LARGE DEFORMATION ANALYSIS OF AXISYMMETRIC INHOMOGENEITIES INCLUDING COUPLED ELASTIC AND PLASTIC ANISOTROPY*

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SUMMARY

A mathematical framework is developed for the study of materials containing axisymmetric inclusions or flaws such as ellipsoidal voids, "penny-shaped" cracks, or fibers of circular cross-section. The general case of nonuniform statistical distributions of such heterogeneities is attacked by first considering a spatially uniform distribution of flaws that are all oriented in the same direction. Assuming an isotropic substrate, the macroscopic material properties of this simpler microstructure naturally should be transversely isotropic. An orthogonal basis for the linear subspace consisting of all double-symmetric transversely-isotropic fourth-order tensors associated with a given material vector is applied to deduce the explicit functional dependence of the material properties of these "aligned" materials on the shared symmetry axis. The aligned and uniform microstructure seems geometrically simple enough that the macroscopic transversely isotropic properties could be derived in closed form. Since the resulting properties are transversely isotropic, the analyst must therefore be able to identify the appropriate coefficients of the transverse basis. Once these functions are identified, a principle of superposition of strain rates may be applied to define an expectation integral for the composite properties of a material containing arbitrary anisotropic distributions of axisymmetric inhomogeneities. A proposal for coupling plastic anisotropy to the elastic anisotropy is presented in which the composite yield surface is interpreted as a distortion of the isotropic substrate yield surface; the distortion directions are coupled to the elastic anisotropy directions. Finally, some commonly assumed properties (such as major symmetry) of the Cauchy tangent stiffness tensor are shown to be inappropriate for large distortions of anisotropic materials.

INTRODUCTION

This paper discusses several tools and viewpoints that have proved useful in the analysis of microcracked bodies and which seem additionally applicable to the analysis of materials containing axisymmetric inhomogeneities such as reinforcing fibers of circular cross-section, ellipsoidal voids, or even cone-shaped inclusions. The analysis to follow assumes that a closed-form solution can be obtained for a contrived material in which all of the inhomogeneities are distributed uniformly *with a common axis of symmetry*. Naturally, the properties of such an aligned material should be transversely isotropic about the shared axis of symmetry. The fourth-order aligned compliance tensor is shown to be expressible as a linear combination of a transversely isotropic integrity basis. Importantly, these base tensors are presented as *analytical*

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DISCLAIMER

functions of the orientation vector, which permits their use in expectation integrals for the compliance of realistic materials containing inhomogeneities distributed anisotropically in many orientations.

Once the arbitrarily anisotropic elastic compliance tensor has been found, it's natural to seek a coupled yield function. If, for example, reinforcing fibers cause a material to become stiffer in some particular direction, then one would expect the yield stress in that direction to change as well. We demonstrate that the yield surface of a composite material may be viewed as a distortion of the yield surface of the underlying substrate. We offer a conjecture for the distortion function that exhibits the qualitatively desirable trait of aligning the plastic anisotropy with the elastic anisotropy.

Finally, this paper closes with an *exact* closed form solution for the stress in a material containing aligned fibers in a weak matrix. This counterexample demonstrates that many commonly adopted assumptions about the nature of the Cauchy elastic stiffness tensor may be inappropriate for large distortions of anisotropic materials. Namely, for this material, the Cauchy stress must be uniaxial in the fiber direction. Consequently, it is shown that the unrotated Cauchy stress (frequently employed in frame-indifferent constitutive laws) must be uniaxial about the unrotated fiber direction. Since the unrotated fiber direction does not generally coincide with the initial fiber direction, the axis of symmetry for the unrotated Cauchy stiffness tensor changes in time. Consequently, the tangent Cauchy stiffness contains *kinematic* terms that are not material properties. Furthermore, the tangent Cauchy stiffness tensor does not even possess the major symmetry commonly assumed in the literature.

