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Fuzzy Fractals, Chaos, and Noise

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Abstract. To distinguish between chaotic and noisy processes, we analyze one- and two-dimensional chaotic mappings, supplemented by the additive noise terms. The predictive power of a fuzzy rule-based system allows one to distinguish ergodic and chaotic time series: in an ergodic series the likelihood of finding large numbers is small compared to the likelihood of finding them in a chaotic series. In the case of two dimensions, we consider the fractal fuzzy sets whose α-cuts are fractals, arising in the context of a quadratic mapping in the extended complex plane. In an example provided by the Julia set, the concept of Hausdorff dimension enables us to decide in favor of chaotic or noisy evolution.

Keywords. Fuzzy sets, chaotic dynamics, fractals

1. Introduction

When studying chaos in fuzzy dynamical systems, one faces two fascinating challenges. The first one, also encountered in crisp dynamical systems, is how to distinguish between chaos and noise. The second one concerns the various ways a fuzzy chaos can be brought about. A systematic study of chaos in fuzzy dynamical systems has been initiated by Kloeden [1], Diamond [2], and Diamond and co-workers [3]. The mathematical definitions of chaos rely on positive topological entropy, sensitive dependence on initial conditions, and positive Liapunov exponents. When transcribed to dynamical fuzzy systems, the definition of chaos invokes the notion of topological entropy; roughly speaking, a system is chaotic if the trajectories are mixing.

Although fuzzy dynamics often applies the extension principle to define mappings of fuzzy sets, other fuzzification schemes allowing one to extend chaotic evolution to the domain of fuzzy sets are possible. For example, Buckley and Hayashi [4] discuss two simple methods, in which one either varies the parameters of the underlying fuzzy set (e.g. three numbers defining a triangular fuzzy set) or chaotically changes the fuzzy set by varying its shape and support. Chaotic dynamic operating on fuzzy truth values has been a subject of an article by Grim [5]. Even within the framework of the extension principle, different fuzzification schemes based on the notion of *s*- and *t*-norms are possible, leading to the mapping of levels of the level sets [6].

The goal of this paper is twofold. First, we apply the predictive ability of a fuzzy controller to distinguish chaotic and noisy behavior in a one-dimensional time series; to this end, we use a well-known example from the number theory. Second, we turn to two-dimensional systems, subject to a quadratic mapping in a complex plane, which leads to a Julia set, an attractor in the space of compact sets with Hausdorff metric [7]. Based on the extension principle, Fridrich [8] investigated the relationship between the initial and asymptotic membership functions for one-dimensional quadratic mappings. Here we apply the extension principle to the inverse function algorithm. Two-dimensional iterated fuzzy set systems have earlier been studied by Cabrell et al. [9] in a different context. If we preserve the notion of level sets, we arrive at a novel description of the Hausdorff dimension, given in terms of a fuzzy set. The membership function of this fuzzy set is a constant when the chaotic system is perturbed by an additive noise and all the points of the iterated system are accounted for; on the other hand, in the absence of noise, the fuzzy set provides a quantitative measure allowing one to distinguish chaotic from noisy mappings.

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2. Ergodic and chaotic time series

Can a chaotic time series be distinguished from a random one? We don't know, but random and ergodic time series are different.

If m is a measure on space E, consider transformations T of E into itself. T is said to be measure preserving, if $m(T^{-1}A) = m(A)$. A is invariant under T in case $T^{-1}A = A$. T is ergodic if it is measure preserving and if each invariant set is trivial in the sense of having a measure of either 0 or 1 [10].

For ergodic transformations, the sequence of iterates of a point is uniformly dense in space. This means that, starting with a point p at time 0, one asks for the frequency with which the iterates of p fall into A. In the limit of infinitely many iterates, for almost every point p this frequency is equal to the relative measure of the region. In mathematical terms, the ergodic theorem reads

$$\lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} I_A(T^i(p)) = \frac{m(A)}{m(E)} , \tag{1}$$

where I_A is the indicator function of A.

