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DOE/ER/54241--158

DOE/ER/542-41-158 FRCR #479

TURBULENT EQUIPARTITIONS IN TWO DIMENSIONAL DRIFT CONVECTION

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JULY 25, 1995

THE UNIVERSITY OF TEXAS

Austin, Texas

DOE/ER/54241--158

Turbulent equipartitions in two dimensional drift convection

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June 28, 1995

Abstract

Unlike the thermodynamic equipartition of energy in conservative systems, turbulent equipartitions (TEP) describe strongly non-equilibrium systems such as turbulent plasmas. In turbulent systems, energy is no longer a good invariant, but one can utilize the conservation of other quantities, such as adiabatic invariants, frozen-in magnetic flux, entropy, or combination thereof, in order to derive new, turbulent quasi-equlibria. These TEP equilibria assume various forms, but in general they sustain spatially inhomogeneous distributions of the usual thermodynamic quantities such as density or temperature. This mechanism explains the effects of particle and energy pinch in tokamaks. The analysis of the relaxed states caused by turbulent mixing is based on the existence of Lagrangian invariants (quantities constant along fluid-particle or other orbits). A turbulent equipartition corresponds to the spatially uniform distribution of relevant Lagrangian invariants. The existence of such turbulent equilibria is demonstrated in the simple model of two dimensional electrostatically turbulent plasma in an inhomogeneous magnetic field. The turbulence is prescribed, and the turbulent transport is assumed to be much stronger than the classical collisional transport. The simplicity of the model makes it possible to derive the equations describing the relaxation to the TEP state in several limits.

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1 Introduction

The term "equipartition" broadly refers to the ergodic property of multi dimensional Hamiltonian systems, which tend to distribute uniformly over the phase-space surface of constant energy. The conservation of energy plays the fundamental role in classical equilibrium thermodynamics. Often, equipartition is also understood in the narrower sense of the same energy, $T/2$, per each degree of freedom, as follows from the ergodic property and the quadratic energy dependence on velocities. The term "turbulent equipartition" was introduced by Yankov (1994b) in order to describe the turbulent relaxed state, in which the system assumes a uniform distribution on the surface of constant invariants respected by the turbulence.

In plasma physics, the best known example of a turbulent equipartition is the quasilinear plateau of the distribution function caused by the nonlinear Landau damping of plasma waves . In a toroidal turbulent plasma, the relevant invariants are given by the frozen-in law (in the fluid limit) or the adiabatic invariants $\mu = m{v_\perp}^2/(2B)$ and $J = \oint m v_\parallel d\ell_\parallel$ and the Liouville theorem (in the collisionless limit), both limits derivable from the more general Poincare invariant . The plasma mixing by low-frequency electrostatic modes in a tokamak, subject to these conservation laws, results in the inhomogeneous density and temperature profiles peaked at the center even in the absence of particle and energy fluxes, thus presenting the underlying mechanism of the pinch effect and the profile consistency in tokamaks

Accounting for both turbulence and particle collisions for relevant tokamak parameters is a complicated problem, and the theoretical predictions of the tokamak density profiles have varied and also suffered uncertainties in their limits of applicability. For this reason, it is desirable to study in some detail the processes of turbulent relaxation in a simplified geometry, thereby avoiding the toroidal complications and making it possible to allow for both turbulent and collisional effects in a regular fashion. We therefore restrict ourselves to the two-dimensional plane geometry, $\mathbf{x} = (x, y)$, $\partial/\partial z = 0$, where the parallel kinetic effects are unimportant, and it is possible to construct a fluid-type system describing the evolution of plasma density $n(\mathbf{x}, t)$ and temperature $T(\mathbf{x}, t)$. The turbulence will be assumed electrostatic, with the potential $\phi(\mathbf{x}, t)$ specified non-self-consistently, although certain self-consistency constraints on ϕ will be discussed. The collision frequency ν will be assumed much greater than relevant turbulent mixing rates so that the Maxwellian distribution function is sufficiently accurately maintained for the given species. More simplified 2D TEP models were discussed by Yankov (1995a) and Isichenko and Petviashvili (1995). There are also real systems, which demonstrate turbulent equipartitions in essentially two dimensions, for example, an ion diode with an externally applied magnetic field (Gordeev and Grechikha, 1995).

Our goal is to demonstrate a nontrivial turbulence-mixed state with inhomogeneous density and temperature. For this to happen, we need an inhomogeneous magnetic field $B = B_z(x, y)$. To avoid the discussion of plasma currents and equilibrium, one can think of external currents creating the field, but this is not essential, because the inhomogeneity can be due to the cylindricity, as in the z-pinch geometry. The TEP density profiles in z-pinches were described by Sasorov (1990) in the framework of magnetohydrodynamics and by Yankov (1994a) in kinetics.

The article is organized as follows. In Sec. 2 we discuss the role of the Liouville theorem and the Lagrangian invariants in turbulent mixing and present the simplest derivation of a turbulent equipartition in two dimensions. In Sec. 3 we review the integral Poincare invariant and show how a kinetic frozen-in law can be derived and applied to the problem of 2D drift convection of guiding centers in the collisionless case. In Sec. 4 we introduce the formal drift-kinetic collisional description, which is used in Secs. 5 and 6 for low- and high-frequency (compared to the collision frequency) turbulence. The results are summarized in Sec. 7.

2 The role of Lagrangian invariants in turbulent mixing

In this section we show that the Liouville theorem underlying Hamiltonian dynamics warrants at least one Lagrangian invariant, and the turbulent mixing drives this invariant to a uniform state, which is an attracting equilibrium.