A CANONICAL "ALIGNED" COMPOSITE

In the next section, we deduce an expression for the compliance of an *arbitrarily* anisotropic cracked or reinforced material. As a starting point, consider a material having a spatially uniform array of axisymmetric inclusions all of a given orientation, $\mathbf{n}$. This imposed geometric symmetry demands that the macroscopic material properties be unchanged upon any rigid rotation about $\mathbf{n}$. In particular, the elastic compliance must be transversely isotropic with respect to $\mathbf{n}$, and (Brannon, 1996a), it must therefore be expressible as a linear combination of a transverse integrity basis $\{\mathbf{B}_1, \ldots, \mathbf{B}_5\}$ defined as follows:

$$(\mathbf{B}_1)_{ijkl} = n_i n_j n_k n_l \tag{1a}$$

$$(\mathbf{B}_2)_{ijkl} = \delta_{ij}\delta_{kl} - n_i n_j \delta_{kl} - \delta_{ij} n_k n_l + n_i n_j n_k n_l \tag{1b}$$

$$(\mathbf{B}_3)_{ijkl} = n_i n_j \delta_{kl} + \delta_{ij} n_k n_l - 2 n_i n_j n_k n_l \tag{1c}$$

$$(\mathbf{B}_4)_{ijkl} = \frac{1}{2}(n_i n_k \delta_{jl} + n_j n_l \delta_{ik} + n_i n_l \delta_{jk} + n_j n_k \delta_{il}) - 2 n_i n_j n_k n_l \tag{1d}$$

$$(\mathbf{B}_5)_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \frac{1}{2}(n_i n_k \delta_{jl} + n_j n_l \delta_{ik} + n_i n_l \delta_{jk} + n_j n_k \delta_{il}) + n_i n_j n_k n_l \ . \tag{1e}$$

In other words, *no matter what physical arguments are used to deduce the composite elastic properties*, there must exist five coefficient functions $\alpha_k(c, \mu, \tilde{N})$ such that the aligned compliance $\mathbf{M}$ can be written in the following form:

$$\mathbf{M}(c, \beta^o, \tilde{N}, \mathbf{n}) = \mathbf{M}^o(\beta^o) + \sum_{K=1}^{5} \alpha_K(c, \beta^o, \tilde{N})\, \mathbf{B}_K(\mathbf{n}) \quad . \tag{2}$$

Here, c symbolically denotes the flaw or inclusion geometry (such as a reinforcing fiber diameter or a crack radius), β^o collectively denotes the elastic properties of the substrate, and $\tilde{N}$ is the number of inhomogeneities per unit mass. The fourth-order tensor $\mathbf{M}^o$ is the isotropic compliance of the underlying substrate matrix material. The expression (2) shows the explicit analytical dependence of the aligned compliance on the alignment orientation $\mathbf{n}$. This expression purposely avoids adopting any *particular* form for the five α_K coefficient functions. Different researchers may derive different — even conflicting — expressions for the compliance of a body having same-orientation inhomogeneities. We merely insist that these coefficients *exist*. All admissible solutions must be expressible in the form (2); they will differ only in the specific forms chosen or derived for the five coefficient functions $\alpha_K(c, \beta^o, \tilde{N})$.

In terms of an orthonormal spatial basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ which is defined such that $\mathbf{e}_3 = \mathbf{n}$ (not necessarily aligned with the laboratory basis), the Euclidean Voigt components of the last term in the compliance expression (2) are

$$\begin{bmatrix} \alpha_2+\alpha_5 & \alpha_2 & \alpha_3 & 0 & 0 & 0 \\ \alpha_2 & \alpha_2+\alpha_5 & \alpha_3 & 0 & 0 & 0 \\ \alpha_3 & \alpha_3 & \alpha_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha_4 \end{bmatrix} \tag{3}$$

The rows and columns correspond to the second-order symmetric ordering {11, 22, 33, 12, 23, 31}. Note that the Euclidean Voigt components of any base tensor $\mathbf{B}_K$ in (1) can be determined from (3) by setting $\alpha_K = 1$ and all other α-coefficients to zero.

The component forms (1) are useful in practice because *they are algebraic functions of the privileged direction* $\mathbf{n}$. They are, however, somewhat difficult to interpret physically. For conceptual discussions, observing how the basis transforms tensors (such as the stress) is far more revealing. Specifically, when the $\mathbf{B}_K$ tensors operate on any symmetric second-order tensor $\underset{\sim}{\sigma}$, and the results are expressed in terms of the material basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 = \mathbf{n}\}$, we see that

$$\mathbf{B}_1 : \underset{\sim}{\sigma} \rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix} \qquad \mathbf{B}_2 : \underset{\sim}{\sigma} \rightarrow \begin{bmatrix} \sigma_{11}+\sigma_{22} & 0 & 0 \\ 0 & \sigma_{11}+\sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{4a,b}$$