Among different applications, the ergodic theorem has found remarkable use in number theory, especially in the context of continued fractions. Let $x \in [0, 1]$, then the continued fraction expansion of x is

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}} , \tag{2}$$

where the partial quotients a_n are positive integers. The transformation

$$T(x) = \frac{1}{x} - \left[\frac{1}{x}\right] , \tag{3}$$

where $[1/x]$ denotes the integer part of $1/x$, gives the fractional part of $1/x$; it preserves the Gauss measure

$$m(A) = \frac{1}{\log 2} \int_A \frac{1}{1+x} dx . \tag{4}$$

Note that the integers $\{a_n\}$ in Eq. (2) can be expressed in terms of the function $T(x)$. If $a(x) = [1/x]$, and $a_n(x) = a(T^{n-1}x)$, $n = 1, 2, \dots$, then $a_1(x)$, $a_2(x)$, ... are just the partial quotients in the continued fraction expansion of x.

The probability of finding an integer k in the sequence $a_1, a_2, a_3, \dots$ is given as

$$p(k) = \frac{1}{\log 2} \log\left[\frac{(k+1)^2}{k(k+2)}\right] \tag{5}$$

Equation (5) is obtained from the ergodic theorem by taking E to interval (0,1); A to be the interval $\left(\frac{1}{k+1}, \frac{1}{k}\right)$, that is the set on which $a_1(x) = k$, and the measure to be Gauss measure, given by Eq. (4). For example, for k = 1, 2, 3, 4, 5, we get $p(1) = 0.4150$, $p(2) = 0.1690$, $p(3) = 0.0931$, $p(4) = 0.0589$, and $p(5) = 0.0406$.

Any sequence of natural numbers drawn form the probability distribution of the quotients of the continued fraction corresponding to an irrational number represents a typical sequence, in the sense that almost all sequences of quotients have this distribution. On the other hand, some numbers lead to sequences of quotients that are not ergodic. For example, the quotients corresponding to $(\sqrt{5}-1)/2$ are all equal to 1. In the same vein, a sequence of uniformly distributed integers will have more large numbers than allowed by the probability distribution of quotients.

These comments lead to a method allowing one to distinguish an ergodic from a random sequence. After constructing the rules based on the ergodic sequence, we register the forecast error for ergodic and random sequences, thus obtaining a classification tool.

In forecasting error using fuzzy rule-based system (FRBS) [11] with lag vector of length 6, rules are obtained for ergodic and random sequences, drawn from a set of random numbers with the probability distribution given by Eq. (5) and from a uniformly distributed set of numbers, respectively.

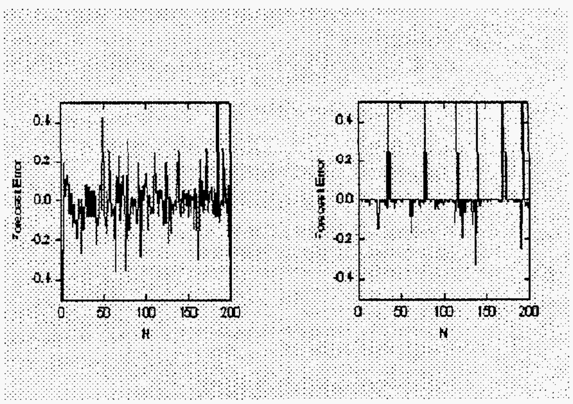

Fig. 1 Ergodic and chaotic time series.

The ergodic sequence (left) is too irregular for FRBS to be captured in rules. The random sequence (right) shows "anomalies" where large integers occur.

3. Spaces of fractals and fuzzy sets

Let (Ω, d) be a complete metric space with metric d. Denote by $\Im(\Omega)$ the space whose points are the compact subsets of Ω, other than the empty set. For $x \in \Omega$ and $B \in \Im(\Omega)$ define the distance from the point x to the set B:

$$d(x, B) = \text{Min}\{d(x, y): y \in B\} , \tag{6}$$

and the separation between the sets $A, B \in \Im(H)$:

$$d(A, B) = \text{Max}\{d(x, B): x \in A\} . \tag{7}$$

Then the Hausdorff metric between A and B, defined as

$$d_H(A, B) = \text{Max}\{d(A, B), d(B, A)\} , \tag{8}$$

yields a metric space of fractals $(\Im(\Omega), d_H)$. If w_i, $i = 1, 2, ..., n$ denotes an iterated function system of contracting mappings of Ω, then the transformation $W: \Im(\Omega) \to \Im(\Omega)$, defined by

$$W(B) = \bigcup_{i=1}^{n} w_i(B) \tag{9}$$

for all $B \in \Im(\Omega)$, is a contraction mapping on $(\Im(\Omega), d_H)$. Its unique fixed point is called an attractor of the iterated function system [12].