The motion $\dot{\mathbf{s}} = \mathbf{w}(\mathbf{s}, t)$ in an arbitrary phase space $\mathbf{s}$ can be always described by the Liouville (or continuity) equation for the phase-space density $f(\mathbf{s}, t)$:

$$\partial_t f + \partial_{\mathbf{s}} \cdot (\mathbf{w} f) = 0, \tag{1}$$

where the velocity $\mathbf{w}$ generally depends on the distribution f via self-consistent fields rendering the problem (1) nonlinear and chaotic. If the motion is Hamiltonian, such that $\mathbf{s} = (\mathbf{p}, \mathbf{q})$ and $\mathbf{w} = (-\partial_{\mathbf{q}} H, \partial_{\mathbf{p}} H)$, where $H(\mathbf{p}, \mathbf{q}, t)$ is the Hamiltonian, the phase-space flow velocity $\mathbf{w}$ is incompressible, $\partial_{\mathbf{s}} \cdot \mathbf{w} = 0$ (the Liouville theorem), and the density f is a Lagrangian invariant:

$$df/dt \equiv (\partial_t + \mathbf{w} \cdot \partial_{\mathbf{s}}) f = 0. \tag{2}$$

Equation (2) states that f is conserved along the phase-space orbit. Since in a chaotic situation the orbits show a mixing behavior and can densely fill the whole accessible phase space volume, the density f assumes an asymptotically uniform distribution. The validity of such a general statement cannot be universal and requires several assumptions.

1. The phase space must be sufficiently multi dimensional. In low dimensions, the mixing is inhibited by KAM tori (Meiss, 1992). The lack of spatial dimensions can be filled by a complicated explicit time dependence of the Hamiltonian: Adding another incommensurate frequency to this dependence is equivalent to adding half a degree of freedom.

2. The accessible phase-space volume should be finite. This is usually warranted by an exact or an approximate conservation law. For example, the conservation of energy E for the multi-particle mechanical Hamiltonian $H(\mathbf{p}, \mathbf{q}) = \mathbf{p}^2/(2m) + V(\mathbf{q})$ with a confining potential V specifies a closed surface $H(\mathbf{p}, \mathbf{q}) = E$ on which the motion takes place.

3. When so, the mixing of f does not occur globally, but it takes place independently on each conservative manifold. In the case of conserved energy, one arrives at the one-parameter microcanonical distribution $f(\mathbf{s}, t \to \infty) = \delta[H(\mathbf{s}) - E]$ on each iso-energetic surface. The Boltzmann-Gibbs exponential distribution for a small subsystem then follows, furnishing the basis of the classical thermodynamics.

4. Equation (2) implies that the values of f are not changed, but only displaced. This seems to contradict the conclusion that f takes on a single, uniform value. For this to happen, one has to introduce a small amount of dissipation, or, equivalently, a small coupling of the system to another, even larger, Hamiltonian system. This will result in additional small diffusion-type terms in Eq. (2), which will smooth out the exponentially growing (Ott and Antonsen, 1989) phase-space gradients of f and thus complete the mixing. Otherwise, if one is interested in a coarse-grain (locally averaged) distribution function, the argument of small dissipation is redundant.

Thus the classical equilibrium statistical mechanics with the equipartition of energy is a consequence of the Lagrangian invariance of the phase-space density subject to the conservation of the total energy. In strongly non-equilibrium systems with sources and sinks energy may not be a good integral of motion; however, there quite may be other integral and/or Lagrangian invariants.

Probably the most famous example of a turbulent equipartition is the convectively turbulent atmosphere (Fermi, 1937). The conductive heat flux is small in comparison with the convective energy flux, which makes the specific entropy of the two-atomic gas (N_2 or O_2) $s = \ln(T^{5/2}/n)$ a good Lagrangian invariant. Here T is the temperature and n the gas particle density in the air. Then, within the range of heights where the vertical convection is important, one concludes that s is a spatial constant. Together with the average hydrostatic equilibrium condition, $d(nT)/dz = -mng$, where m is the average molecule mass and g the gravity, we find the average vertical temperature gradient $dT/dz = -(2/7)mg \simeq -9.8^0\text{C/km}$. The steady temperature gradient indicates the absence of thermal equilibrium and is a simple example of a turbulent equipartition. Various additional effects, such as water condensation and radiation transport, introduce corrections to the isentropic atmosphere model, which, nevertheless, remains a starting point much better than the model of an isothermal atmosphere.

A somewhat more complicated example of turbulent equipartition is found in the theory of zonal winds on the Jupiter (Marcus, 1993), in which the Lagrangian invariant of potential vorticity assumes a step-like structure with several zones of good turbulent mixing, which corresponds to the zonal wind pattern close to the observations.

In collisionless plasmas, the quasilinear plateau results from the turbulent mixing of the particle distribution function $f(\mathbf{x}, \mathbf{v})$, which is a Lagrangian invariant of the Vlasov equation. A more complicated example of quasilinear plateau with an inhomogeneous plasma density in an electrostatically plugged magnetic mirror was studied by Pastukhov (1980).