$$\mathbf{B}_3:\underset{\sim}{\sigma} \rightarrow \begin{bmatrix} \sigma_{33} & 0 & 0 \\ 0 & \sigma_{33} & 0 \\ 0 & 0 & \sigma_{11}+\sigma_{22} \end{bmatrix} \tag{4c}$$

$$\mathbf{B}_4:\underset{\sim}{\sigma} \rightarrow \begin{bmatrix} 0 & 0 & \sigma_{13} \\ 0 & 0 & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & 0 \end{bmatrix} \qquad \mathbf{B}_5:\underset{\sim}{\sigma} \rightarrow \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{4d,e}$$

Note that $\mathbf{B}_1$ extracts the part of $\underset{\sim}{\sigma}$ that acts in the direction of the axis of symmetry $\mathbf{n}$ while $\mathbf{B}_4$ projects $\underset{\sim}{\sigma}$ to its part that tends to shear the axis of symmetry. The base tensor $\mathbf{B}_5$ extracts the transverse part of $\underset{\sim}{\sigma}$ (i.e., the part that lies in the plane perpendicular to $\mathbf{n}$), while $\mathbf{B}_2$ yields the transverse trace $\sigma_{11}+\sigma_{22}$ times the transverse identity. Unlike the other base tensors which essentially *project* $\underset{\sim}{\sigma}$ in particular directions, the base tensor $\mathbf{B}_3$ maps the *axial* component σ_{33} to the *transverse* 11 and 22 positions; furthermore, $\mathbf{B}_3$ maps the *transverse* trace $\sigma_{11}+\sigma_{22}$ to the *axial* 33 position. Thus $\mathbf{B}_3$ represents the *only* means of allowing *axial* input to affect *lateral* response, or vice versa. It would play a pivotal role, for example, in predicting "crack buckling," in which cracks pop open due to lateral compression in their own plane.

The transverse integrity basis (1) is distinguished from so-called fabric tensors because it has nothing to do with material constants such as Poisson's ratio. It is a purely mathematical construct resulting from transverse symmetry only. If desired, the basis can be orthogonalized (in fourth-order tensor space) by replacing $\mathbf{B}_5$ by $\mathbf{B}_5-\frac{1}{2}\mathbf{B}_2$ and $\mathbf{B}_2$ by $\frac{1}{2}\mathbf{B}_2$, which corresponds to decomposing the *transverse* projection of $\underset{\sim}{\sigma}$ into its *planar* isotropic and deviatoric parts.

COMPLIANCE FOR RANDOM-SIZE RANDOM-ORIENTATION INHOMOGENEITIES

Real materials do not contain same-size same-orientation inhomogeneities. We now introduce some tools that will allow us to extend the general solution (2) for the compliance of a body containing same-orientation inhomogeneities to a realistic body having inclusions or flaws of many sizes and many orientations. The orientation of an axisymmetric flaw or inclusion is described by the unit normal $\mathbf{n}$ parallel to the axis of symmetry or, alternatively, by corresponding points on the unit sphere. For example, the unit normal $\mathbf{n}=\{1,0,0\}$ corresponds to the point (1,0,0) on the sphere. Many axisymmetric inhomogeneities are "neutrally oriented," meaning that diametrically opposite points on the unit sphere are equivalent. For example, a crack with normal $\mathbf{n}$ behaves the same as a crack with normal $-\mathbf{n}$. Put differently, any material function $f(\mathbf{n})$ for neutrally oriented inhomogeneities should have the property that $f(-\mathbf{n})=f(\mathbf{n})$. Oriented axisymmetric inhomogeneities (such as cones) are not bound by this additional symmetry restriction. Any function of orientation $\mathbf{n}$ may be regarded as a function on the unit sphere. Any contiguous *set* of orientations may be described by a contiguous patch of area on the unit sphere, hereafter referred to as a "**solid angle**" (which includes the diametrically opposite area for neutrally-oriented inhomogeneities).