The space of fuzzy sets $D(\Omega)$ on Ω is defined as the set of upper semicontinuous and normal mappings $u: \Omega \to [0, 1]$. For $0 < \alpha \le 1$, the level set is a crisp set $[u]^\alpha = \{x \in \Omega: u(x) \ge \alpha\}$; a metric on $D(\Omega)$ can be defined as

$$d_\alpha(u, v) = d_H([u]^\alpha, [v]^\alpha) . \tag{10}$$

When the contractive operator is defined on $(D(\Omega), d_\alpha)$, its unique fixed point defines a fuzzy set attractor [9].

In fuzzy sets theory, a fuzzy set defined on a Cartesian product of crisp set is called a fuzzy relation. In the case of a Cartesian product of two crisp sets, an image analogy has proved fruitful in visualization of fuzzy sets: grey or color levels of an image admit a natural representation in terms of fuzzy sets.

4. Julia set

Quadratic Julia sets arise from sequences of complex numbers defined iteratively by the relation

$$z_{n+1} = z_n^2 + c , \tag{11}$$

where c is a complex number. Fixing c, while varying the initial point z_0, we may look for the values of z_0, for which the sequence z_n remains bounded. These values form the filled Julia set (or prisoner's set) K_c; the Julia J_c set consists of the boundary points of K_c. Equivalently, J_c can be defined as the closure of the set of repelling periodic points of the mapping

$$F(z) = z^2 + c \tag{12}$$

associated with Eq. (11).

A simple algorithm that produces a Julia set of the quadratic mapping (12) relies on the fact that the Julia set for is an attractor of an iterated function system, consisting of two functions $f_1(z) = \sqrt{z-c}$ and $f_2(z) = -\sqrt{z-c}$, which are just the functions inverse to F. In the inverse iteration method, a fixed initial point z_0 is iterated by selecting either f_1 or f_2 with equal probability of 0.5; the sequence $\{z_n: n = 0, 1, 2, ...\}$ converges to the attractor of the iterated function system. We use the inverse iteration method to study the transformation of an initial fuzzy set of a simple form.

The escape set, defined as the complement of the prisoner's set, can be divided into equipotential sets by using an escape time algorithm: the escape time for a point z outside of K_c is the first n for which z_n in Eq. (10) has modulus greater than a given radius R. Whereas an equipotential set is by definition an invariant of the iterative mapping (12), it is no longer so when Eq. (11) is supplemented by an additive noise term. This leads to a different mapping in the space of fuzzy sets than the mapping resulting from the extension principle.

4.1. Julia set: Inverse iteration method

In the simplest formulation, based on the classic extension principle [1], a mapping $f: \Omega \to \Omega$ induces a fuzzification $\bar{f}: D(\Omega) \to D(\Omega)$, defined on the space $D(\Omega)$ of fuzzy sets on a set Ω, as

$$(\bar{f}u)(y) = \sup\{u(x):\ x \in f^{-1}(y)\}. \qquad (13)$$

A simple fuzzification scheme relies on the definition of level sets and the resolution theorem, by virtue of which, any fuzzy set, $X = \sum u(x)/x$, can be represented as

$$X = \bigcup_{\alpha \in [0,1]} \alpha[u]^{\alpha} \qquad (14)$$

In Eq. (14), $[u]^{\alpha}$ is interpreted as a fuzzy set with a membership function whose value is unity. The resolution theorem reduces the transformations of fuzzy sets to interval arithmetic. Under broad conditions, spelled out in Ref. [2], the level sets satisfy the following transformation rule

$$[\bar{f}u]^{\alpha} = f([u]^{\alpha}) . \qquad (15)$$

Equation (15) remains valid even for a more general fuzzification schemes, known as Γ-fuzzification [2].