Our specific interest lies in the turbulent transport in tokamaks. The tokamak density and temperature profiles have been experimentally found to be peaked at the axis and sufficiently resilient to the attempts to alter the profiles by external perturbations. The empirical paradigm of the "profile consistency" (Coppi, 1980) is conspicuously similar to the example of isentropic atmosphere. The large-scale trapped-ion modes (Tang and Rewoldt, 1992) can play the role of the convective turbulence. The concepts of turbulent diffusion, constrained by adiabatic invariants and the Liouville theorem, and resulting inhomogeneous profiles have long been used for the Van Allen radiation belts in the Earth's magnetosphere (Dungey, 1965; Birmingham et al., 1967). However, apart from a few notable exceptions, (Hasegawa, 1987; Kadomtsev, 1995), this connection has been largely missing from the fusion theory literature. Also, in a tokamak, the situation is more complicated because of the presence of both the poloidal and the toroidal magnetic fields and also due to the important role of collisions.

For plasma transport, the simplest prototypical model demonstrating the physics of turbulent equipartitions is the 2D $\mathbf{E} \times \mathbf{B}$ drift motion of charged particles in a steady inhomogeneous magnetic field $B = B_z(x, y)$:

$$\mathbf{v} = -\frac{c}{B}\nabla\phi(x,y,t) \times \hat{\mathbf{z}}, \tag{3}$$

where $\phi(x,y,t)$ is the electrostatic potential, and the magnetic drift is neglected. The continuity equation $\partial_t n + \nabla \cdot (n\mathbf{v}) = 0$ for the particle density n can be written

$$\left(\partial_t - \frac{c}{B}\nabla\phi(x,y,t) \times \hat{\mathbf{z}} \cdot \nabla\right)\frac{n}{B} = 0. \tag{4}$$

We thus find that the ratio n/B is a Lagrangian invariant, and the $\mathbf{E} \times \mathbf{B}$ turbulent mixing of particles will result in the turbulent equipartition in which the particle density is proportional to the magnetic field.

In the following sections, we gradually add to this simple picture several additional effects, such as magnetic drifts and particle collisions. As we do so, we introduce new concepts and techniques.

3 Poincare invariant, frozen-in law, and collisionless "μ-hydrodynamics" of guiding centers

In this section we review the relation between the relative integral Poincare invariant and the frozen-in law in a charged fluid and collisionless kinetics and re-establish the turbulent equipartition $n(x,y)/B(x,y) = \text{const}$ from a more general standpoint.

The Lagrangian invariant n/B discussed in the previous section reminds the frozen-in law in ideal magnetohydrodynamics (MHD), where magnetic field is not fixed but allowed to evolve. Therefore, the result obtained by a calculation for a particular case could be a consequence of a more general conservation law. Let us briefly review the argument (Yankov, 1995b) pertaining to the general Hamiltonian origin of the frozen-in laws in hydrodynamics.

Consider the particle motion with the Hamiltonian $H(\mathbf{p}, \mathbf{q}, t)$ defining the phase-space flow as follows:

$$\dot{\mathbf{p}} = -\partial_{\mathbf{q}} H, \quad \dot{\mathbf{q}} = \partial_{\mathbf{p}} H. \tag{5}$$

Since the flow (5) is incompressible in each phase plane (p_i, q_i), the area within any contour $\gamma_i(t)$, moving with the flow in the plane, is conserved:

$$I_i = \oint_{\gamma_i(t)} p_i dq_i = \text{const}. \tag{6}$$

If the contours $\gamma_i(t)$ are the 2D projections of a single 6D contour $\gamma(t)$ moving with the flow (5), the summation of the constants (6) yields the integral Poincare invariant (Arnold, 1978):

$$I = \oint_{\gamma(t)} \mathbf{p} \cdot d\mathbf{q} = \text{const.} \tag{7}$$

Consider a charged fluid in which one can introduce the local fluid momentum $\mathbf{p}(\mathbf{q}) = m\mathbf{v} + (e/c)\mathbf{A}$, m and e being the mass and the charge of the fluid element and $\mathbf{A}$ the vector potential. The "one point-one momentum," or $\mathbf{p} = \mathbf{p}(\mathbf{q})$ approximation is the key property of usual hydrodynamics. In this case the Poincare invariant (7) reduces to the integration around a contour in the 3D space of $\mathbf{q}$ and can be rewritten by the Stokes lemma into a flux of the (generalized) vorticity

$$\boldsymbol{\Omega} = \nabla \times \mathbf{p} = \nabla \times m\mathbf{v} + (e/c)\mathbf{B} \tag{8}$$

through the contour moving with the fluid:

$$I = \oint \mathbf{p}(\mathbf{q}) \cdot d\mathbf{q} = \int \boldsymbol{\Omega} \cdot d\mathbf{S} = \text{const.} \tag{9}$$

In a differential form, this frozen-in law has the familiar form (Buneman, 1952; Braginskii, 1965; Lynden-Bell, 1967; Sudan, 1979)

$$\partial_t \boldsymbol{\Omega} = \nabla \times (\mathbf{v} \times \boldsymbol{\Omega}) \tag{10}$$

indicating that the vorticity field $\boldsymbol{\Omega}$ is frozen into the fluid and hence conserves topology. In the absence of the magnetic part of $\boldsymbol{\Omega}$, this is equivalent to the Kelvin theorem for ideal fluid; the opposite limit is the MHD freezing-in of the magnetic field lines into the fluid.