Consider a body of *unit mass* containing a *total* of $\tilde{N}$ inhomogeneities. In a realistic model, a randomly selected flaw could have any orientation $\mathbf{n}$ and any geometrical attribute c which we will take in this section to be a characteristic size such as a crack radius or fiber diameter. As with fiber-reinforced composites, oil shale, or partially spalled metal, some sizes and some orientations may be more common than others. This

variation is described through the use of a **joint distribution function**, $p(c,\mathbf{n})$, defined by its interpretation when integrated:

> The fraction of inhomogeneities that have a size c between a and b and an orientation $\mathbf{n}$ within some solid angle $\Delta\Omega$ on the unit sphere is
>
> $$\int_{\Delta\Omega}\int_{a}^{b} p(c,\mathbf{n})\,dc\,d\Omega \quad . \tag{5}$$

To account for clustering, the function $p(c,\mathbf{n})$ might conditionally depend on the spatial location. Since flaws can nucleate, grow, and coalesce, $p(c,\mathbf{n})$ might also be a function of time. Of course, the integral over all possible inhomogeneity sizes and orientations (i.e. over the entire unit sphere and from c=0 to ∞) is unity. Flaws having more than one geometrical attribute (such as conical flaws defined by height and radius) would be described by a similar integral over more independent variables.

The size and orientation distribution function $p(c,\mathbf{n})$ will now be combined with the aligned compliance (2) to derive the compliance for a body containing flaws of many sizes and orientations. Consider a body containing an array of random-size and random-orientation flaws. Applying a principle of superposition of strain rates for a dilute array of cracks, the increase in compliance is taken to be the sum of the increases for each possible crack size and orientation times the fraction of cracks of that specified size and orientation. For a continuous distribution, the compliance $\mathbf{M}$ is computed by the expectation integral

$$\mathbf{M} = \int_{\Omega}\int_{0}^{\infty} \mathbf{M}(c,\beta^o,\tilde{N},\mathbf{n})\,p(c,\mathbf{n})\,dc\,d\Omega, \tag{6}$$

where $\mathbf{M}(c,\beta^o,\tilde{N},\mathbf{n})$ is given by (2), so that (6) may be written

$$\mathbf{M} = \mathbf{M}^o(\beta^o) + \sum_{K=1}^{5}\int_{\Omega}\mathbf{B}_K(\mathbf{n})\,\bar{\alpha}_K(\mathbf{n},\tilde{N},\beta^o)d\Omega \quad , \tag{7}$$

$$\textit{where} \quad \bar{\alpha}_K(\mathbf{n},\tilde{N},\beta^o) \equiv \int_{0}^{\infty}\alpha_K(c,\beta^o,\tilde{N})\,p(c,\mathbf{n})dc \quad . \tag{8}$$

For nondilute arrays of flaws, it seems reasonable to expect the compliance to nevertheless be of the above form, with the dependence on $\tilde{N}$ being nonlinear to account for crack interaction (which is sometimes surprisingly inconsequential due to competing effects of amplification and shielding). The expression (7) shows that *if the solution for aligned flaws is known*, then so is the solution for randomly distributed flaws, provided of course that flaw interactions permit a superposition of strain rates. Recent work (Brannon, 1996a) has shown that various choices for (or approximations of) the distribution function can lead to heretofore diverse classes of material models. For example, if the flaws have no preferred orientation, the distribution function $p(c,\mathbf{n})$ becomes independent of $\mathbf{n}$. Therefore, $\bar{\alpha}_K$ in (8) becomes independent of $\mathbf{n}$. The integral over $\mathbf{n}$ in (7) reduces to integrating each $\mathbf{B}_K(\mathbf{n})$, which can be done in closed form given the

analytical expressions in (1). The final result for the compliance is isotropic with composite shear and bulk moduli, μ and K, being related to the associated substrate matrix moduli, μ_o and K_o by

$$\frac{1}{\mu} = \frac{1}{\mu_o} + \frac{2}{15}(2\bar{\alpha}_1 + 2\bar{\alpha}_2 - 4\bar{\alpha}_3 + 6\bar{\alpha}_4 + 7\bar{\alpha}_5) \tag{9a}$$

$$\frac{1}{K} = \frac{1}{K_o} + \bar{\alpha}_1 + 4\bar{\alpha}_2 + 4\bar{\alpha}_3 + 2\bar{\alpha}_5 \tag{9b}$$

The expressions (9) are very general in that they hold for a random distribution of any type of axisymmetric inhomogeneity. The analyst need only determine the appropriate coefficient functions for an *aligned* microstructure in order to immediately know the associated elastic properties for uniformly oriented inhomogeneities.

