAIMER

The Advanced Mean Value (AMV) Method for Probabilistic System Analysis

Let Y be a scalar random variable defined as follows:

$$Y = g(X), \tag{1}$$

where X is an n-variable random vector with arbitrary joint probability distribution, characterized by the joint PDF, $f_X(x)$, and $g(\cdot)$ is a deterministic function. The probability distribution of the random variable Y can be characterized with the CDF of Y, $F_Y(y)$, for various values of the arbitrary scalar y. By definition

$$F_Y(y) = P(Y \le y) = P(g(X) \le y) = \int dx_1 \underset{g(x) \le y}{\cdots} \int dx_n f_X(x), \; -\infty < y < \infty, \tag{2}$$

where the fact that the random vector X is n-dimensional has been used. The exact solution is an n-fold improper integral of the joint PDF $f_X(x)$ over a subset of the domain of definition of the underlying random variable. The integral can only be solved in closed form for a very limited number of cases, and many more general numerical approximations are quite expensive. In view of this, it is clear that less cumbersome, approximate solution approaches are particularly attractive.

To approximate Eq. (2), the AMV method can be employed (see [1] for a detailed discussion of AMV in this framework). To summarize, first a linear approximation to the function in Eq. (1) is computed about an estimate of its mean using a truncated Taylor series. A second step then approximates the n-dimensional integral associated with Eq. (2) with a one-dimensional integral in a well-known probability space. Note, however, that the AMV method does not yield the result $F_Y(y_i)$, $i = 1, \ldots, m$. Rather, it estimates the CDF of the random variable Y at $\hat{y}_i$, $i = 1, \ldots, m$, where in general $y \neq \hat{y}$. The AMV+ method performs these steps in an iterative loop to give a more accurate estimate of the CDF of Y.

CDF/PDF Approximation Methods

The fact that the abscissa locations $\hat{y}$ differ, in general, from the points y in the AMV method precludes approximation of the CDF or PDF at an arbitrary point. To overcome this difficulty, an approximation to the CDF is formulated as follows

$$F_Y(y) \cong \Phi\left[w\left(\frac{y-a}{b}\right)\right] = \Phi[w(z)]. \tag{3}$$

Here, a and b are estimates of the mean and standard deviation of Y, and $\Phi(\cdot)$ is the CDF of a standard normal random variable. The function $w(z)$ is an interpolating function with

$$w(z_i) = \Phi^{-1}[F_Y(y_i)], \tag{4}$$

where the points $(y_i, F_Y(y_i))$, $i = 1, \ldots, m$, are the output of the AMV analysis. It follows that the corresponding estimate to the PDF of Y is

$$f_Y(y) \cong \phi\left[w\left(\frac{y-a}{b}\right)\right]\frac{dw}{dy}, \tag{5}$$

where $\phi(\cdot)$ denotes the standard normal PDF function.

A least squares formulation for $w(z)$, first suggested by Wu and Burnside [6], was explored in a previous paper [1]. Unfortunately, that approach was insufficient for cases where Y exhibited highly non-Gaussian behavior. For this reason, a more general methodology is considered which uses two different schemes for estimating $w(z)$: the Artificial Neural Network and cubic spline interpolating polynomial.

Artificial Neural Network Approximation to $w(z)$

An Artificial Neural Network (ANN) is one means of approximating the functional expression in the previous section. An attractive feature is that it can provide robust, accurate, and efficient interpolators for functions known only via examples of their correct behavior. One ANN that does not appear in the literature, but is a direct extension of the Connectionist Normalized Linear Spline (see [3]) is the Univariate Polynomial Spline (UPS) network. In this particular network, an approximation to the function $v = w(z)$, known only through its realizations at z_i, $i = 1, \ldots, m$, is needed. The global approximation is to be constructed from components that are accurate in local vicinities.

To summarize the development of the UPS network in this framework, consider the identity

$$v\theta(z, c_j, \beta) = w(z)\theta(z, c_j, \beta), \; j = 1, \ldots, N, \tag{6}$$

where $\theta(\cdot)$ is a radial basis function, centered at c_j, with width parameter β. This expression is created at N locations. One can approximate $w(z)$ on the right-hand side using the first three terms of its Taylor series expansion, then superimpose the set of approximations and normalize, to give

$$v \cong q(z, \mathbf{a}, \beta, \mathbf{c}) = \frac{\sum_j [(a_{0j} + a_{1j}(z - c_j) + a_{2j}(z - c_j)^2)]\theta(z, c_j, \beta)}{\sum_j \theta(z, c_j, \beta)}. \tag{7}$$

The function $q(\cdot)$ is an ANN if its parameters can be estimated adaptively. Its specific mathematical form is the expression given on the right. The parameters c_j, $j = 1, \ldots, N$, are pre-selected. The parameters a_{0j}, a_{1j}, a_{2j}, $j = 1, \ldots, N$, and β are identified using a least squares approach with the data $(z_i, w(z_i))$, $i = 1, \ldots, m$. A frequently used form of radial basis function, and the one used herein because it has the same structure as the Gaussian kernel, is the quadratic exponential.

Cubic Spline Interpolation of $w(z)$

To approximate the CDF with a cubic spline interpolating function, the discrete points $(z_i, w(z_i))$, $i = 1, \ldots, m$, are connected via $m - 1$ distinct third-order polynomials that satisfy [2]

$$w''_{i-1}(z_i) = w''_i(z_i), \; i = 2, \ldots, m-1. \tag{8}$$

This, along with conditions at the boundaries z_1 and z_m, results in an m-dimensional tridiagonal system of linear equations that can readily be solved to indirectly arrive at the coefficients of the cubic interpolating polynomials.