The above derivation assumed that the motion of a fluid element is described by a Hamiltonian equation. This is not always the case. Consider the momentum balance equation

$$m\frac{d\mathbf{v}}{dt} = -e\left(\nabla\phi + \frac{1}{c}\partial_t\mathbf{A}\right) + \frac{e}{c}\mathbf{v} \times \mathbf{B} - \frac{\nabla p}{n}. \tag{11}$$

If the pressure p is a function of the density n, then the pressure term can be represented as the gradient of enthalpy $w = \int dp(n)/n$ and thus included in the electrostatic potential ϕ. Then the fluid element obeys the Hamilton equation with the Hamiltonian

$$H(\mathbf{p}, \mathbf{q}, t) = e\phi(\mathbf{q}, t) + w(\mathbf{q}, t) + \frac{1}{2m}\left[\mathbf{p} - \frac{e}{c}\mathbf{A}(\mathbf{q}, t)\right]^2. \tag{12}$$

If the pressure is not a function of density, the pressure force contains a solenoidal component, which makes the fluid element motion not Hamiltonian in the usual sense. This also breaks the Poincare invariant (7) and introduces the topology breaking term in the vorticity equation (10). Upon applying curl to Eq. (11), we obtain instead of (10):

$$\partial_t \boldsymbol{\Omega} = \nabla \times (\mathrm{v} \times \boldsymbol{\Omega}) - \nabla \frac{1}{n} \times \nabla p. \tag{13}$$

The topology breaking term $\nabla n \times \nabla p$ is important in both fluid and plasma dynamics. In fluids, it describes the vorticity generation by buoyancy. In plasmas, it is known as thermal EMF or battery effect, a mechanism of the magnetic field generation.

In some cases, however, the loss of the topology and of Lagrangian invariants is only partial, and the invariants can be recovered in a new form, because of the flexibility offered by the Poincare invariant. For example, one can use a contour drawn on a surface of constant specific entropy s. Then, the solenoidal part of the pressure term in (11), $\partial_s p(n,s)\,\nabla s$, is perpendicular to the surface, and we have a "reduced" Hamiltonian motion of the fluid element on the surface $s=$ const. The Poincare invariant (9) is newly recovered in which the vorticity component parallel to ∇s only remains. For the case of the Lagrangian invariance of the entropy s (no viscous dissipation or heat conduction), this leads to the new Lagrangian invariant $L = \boldsymbol{\Omega}\cdot\nabla s/n$, a statement known as the Ertel theorem (Landau and Lifshitz, 1987). A different example of non-trivial Lagrangian invariants is presented in Sec. 5.

In a collisionless Vlasov plasma, the Poincare invariant also exists, but it is not in general reducible to a 3D form, and the contour of the integration remains in 6D. A simplification (Yankov, 1995b) is achieved in the drift approximation by imposing the conservation of the adiabatic invariant $\mu = m v_\perp{}^2/(2B)$ and ignoring three more variables: the gyro angle α_μ and, in a 2D geometry, the coordinate z and the velocity v_z. Then the contour can be placed on the remaining $(6-4) = 2$ dimensional manifold at constant μ, where both the magnetic flux $\int B dx dy$ (the mechanical part of the vorticity $\boldsymbol{\Omega}$ is small compared to its magnetic part) and the number of particles $\int f(x,y,\mu) dx dy$ inside the contour are conserved. Here $f(x,y,\mu)$ denotes the guiding center density with the given μ. We thus arrive to the more general Lagrangian invariant $L_\mu(x,y,t) = f(x,y,\mu,t)/B(x,y,t)$, which is valid for any 2D collisionless plasma motions conserving the adiabatic invariant μ. The presence of the invariant L_μ makes the 2D drift kinetics akin to a hydrodynamic description, which we call "μ-hydrodynamics." We thus infer the turbulent equipartition, in which L_μ is spatially uniform:

$$\lim_{t\to\infty} \frac{f(x,y,\mu,t)}{B(x,y,t)} = L_{\mu 0}. \tag{14}$$

The integration over μ results in the density profile

$$\lim_{t\to\infty} \frac{n(x,y,t)}{B(x,y,t)} = \text{const}, \tag{15}$$

as found in Sec. 2 for a cold plasma in a time-independent magnetic field. The thermal effects do not modify the turbulent equipartition, because the magnetic drifts occur along the lines $B(x,y) = \text{const}$ and do thus not alter the distribution (15).

Since during the relaxation process the density, the temperature, and the magnetic field are generally not functions of each other, a fluid equation like (13) would suggest that there is no frozen-in law, whereas we saw the existence of the Lagrangian invariant $L = f(x,y,\mu,t)/B(x,y,t)$. This apparent contradiction is resolved by noting that L is conserved along orbit moving with the guiding center velocity, which is different from the fluid velocity. In other words, the "topology breaking" term in (13) can be absorbed into the "topology preserving" term by renormalizing the velocity $\mathbf{v}$.

Since we now have a TEP prediction for the distribution function f, the profile of the perpendicular temperature $T_\perp = \langle \mu B \rangle$ is also readily available:

$$T_\perp(x,y,t\to\infty)/B(x,y) = \text{const}. \tag{16}$$

As no collisional coupling between the perpendicular and the parallel velocities is present in this model, the parallel temperature $T_\| = \left\langle m{v_\|}^2/2 \right\rangle$ is a Lagrangian invariant by itself and hence assumes the uniform distribution

$$T_\|(x,y,t\to\infty) = \text{const}. \tag{17}$$

In a collisional plasma, we expect a turbulent equipartition with an isotropic temperature showing an intermediate behavior between (16) and (17). The effect of collisions is addressed in the remaining sections.

4 Drift kinetic equation

In this section we establish the collisional drift kinetic equation to be used in Secs. 5 and 6.