As in the work of Seaman (1985), one could use (6) to analyze the effect of a predominant inclusion or flaw orientation embedded in a substrate containing uniformly oriented flaws. The present analysis could be easily generalized to allow the predominant flaw to be of a different type than the substrate flaw (as for spall of a dilutely fibrous material). However, *general* anisotropy can be modeled only by allowing arbitrary dependence of the distribution on $\mathbf{n}$, in which case the integral over orientation in (7) can be evaluated numerically by discretizing the unit sphere. Dienes (1987) simplifies this procedure by taking only two of the five α_K functions in (2) to be nonzero and by assuming an analytical form for the size dependence of the distribution function so that the integral in (8) can be computed *a priori* in closed form.

COUPLING ELASTIC AND PLASTIC ANISOTROPY

In the absence of better approaches, sophisticated treatments of elastic anisotropy are often inappropriately paired with far more simplistic isotropic yield models or, at best, with kinematic hardening models that only shift a nominal isotropic yield surface in stress space. Since a yield surface can be defined as the boundary of stresses attainable from the current stress via a recoverable elastic path, it seems natural for the nominal yield surface to inherit the anisotropy of the elastic properties. If, for example, the elastic compliance is orthotropic, then the yield surface should also be orthotropic with respect to the same privileged directions. Here we extend the conjecture of Schreyer and Zuo (1995) that the yield behavior of a material may be coupled to the spectral directions of the elasticity tensor. In the discussion to follow, it is important to recognize that two *different* yield functions $f_1(\underline{\sigma})$ and $f_2(\underline{\sigma})$ are equivalent if the inequalities $f_1(\underline{\sigma}) < 0$ and $f_2(\underline{\sigma}) < 0$ describe the same set of stresses. For example, $f_1(\underline{\sigma}) = \underline{\sigma}:\underline{\sigma} - k^2$ is equivalent to $f_2(\underline{\sigma}) = (\underline{\sigma}:\underline{\sigma})/k^2 - 1$.

Consider a material consisting of an isotropic substrate having a nominal yield function $f_o(\underline{\sigma})$. Upon the addition of a *small* amount of adulterants such as inclusions or voids, the new yield surface defined by $f(\underline{\sigma}) = 0$ should be only a slight distortion of the substrate yield surface defined by $f_o(\underline{\sigma}) = 0$. Assuming the set of elastically attainable stresses (i.e., the interior of the yield surface) always remains simply connected, the yield surface for *any* amount of adulterant can be viewed as a distortion of the yield surface of the underlying substrate. Mathematically, there must exist a simple distortion function g such that $f(\underline{\sigma})$ is *equivalent* to $f_o(g(\underline{\sigma}))$. The distortion function g captures the change in the yield surface due to a change in the *elastically* attainable stresses, so it seems reasonable to assume that any changes in the elastic properties should produce consonant changes in the plastic properties. At least qualitatively, this coupling

could be described by a distortion function of the form $g(\underset{\sim}{\sigma}) = \mathbf{L}:\underset{\sim}{\sigma}$ where $\mathbf{L}$ is a fourth-order distortion operator *that possesses the same eigenprojectors as the elastic compliance* $\mathbf{M}$. The eigenproblem for the compliance $\mathbf{M}$ is defined

$$M_{ijkl}A_{kl} = \lambda A_{ij}. \tag{10}$$

Suppose $\mathbf{M}$ has m distinct eigenvalues $\{\lambda_1, \ldots, \lambda_m\}$. If an eigenvalue λ_α has multiplicity n_α, major symmetry of $\mathbf{M}$ guarantees the existence of n_α associated *orthonormal* eigentensors $\{\mathbf{A}^1, \ldots, \mathbf{A}^{n_\alpha}\}$. The eigenprojector $\mathbf{P}_\alpha$ associated with λ_α is defined

$$(P_\alpha)_{ijkl} = A^1_{ij}A^1_{kl} + \ldots + A^{n_\alpha}_{ij}A^{n_\alpha}_{kl} \tag{11}$$

Any $\mathbf{M}$ may be written in terms of its eigenspace as

$$\mathbf{M} = \lambda_1\mathbf{P}_1 + \ldots + \lambda_m\mathbf{P}_m. \tag{12}$$