In practice, this method works very well when the set of interpolating points from AMV sufficiently spans the distribution of the response. A poor approximation may result, however, if the interpolating points span only a segment of $F_Y(y)$ because the cubic spline approximation is unable to extrapolate to values not contained in $[y_1, y_m]$. One way to allow for extrapolation is to assume

$$F_Y(y) = \begin{cases} 0 & y < y_1 \\ 1 & y > y_m \end{cases}, \tag{9}$$

but this can lead to obvious inaccuracies in the moment computations if $F_Y(y)$ is not sufficiently close to 0 or 1 at the endpoints.

Numerical Integration Techniques

The PDF of Y is obtained through analytical differentiation of the expression for the CDF of Y. With this approximation to $f_Y(y)$, the moments of Y can be calculated directly utilizing a numerical integration scheme. To compute the moments of $Y = g(X)$, form the Mth moment of Y as

$$E[Y^M] = \int_{-\infty}^{\infty} y^M f_Y(y) dy, \; M = 1, 2, \dots. \tag{10}$$

As shown in [1], the Gauss-Hermite quadrature formulation converges very quickly when considering nearly-Gaussian response variables because it can accurately account for the tails of $f_Y(y)$. When considering PDFs that do not exhibit this characteristic, however, slower rates of convergence and less accurate estimates of the statistics can be expected.

Example Problem

This example illustrates the synergetic operation of the three steps previously discussed to estimate response statistics. Consider the simple function

$$Y = g(X_1, X_2) = X_1 / X_2^2. \tag{11}$$

In addition, assume the random variables X_1 and X_2 to be independent and lognormally distributed, with means $\mu_{X_1} = \mu_{X_2} = 1$ and standard deviations $\sigma_{X_1} = \sigma_{X_2} = 0.2$.

Closed-form expressions for the CDF, PDF, and moments of Y were derived in [1] and [4]. Using these analytical expressions, the first 3 moments of Y can be computed, as listed in Table 1. Estimates of these moments and the associated relative error using the numerical techniques outlined in this paper are also shown. Results indicate that the method does a good job of estimating the first three moments. The fact that the errors remain small for higher moments when using the cubic spline is indicative that the cubic spline does a slightly better job of approximating the tail of $f_Y(y)$ than does this particular ANN.

Table 1: Central moments of $Y = g(X_1, X_2)$ with $\sigma_{X_1} = \sigma_{X_2} = 0.2$.

Moment	Exact	CSI (% error)	ANN (% error)
$\mu_Y = E[Y]$	1.1249	1.1225 (-0.213)	1.1264 (0.133)
$E[(Y-\mu_Y)^2]$	0.2741	0.2761 (0.730)	0.2703 (-1.39)
$E[(Y-\mu_Y)^3]$	0.2149	0.2142 (-0.326)	0.1998 (-7.03)

The accuracy of the moment estimates is sensitive to some key elements of the method. For example, the maximum and minimum abscissa values in the AMV+ analysis, y_1 and y_m, must sufficiently span the distribution of Y. This can be disadvantageous because, when using this method in a typical application, one will not generally know the general character of the output distribution.

Some simplifying assumptions have been made to facilitate this example. First, X_1 and X_2 are specified independent; the method discussed here is not restricted to problems with independent underlying random variables. Second, AMV was developed to address problems where no explicit knowledge of the functional relationship $g(X)$ exists. This method also functions in this situation, making it an ideal tool to be used in conjunction with finite element analysis to estimate the statistical nature of systems with uncertain parameters.

Conclusions and Future Work

A technique for using Gauss-Hermite numerical quadrature coupled with the AMV+ method for the computation of the moments of the response of mechanical and other analytical systems has been developed. The method has been applied to an example problem in a practical framework with promising results. These techniques are efficient and well-suited for use with general finite element analysis codes.

References

[1] Field, R.V. Jr., T.L. Paez, and J.R. Red-Horse (1999), "Utilizing Gauss-Hermite Quadrature to Evaluate Uncertainty in Dynamic System Response," *Proceedings of the 17th International Modal Analysis Conference*, Orlando, FL, pp. 1856-1861.

[2] Gautschi, W. (1997), *Numerical Analysis: An Introduction*, Birkhäuser.

[3] Jones, R. D., et al., (1990), "Nonlinear Adaptive Networks: A Little Theory, A Few Applications," *Cognitive Modeling in System Control*, The Santa Fe Institute.

[4] Red-Horse, J.R. and T.L. Paez (1998), "Uncertainty Evaluation in Dynamic System Response," *Proceedings of the 16th International Modal Analysis Conference*, Santa Barbara, CA, pp. 1206-1212.

[5] Wu, Y.-T. and P.H. Wirshing (1989), "New Algorithm for Structural Reliability Estimation," *ASCE Journal of Engineering Mechanics*, Vol. 113, pp. 1319-1334.

[6] Wu, Y.-T. and O.H. Burnside (1988), "Efficient Probabilistic Structural Analysis Using an Advanced Mean Value Method," *Probabilistic Methods in Civil Engineering, Proceedings of the 5th ASCE Specialty Conference, Engineering Mechanics Division and Geotechnical Division*, ASCE, P.D. Spanos, ed., VPI&SU, Blacksburg, VA, pp. 492-495.