Ignoring the gyro-angle degree of freedom, we use the drift kinetic equation for the distribution function $f(x,y,v_\|,\mu,t)$ understood as the guiding center density in the space of $(\mathbf{x},v_\|,\mu)$. The collisionless motion in the phase space being given by

$$\begin{aligned} \dot{\mathbf{x}} &= -\frac{c}{B}\nabla\left(\phi + \frac{\mu}{e}B\right)\times\hat{\mathbf{z}}, \\ \dot{\mu} &= 0, \quad \dot{v_\|} = 0, \end{aligned} \tag{18}$$

the drift kinetic equation takes the form:

$$\partial_t f + \frac{c}{e}\left[e\phi + \mu B, \frac{f}{B}\right] = C(f). \tag{19}$$

Here $[A, B] \equiv \nabla A \times \nabla B \cdot \hat{\mathbf{z}}$ is the Jacobian in the (x, y) plane and $C(f)$ is the collision operator for the guiding centers:

$$C(f) = -\partial_{\mathbf{x}}\mathbf{\Gamma}^{\mathbf{x}} - \partial_{\mu}\Gamma^{\mu} - \partial_{v_\parallel}\Gamma^{v_\parallel}, \tag{20}$$

$$\Gamma^i = -D^{ij}\partial f/\partial x^j + U^i f \tag{21}$$

where the coefficients of diffusion D^{ij} and of dynamical friction U^i are linear integral operators on f (Hazeltine and Meiss, 1992). The collision operator turns to zero if and only if the distribution function is a Maxwellian:

$$f_M(\mathbf{x}, \mu, v_\parallel) = \sqrt{\frac{m}{2\pi}}\frac{n(\mathbf{x})B(\mathbf{x})}{T^{3/2}(\mathbf{x})}\exp\left(-\frac{\mu B(\mathbf{x}) + m{v_\parallel}^2/2}{T(\mathbf{x})}\right). \tag{22}$$

Strictly speaking, the equilibrium density and the temperature in (22) must be uniform for the condition $C(f) = 0$ to hold exactly. However, the spatial component $\mathbf{\Gamma}^{\mathbf{x}}$ of the collisional flux signifying the classical collisional transport is much smaller than the velocity-space fluxes:

$$\frac{\partial_{\mathbf{x}}\mathbf{\Gamma}^{\mathbf{x}}}{\partial_{v_\parallel}\Gamma^{v_\parallel}} \sim \frac{\partial_{\mathbf{x}}\mathbf{\Gamma}^{\mathbf{x}}}{\partial_{\mu}\Gamma^{\mu}} \sim (\nu\tau_{class})^{-1} \sim \left(\frac{\rho}{\lambda}\right)^2 \ll 1, \tag{23}$$

where $\tau_{class} \sim \lambda^2/D_{class}$ is the classical diffusion time through the characteristic scale λ with the diffusivity $D_{class} \sim \rho^2\nu$, ρ is the gyro radius and ν the collision frequency for the given species. Therefore, on the time scales $\nu^{-1} \ll t \ll \tau_{class}$, we have a local Maxwellian with slowly varying $n(\mathbf{x}, t)$ and $T(\mathbf{x}, t)$. In the next section assume that the electrostatic turbulence $\phi(\mathbf{x}, t)$ acts on these time scales and thus governs the evolution of the plasma density and temperature.

5 Plasma convection in the low-frequency turbulence: $\omega \ll \nu$

In this section we derive the equations describing the slow drift convection of magnetized plasma and find a new kind of Lagrangian invariants, which are hybrids of the frozen-in quantity n/B and the entropy s.

To derive the equations for n and T, we use the particle and the energy conservation laws in the form

$$\int C(f)\, d\mu dv_{\parallel} = -\nabla \cdot \boldsymbol{\Gamma}_{class} \simeq 0, \tag{24}$$

$$\int (\mu B + m{v_{\parallel}}^2/2) C(f)\, d\mu dv_{\parallel} = -\nabla \cdot \mathbf{q}_{class} \simeq 0, \tag{25}$$

where the classical collisional particle and energy fluxes $\boldsymbol{\Gamma}_{class}$ and $\mathbf{q}_{class}$, respectively, are negligible with respect to their turbulent counterparts.

The desired equations can be obtained by integrating Eq. (19), or

$$\partial_t f + c\left[\phi, \frac{f}{B}\right] + \frac{c}{e}\left[\ln B, \mu f\right] = C(f), \tag{26}$$

over μ and $v_{\parallel}$, weighted by 1 and $(\mu B + m{v_{\parallel}}^2/2)$. Using the fact the distribution function is very close to the local Maxwellian (22), so that

$$\left\langle \frac{m{v_{\parallel}}^2}{2} \right\rangle = \frac{T}{2}, \quad \langle \mu \rangle = \frac{T}{B},$$

$$\left\langle \frac{m{v_{\parallel}}^2}{2}\mu \right\rangle = \frac{T^2}{2B}, \quad \langle \mu^2 \rangle = \frac{2T^2}{B^2},$$

we obtain:

$$\partial_t n + c\left[\phi, \frac{n}{B}\right] - \frac{c}{e}\left[\frac{1}{B}, nT\right] = 0, \tag{27}$$

$$\frac{3}{2}\partial_t T + \frac{c}{B}\left[\phi, \ln\frac{T^{3/2}}{B}\right] - \frac{cT}{e}\left[\frac{1}{B}, \ln(nT^{7/2})\right] = 0. \tag{28}$$

Since the magnetic drift term in (27) is different for the electrons and the ions, the quasi-neutrality constraint $n_e = n_i = n$ reads that the total plasma pressure be constant on the lines of constant B:

$$[n(T_e + T_i), B] = 0. \tag{29}$$

For a self-consistent description of the turbulent potential, one needs to take into account the ion inertia (polarization drift), which will result in a small angle between the contours of plasma pressure and the magnetic field. In the following analysis, we assume that the electric field is small enough so that the inertial corrections are important only in the equation for ϕ, but can be neglected in the evolution of n and T, as was done in Eqs. (27) and (28).