Hence the proposed yield distortion tensor $\mathbf{L}$ would be expressible as a similar combination of the same projectors. That is,

$$f(\underset{\sim}{\sigma}) = f_o(\mathbf{L}:\underset{\sim}{\sigma}) \tag{13}$$

$$\textit{where} \qquad \mathbf{L} = \mathcal{L}_1\mathbf{P}_1 + \ldots + \mathcal{L}_m\mathbf{P}_m \tag{14}$$

The $\mathcal{L}_j$ coefficients would be measured in the laboratory. The form (14) here proposed for the yield surface distortion is based on phenomenological symmetry arguments. Karafillis and Boyce (1993) use an expression of the form (13) in the context of metal plasticity, but the quantity $\mathbf{L}:\underset{\sim}{\sigma}$ (there called an "equivalent isotropic stress") is *not* specifically coupled to the elastic properties. A yield function of the form (13) appears in a proposal by Tsai and Wu (1971) for transversely isotropic materials, where (in effect) they use a Mises nominal function f_o. They require $\mathbf{L}$ to be transversely isotropic, but they do *not* impose the restriction (14). Hence their yield function requires *five* experimentally measured yield stresses, one of which they warn *cannot be determined uniquely*. Since transversely isotropic tensors can have no more than *four* distinct eigenvalues, Tsai and Wu's indeterminacy might be resolved by imposing the restriction (14).

Extensive experiments to determine complete yield surfaces are often costly, so it seems worthwhile to pursue phenomenological theories that could *predict* the $\mathcal{L}_j$ coefficients in (14). For small volume fractions of inhomogeneities (not necessarily small number fractions), the $\mathcal{L}_j$ coefficients might be given by a function G of the change in the elastic properties in the corresponding direction:

$$\mathcal{L}_j = G(\lambda_j/\lambda^o_j), \qquad \textit{where} \qquad \lambda^o_\alpha \equiv M^o_{ijkl}(P_\alpha)_{ijkl} \tag{15}$$

The form of the unprescribed function G would be dictated by microstructural arguments. The expression (15) couples elastic and plastic anisotropy in an intuitively appealing way that is surely superior to an isotropic yield model. It can be justified for certain simplistic yield criteria. Consider, for example, an isotropic material that is dilutely reinforced with a substance having a comparatively high yield strength. This material could be said to have yielded once the matrix material has yielded. If the reinforcing fibers are

distributed isotropically, then both the composite and the matrix compliances are isotropic, and therefore a spectral analysis reveals that

$$\mathbf{M} = \lambda_1 \mathbf{P}_1 + \lambda_2 \mathbf{P}_2 \qquad \textit{where} \quad \lambda_1 = \frac{1}{2\mu}, \quad \lambda_2 = \frac{1}{3K},$$

$$(\mathbf{P}_1)_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \frac{1}{3}(\delta_{ij}\delta_{kl}), \quad \textit{and} \quad (\mathbf{P}_2)_{ijkl} = \frac{1}{3}(\delta_{ij}\delta_{kl}) \tag{16}$$

Here, μ and K are the composite shear and bulk moduli, respectively. For an isotropic matrix material described by a Von Mises criterion, yield occurs when the magnitude of the stress deviator reaches a critical value κ_o, or equivalently, when the magnitude of the strain deviator reaches a critical value $\kappa_o/2\mu_o$, where μ_o is the shear modulus of the matrix material. If the same critical strain were to be taken as the yield strain for the reinforced material, the magnitude of the stress deviator at yield would be $\kappa = \kappa_o(\mu/\mu_o) = \kappa_o(\lambda_1^o/\lambda_1)$ and therefore, the function G in (15) would be given by $G(x) = x$. Alternatively, if one were to advocate an energy criterion for yield, similar arguments would lead to the conclusion that $G(x) = x^{1/2}$. More sophisticated arguments that account for specific microstructures would certainly lead to different forms for the function G. Again, the special form (15) seems reasonable only if the inhomogeneities comprise a small fraction of the total composite volume (as for cracks, fine whiskers, or thin laminates). The general conjecture (14) seems to have much broader applicability since it is based solely on the elastic material directions.

LARGE DEFORMATION ISSUES

In this section, we study a simple microstructure for which an *exact* solution for the stress can be derived as a function of the deformation gradient $\mathbf{F}$, no matter how large. This simple analysis disproves many common assumptions about the nature of constitutive functions. The counterexample to follow proves that the tangent Cauchy stiffness tensor might contain many unexpected terms and is not even major-symmetric.

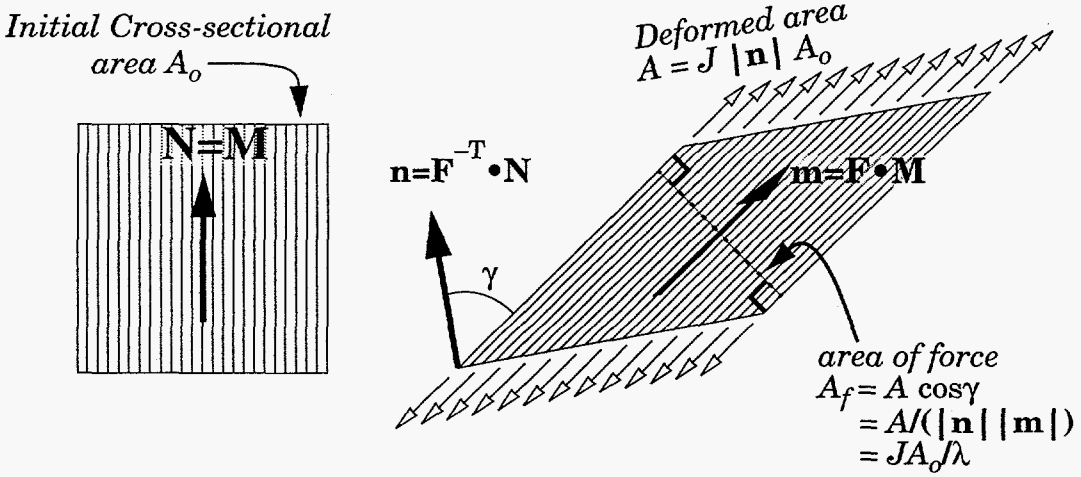

Figure 1: An idealized array of fibers in a negligibly stiff matrix (e.g., air). The fibers distort to a new orientation parallel to $\mathbf{m}$. The cross-sectional area A_o originally normal to the fiber direction distorts to a new orientation parallel to $\mathbf{n}$ (not a unit vector). The right side of the figure shows the distorted shape as seen in the plane spanned by $\mathbf{n}$ and $\mathbf{m}$.

Consider a material consisting of stiff fibers uniformly distributed in a very weak matrix (air). *Single* fibers are presumed well-characterized. That is, if a single fiber is stretched so that its current length divided by its initial length is λ, then the force in that fiber is given by some *known* function $\mathcal{f}(\lambda)$ satisfying $\mathcal{f}(1)=0$. Suppose all the fibers have an initial orientation parallel to a unit vector $\mathbf{M}$, and they are distributed uniformly with ν fibers per unit *initial* cross-sectional area. Then the representative volume element sketched in Fig. 1 contains a total of νA_o fibers. A homogeneous deformation $\mathbf{F}$ will distort those fibers to a new orientation parallel to

$$\mathbf{m} \equiv \mathbf{F} \bullet \mathbf{M} \ . \tag{17}$$

The fiber stretch λ is simply the magnitude of $\mathbf{m}$:

$$\lambda \equiv \sqrt{\mathbf{M} \bullet \mathbf{F}^T \bullet \mathbf{F} \bullet \mathbf{M}} = \sqrt{\mathbf{m} \bullet \mathbf{m}} \ . \tag{18}$$

For fibers with negligible bending stiffness, the macroscopic Cauchy stress $\underset{\sim}{\sigma}$ must be uniaxial in the deformed fiber direction. The stress equals the force per fiber $\mathcal{f}(\lambda)$ times the number of fibers νA_o, divided by the deformed cross-sectional area perpendicular to the fiber direction, JA_o/λ, where J is the determinant of the deformation gradient $\mathbf{F}$. Utilizing Nanson's relation between initial and deformed areas (see Malvern, 1969), it can be shown (Brannon, 1996b) that the *exact* solution for the Cauchy stress is

$$\sigma_{ij} = \frac{\mathcal{G}(\lambda)}{J} m_i m_j \ , \quad \textit{where} \quad \mathcal{G}(\lambda) = \nu \frac{\mathcal{f}(\lambda)}{\lambda} \ . \tag{19}$$

and, therefore, the exact solution for the so-called unrotated Cauchy stress ($\bar{\sigma}_{ij} \equiv R^T_{ip} R^T_{jq} \sigma_{pq}$) is

$$\boxed{\bar{\sigma}_{ij} = \frac{\mathcal{G}(\lambda)}{J} \bar{m}_i \bar{m}_j} \quad \textit{where} \quad \bar{m}_i = R^T_{ip} m_p = \bar{V}_{ij} M_j \tag{20}$$