The realizability of this approximation is not clear; however, the results obtained below are compatible with the assumption.

Upon introducing the fluid velocity,

$$\mathbf{u}(\mathbf{x},t) = \left\langle -\frac{c}{B}\nabla\left(\phi + \frac{\mu}{e}B\right)\times\hat{\mathbf{z}}\right\rangle - \frac{1}{n}\nabla\frac{cnT}{eB}\times\hat{\mathbf{z}} = -\frac{c}{B}\left(\nabla\phi + \frac{\nabla(nT)}{en}\right)\times\hat{\mathbf{z}}, \qquad (30)$$

consisting of the average guiding center drift velocity and the diamagnetic velocity, Eq. (27) and (28) can be rewritten in terms of the Lagrangian derivative $d_t = \partial_t + \mathbf{u}\cdot\nabla$. Introducing the quantities potentially suspicious for conservation, $r = \ln(n/B)$ and the entropy $s = \ln(T^{3/2}/n)$, we obtain:

$$d_t r = -\frac{c}{eB}[T,s], \qquad (31)$$

$$d_t s = -\frac{5}{2}\frac{c}{eB}[T,r]. \qquad (32)$$

Since we stay in a highly collisional regime, the above equations should be directly derivable from the Braginskii (1965) fluid equations. By neglecting inertial terms in Braginskii's momentum balance equation and viscous entropy production, in the limit of $\omega_c\tau \gg 1$, Eqs. (31) and (32) indeed follow. The only Braginskii transport term relevant in this limit is the skew thermal flux $\mathbf{q}_{\wedge T} = (5/2)(cnT/eB)\nabla T \times \hat{\mathbf{z}}$ resulting from the diamagnetic flow of particles. The formally much greater parallel flux vanishes in our geometry because of $\partial/\partial z = 0$; however, if a finite parallel inhomogeneity scale $L_{\parallel}$ were allowed, the parallel transport would be smaller than diamagnetic (skew) transport for $L_{\parallel} \gg \omega_c\tau L_{\perp}$, where $L_{\perp}$ is the inhomogeneity scale in the (x,y) plane and $\tau = 1/\nu$ is the collision time. The last applicability condition corresponds to the neglect of gyro viscosity in comparison with the retained diamagnetic fluxes; this results in $L_{\perp} \gg (\rho l)^{1/2}$, where $\rho = v_T/\omega_c$ is the gyroradius and $l = v_T\tau$ is the parallel mean free path.

The meaning of Eq. (31) is the generation of the vorticity (or the frozen-in quantity r) by the non-parallel gradients of density and temperature. Due to the magnetic drifts, we no longer have the usual fluid picture of "one point-one velocity," and in (32) there also appears the entropy generation term $\nabla T \times \nabla B$. Thus the entropy s is no longer a Lagrangian invariant as it was in the atmosphere. However, the underlying kinetic frozen-in law (Sec. 3) indicates that the invariants are likely to be modified rather than destroyed.

In the considered model, this is indeed the case. The symmetry of Eqs. (31) and (32) suggests the ansatz $L = \alpha r + s$:

$$d_t L = -\alpha\frac{c}{eB}\left[T, \frac{5}{2\alpha}r + s\right]. \qquad (33)$$

By choosing $\alpha = 5/(2\alpha)$, or $\alpha = \pm\sqrt{5/2}$, we infer the new Lagrangian invariants

$$L_\pm = \pm\sqrt{5/2}\, r + s = \ln\left(\frac{T^{3/2}}{n}\left(\frac{n}{B}\right)^{\pm\sqrt{5/2}}\right) \tag{34}$$

conserved along the new trajectories with the velocities

$$\mathbf{u}_\pm = \mathbf{u} \mp \sqrt{\frac{5}{2}}\frac{c}{eB}\boldsymbol{\nabla}T \times \hat{\mathbf{z}}. \tag{35}$$

These velocities are neither the guiding center nor the fluid velocities, but rather Riemann-like characteristics.

Thus the turbulent relaxation can be described in terms of the mixing of the two Riemann-type invariants $L_\pm$. The turbulent equipartition then corresponds to the spatially homogeneous distribution of $L_\pm$. In terms of the density and the temperature, this means

$$\frac{n(x,y,t\to\infty)}{B(x,y)} = \text{const}, \quad \frac{T^{3/2}(x,y,t\to\infty)}{B(x,y)} = \text{const}. \tag{36}$$

Result (36) was presented in (Yankov, 1995a) without derivation.

For the purpose of 2D gyro-advection, where the turbulence energy is drawn upon the same pool of the plasma thermal energy, a necessary (yet hardly sufficient) self-consistency condition implies that the mixing does not produce any new energy. Combining Eqs. (27) and (28) into

$$\frac{3}{2}\partial_t(nT) + \frac{c}{B}\left[\phi, \ln\frac{(nT)^{3/2}}{B^{5/2}}\right] + \frac{5cT}{2eB^2}\left[B, \ln(nT^2)\right] = 0, \tag{37}$$

the energy balance is written

$$\int\left(e\phi\left[\frac{1}{B}, \ln(nT)\right] + \frac{5}{3}T\left[\frac{1}{B}, \ln(nT^2)\right]\right)dxdy = 0. \tag{38}$$

Conditions like (29) and (38) will be automatically met for a self-consistent system involving both the plasma and the turbulence dynamics. Since the relaxed state (36) is consistent with the quasineutrality condition (29) and the energy conservation (38), one can hope that the principal features of the plasma relaxation to the TEP distribution (36) will survive for a reasonably general self-consistent turbulent evolution.