Here, $\mathbf{R}$ and $\bar{\mathbf{V}}$ are the rotation and stretch from the polar decomposition, $\mathbf{F} = \mathbf{R} \bullet \bar{\mathbf{V}}$. The exact solution for the second Piola-Kirchhoff stress ($\bar{s}_{ij} \equiv J F^{-1}_{ip} \sigma_{pq} F^{-T}_{qj}$) is

$$\boxed{\bar{s}_{ij} = \mathcal{G}(\lambda) M_i M_j} \ . \tag{21}$$

These exact solutions demonstrate that the stress is a nonlinear function of strain even if the fiber force function $\mathcal{f}(\lambda)$ is linear (affine). In other words, nonlinearity of large-distortion constitutive laws arises as much from kinematics as from inherent material nonlinearities. An implicit goal of special stress and strain measures is to (approximately) capture these kinematic contributions. Taking rates of these exact solutions, it can be shown that there indeed exist *exact* fourth-order tensors $\bar{E}_{ijkl}$ and $\bar{\xi}_{ijkl}$ such that

$$\dot{\bar{s}}_{ij} = \bar{E}_{ijkl} \dot{\bar{\varepsilon}}_{kl} \quad \textit{and} \quad \dot{\bar{\sigma}}_{ij} = \bar{\xi}_{ijkl} \bar{D}_{kl} \ , \tag{22}$$

where $\dot{\bar{\varepsilon}}$ is the rate of the Lagrange strain and $\bar{\mathbf{D}}$ is the unrotated symmetric part of the velocity gradient. Taking the rate of (21) shows that the *exact* tangent Piola-Kirchhoff stiffness tensor $\bar{\mathbf{E}}$ is

$$\bar{E}_{ijkl} = \frac{\mathcal{G}'(\lambda)}{\lambda} M_i M_j M_k M_l \tag{23}$$

This tensor is transversely isotropic with respect to the initial fiber orientation, and it possesses only one nonzero component in the fiber direction (an intuitive result). In contrast, the *exact* Cauchy stiffness tensor $\bar{\xi}_{ijkl}$ (given by a very complicated expression) turns out to be transversely isotropic with respect to the unrotated *but time-dependent* fiber orientation, $\overline{\mathbf{m}}$. The Cauchy stiffness does contain a nonzero term in the fiber direction; but (unlike the Piola-Kirchhoff stiffness) it additionally contains unexpected nonzero components. Distortion causes the axis of symmetry $\overline{\mathbf{m}}$ to rotate in the unrotated frame. Hence, because the Cauchy stress is always uniaxial, its rate is *not* uniaxial, which leads to additional nonaxial kinematic components in the Cauchy tangent stiffness tensor. Finally, the exact Cauchy stiffness tensor *isn't even major-symmetric*, as is commonly assumed in the literature. The lack of major-symmetry arises from the rate of the Jacobian J, which contributes nonsymmetric terms of the form $\sigma_{ij}\delta_{kl}$.

The Piola-Kirchhoff tensors are much better behaved from the standpoint of material directions, so they might be a better choice for problems involving severe distortion of highly anisotropic materials. In particular, the vector $\mathbf{n}$ in the second section of this paper may best be interpreted as the reference vector $\mathbf{N}$ with Eq. (6) being the Piola-Kirchhoff compliance. Even so, nonlinearities in the stiffness magnitude (whether kinematical or mechanical) *must* be incorporated into any sensible theory. If the tensor (23) were cavalierly assumed constant and then *inappropriately* applied to problems with large stretch, the uniaxial force-displacement curve could contain a zero slope, causing numerical instability. This behavior may account for the popularity of the Cauchy stress in numerical calculations. If the Cauchy stiffness $\bar{\xi}_{ijkl}$ were assumed constant and then naively applied to large *uniaxial* stretches, the resulting force-displacement curve would be logarithmic, in *fortuitous* agreement with behavior typically observed in the laboratory. We have shown that the same cavalier Cauchy stress law would not fare so well in anisotropic distortion experiments.

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