6 Plasma diffusion in the high-frequency turbulence: $\omega \gg \nu$

In this section we address the limit in which the effect of plasma turbulence can be described by the collisionless physics of the action-space diffusion, yet the collisions remain important on a longer time scale. We derive equations for nonlocal transport, which have nontrivial steady-state solutions, but we are not able to identify new invariants whose turbulent equipartition corresponds to the found solutions.

Now suppose the turbulence frequency is greater than the collision frequency. Then the instantaneous Maxwellian plasma response (27)–(28) to the turbulent mixing is no longer valid, and one needs another simplification. This one comes in the form of the action diffusion (Birmingham et al., 1967; Liu et al., 1971; Kaufman, 1972; Morozov et al., 1988; Isichenko et al., 1995), which is set on the time scale longer than the turbulence correlation time $1/\omega$ but shorter than the collision time $1/\nu$.

By introducing the magnetic Clebsch variables $\boldsymbol{\alpha} \equiv (\alpha, \beta)$ such that the Jacobian of the transformation is $[\alpha, \beta] = B$ and the new distribution function $F(\alpha, \beta, \mu, v_{\|}) = f(x, y, \mu, v_{\|})/B$, we infer that the effect of the turbulence on the smoothed distribution function F is pure diffusion in the $\boldsymbol{\alpha}$ plane (Isichenko et al., 1995; Isichenko and Petviashvili, 1995):

$$\partial_t F = \partial_{\boldsymbol{\alpha}} \cdot D^{\boldsymbol{\alpha\alpha}} \cdot \partial_{\boldsymbol{\alpha}} F + C(F). \tag{39}$$

Returning back to the Cartesian coordinates, Eq. (39) is written

$$\partial_t f = \boldsymbol{\nabla} \cdot \left(B D^{\mathbf{xx}} \cdot \boldsymbol{\nabla} \frac{f}{B} \right) + C(f). \tag{40}$$

Here again we assume that the the turbulent transport time scale $\lambda^2/D^{\mathbf{xx}}$ is much shorter than the classical transport time but much longer than the collision time. Then f is close to a local Maxwellian, and we can apply to Eq. (40) the same procedure that was used for Eq. (19).

The particle and energy moments of Eq. (40),

$$\partial_t n = \boldsymbol{\nabla} \cdot \int B D \cdot \boldsymbol{\nabla}(f/B)\, d\mu dv_{\|}, \tag{41}$$

$$\frac{3}{2}\partial_t(nT) = \int (\mu B + \frac{m v_{\|}^2}{2}) \boldsymbol{\nabla} \cdot \left(B D \cdot \boldsymbol{\nabla} \frac{f}{B} \right) d\mu dv_{\|}, \tag{42}$$

depend on assumptions about the diffusion tensor $D = D^{\mathbf{xx}}$. In general, the diffusion tensor depends on both $\mathbf{x}$ and μ, and the result of the integration is not explicit. The integrals can be solved by Taylor expanding the diffusivity:

$$D(\mathbf{x}, \mu) = \sum_{j=0}^{\infty} D_j(\mathbf{x}) \frac{\mu^j}{j!}. \tag{43}$$

Then the transport equations can be written as

$$\partial_t n = -\nabla \cdot \Gamma, \tag{44}$$

$$\frac{3}{2}\partial_t(nT) = -\nabla \cdot \mathbf{q} + Q, \tag{45}$$

where the particle flux Γ, the energy flux $\mathbf{q}$ and the source Q are as follows:

$$\Gamma = -\sum_{j=0}^{\infty} BD_j \cdot \nabla \left[\frac{n}{B}\left(\frac{T}{B}\right)^j\right], \tag{46}$$

$$\mathbf{q} = -B^2 \sum_{j=0}^{\infty} \left(j+\frac{3}{2}\right) D_j \cdot \nabla \left[\frac{n}{B}\left(\frac{T}{B}\right)^{j+1}\right] - \frac{n}{2}\sum_{j=0}^{\infty} D_j \left(\frac{T}{B}\right)^{j+1} \cdot \nabla B, \tag{47}$$

$$Q = -B\nabla B \cdot \sum_{j=0}^{\infty}(j+1)D_j \cdot \nabla \left[\frac{n}{B}\left(\frac{T}{B}\right)^{j+1}\right]. \tag{48}$$

The quantity Q in (44) can be interpreted as the energy exchange between the particles and the turbulence; however, as there is very little energy in the turbulence itself, we have a self-consistency condition similar to (38),

$$\int Q dx dy = 0 \tag{49}$$

This energy exchange appears as a nonlocal energy transfer via turbulence by emission and absorption of waves (Mattor and Diamond, 1994). Note that the definitions of the energy flux $\mathbf{q}$ and the energy exchange Q are not unique, and one may use any gauge conserving the local rate of the energy change, $Q - \nabla \cdot \mathbf{q}$. Our choice of the gauge is such that $Q = 0$ for a homogeneous magnetic field.

In the limit when the $\mathbf{E} \times \mathbf{B}$ drift is much faster than the ∇B drift, the diffusivity D is a weak function of μ, and only D_0 can be considered different from zero. Then we have

$$\Gamma = -BD_0 \cdot \nabla \frac{n}{B}, \tag{50}$$

$$\mathbf{q} = -\frac{nT}{2} D_0 \cdot \nabla \ln \frac{(nT)^3}{B^5}, \tag{51}$$

$$Q = -B\nabla B \cdot D_0 \cdot \nabla \frac{nT}{B^2}. \tag{52}$$

In this limit, the TEP density distribution remains the same as in the case of a low-frequency turbulent convection: the particle flux (50) is zero for the profile $n/B = \text{const}$.

The energy balance (44) implies that the temperature profile is determined non-locally by the condition

$$Q - \nabla \mathbf{q} = 0 \tag{53}$$

involving both the magnetic field $B(x, y)$ and the turbulence distribution via $D_0(x, y)$.

The meaning of Eq. (53) can be two-fold. On the one hand, one can consider Eq. (53) as a condition on the turbulence profile for the given plasma profiles. It is, however, unlikely that an arbitrary temperature profile can be steadily sustained by means of the turbulence adjustment to the profiles of B and T, because the turbulence distribution is determined by the profiles via additional equations for the electric field, which we did not derive nor analyzed here. On the other hand, Eq. (53) can be interpreted as a condition for the temperature profile for the given or the temperature-profile-dependent turbulent diffusivity tensor D_0.

The self-consistent case is nonlinear and difficult to analyze, so we make the somewhat arbitrary assumption that the diffusivity is homogeneous and isotropic. Then the relaxed temperature profile satisfies

$$3\nabla^2 T - \nabla T \cdot \nabla \ln B - 2T\nabla^2 \ln B = 0. \tag{54}$$

Due to the nonlocal energy transfer, the steady-state temperature profile $T(x, y)$ is determined by the magnetic field non-locally. A qualitative feeling of the effect of the magnetic geometry on the temperature can be obtained for the simplified case of an exponential magnetic field profile, $\ln B = kx$, $k = \text{const}$. Then Eq. (54) has two solutions: $T = \text{const}$ and $T \propto B^{1/3}$. The former seems to be an artifact of the exponential B profile, whereas the latter shows the adiabatic-type plasma heating with $T \propto n^{1/3}$; however, the rate of this heating is slower than in the turbulent equipartition (36) characteristic of the slow turbulent drift convection.

Although the entropy is no longer a good invariant for the high-frequency turbulent mixing, and it is difficult to indicate which invariants support the steady-state temperature profile, these invariants may exist for various collisional regimes defining non-trivial turbulent equipartitions like (54).

7 Conclusion

The analysis of the behavior of complex systems, such as turbulent fluids and plasmas, remains a challenge for theory, and one needs to look for either global or local conservation laws which are of a great help in such an analysis. The presence of invariants other than energy suggests that the system can relax to a state determined by the equipartition on the surface to which the system motion is constrained by these invariants (turbulent equipartition). More specifically, the turbulent equipartition is determined by the spatially uniform distribution of the applicable Lagrangian invariants. In terms of various thermodynamic quantities, such as density, temperature, and equilibrium flow, the TEP states need not be uniform; in the discussed examples of atmosphere, Jovian winds, and magnetized plasmas, these quantities were found inhomogeneous, with the profiles sensitive to the external fields such as gravity, inhomogeneous Coriolis force, or inhomogeneous magnetic field.

While being an effectively paradigm for several problems in geophysics and solar physics, the method of turbulent equipartition has only recently appeared in the context magnetic fusion, where the inhomogeneous density and temperature profiles in the absence of sufficient particle and energy sources has long been empirically described in terms of pinch effect and profile consistency. The pinch effect in tokamak is associated with the trapped particles strongly affected by both plasma turbulence and collisions in a regime with no apparent Lagrangian invariants, a situation similar to the one discussed in Sec. 6. This makes the problem sufficiently complicated, and these authors do not agree on some theory details of the of tokamak pinch effect (Yankov, 1994a; Isichenko et al., 1995; Nycander and Yankov, 1995), although the importance of the physics of turbulent equipartitions is quite clear.

In this work, we have specialized in the simpler example of 2D turbulent plasmas, which demonstrated the TEP density distribution $n \propto B(x,y)$, due to the conservation of the frozen-in invariant(s) in various regimes of collisionality. The question of the turbulent equipartition of plasma temperature is related to the second Lagrangian invariant identical to or derived from the specific entropy s. The existence of such invariants was shown for the case collisionless plasma and in the regime of very frequent collisions (higher than the turbulence frequency). This resulted in the TEP temperature profiles $T_\perp \propto B(x,y)$ and $T \propto B^{2/3}(x,y)$, respectively. In the intermediate regime, when the collision frequency is smaller than the turbulence frequency, a kinetic formalism was developed which predicted less universal steady-state temperature profiles depending on the magnetic field non-locally. It is not clear whether or not this result represents a true turbulent equipartition supported by underlying local invariants. These invariants could exist, and the formalism of the Poincare invariant and the associated kinetic frozen-in law may prove useful in the search for applicable constraints for this and more complicated systems.

Acknowledgments

The authors wish to thank R. N. Sudan and S. C. Cowley, the organizers of the NSF Program on Intermittent and Turbulent Processes in Plasma, and the staff of the Institute for Theoretical Physics at UCSB for putting together lots of efforts and making this program possible. Helpful discussions with J. D. Callen, J. Nycander, A. I. Smolyakov, and P. N. Yushmanov are also gratefully acknowledged. This research was supported in part by the US Department of Energy under Grant No. DE-FG05-80ET53266, by the Russian Fundamental Science Foundation Grant No. 94-02-05921a, by the Russian State Program "Fundamental Problems of Nonlinear Dynamics", and by the National Science Foundation under Grant No. PHY94-07194.

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