The chaotic properties of a dynamical systems can be quantified in terms of the Hausdorff dimension, whose fractional values indicate the existence of chaos. The dimension may be viewed as a measure of information necessary to specify the location within a given accuracy [13]. Mathematically, if $N(\varepsilon)$ is the number of cubes of side ε in a p-dimensional space needed to cover the set, the Hausdorff dimension h is

$$h = \lim_{\varepsilon \to 0} \log N(\varepsilon)/\log(1/\varepsilon) . \qquad (16)$$

In the following we use the Hausdorff dimension to characterize the chaotic behavior of different level sets of a fuzzy set.

For $c = 0.238489 + i\ 0.519198$ ($i = \sqrt{-1}$) the resulting Julia set is shown in Fig 2. In the absence of noise, the attractor resembles a one-dimensional curve with the Hausdorff dimension close to unity. We now supplement the iterative equations of the inverse iteration method with an additive uniform noise of strength q. This implies that, at each step, the real and imaginary parts of the square root are supplemented by a random number from the interval (0, 1) multiplied by q. As the iteration process is more and more perturbed by noise, the attractor is gradually being filled, with its Hausdorff dimension approaching the value of 2; see Fig. 3.

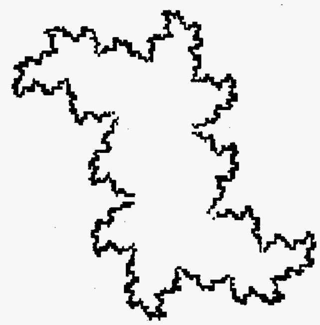

Fig. 2 Julia set for the complex parameter c = 0.238489 + i 0.51919800.

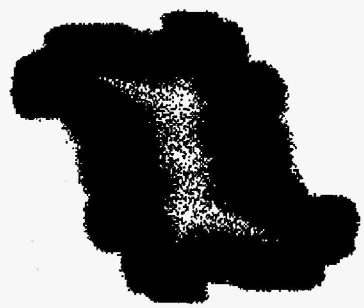

Fig. 3 Noise-perturbed Julia set. The noise strength q = 0.2; c as in Fig. 2.

As in Ref. [6], we now construct an initial fuzzy set X having the form of a quadrilateral pyramid with base area in the form of a square with half-side of 2.0 around the origin; the apex of the pyramid is a point with coordinates (0, 0, 1) in the x,y,z-space. This fuzzy set is subject to the quadratic transformation given by Eq. (12), applied to X by virtue of the extension principle. The extension principle is used in the form of Eq. (15), which tells us to apply the quadratic transformation to the level sets. The Hausdorff dimension of the resulting level sets is shown in Fig. 4 as a function of the level set size for different noise strengths. The α-cut size is expressed in terms of the half-size of the underlying square, whereas the noise strength q refers, once again, to the multiplicative factor scaling the additive white noise. During the course of the iterating process, we drop the points falling outside of the level set boundaries. For this reason, for small values of α, the Hausdorff dimension approaches zero: after many iterations there are only few points left.

We note that, for $q = 0$, the inverse iteration results in a fuzzy set whose each level set is a fractal. Such a fuzzy set can naturally be termed a fractal fuzzy set. As q increases, the α-cuts become uniformly filled; the chaotic properties of the attractor become truncated [14].

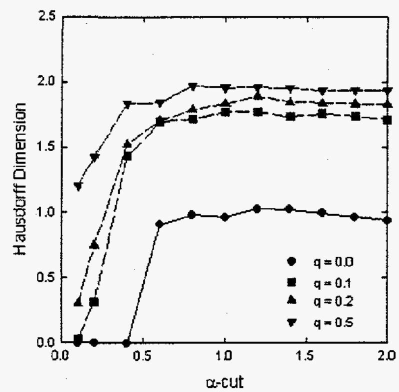

Fig. 4 Hausdorff dimension for the level sets resulting from the inverse iteration method as a function of the α-cut size for different noise values.

4.2. Julia set: forward iteration

The inverse iteration method relies on the backward iteration: one selects one of the two possible preimages of a given point in the complex plane. The forward iteration maps each point either to the interior of the Julia set (prisoner's set) or to its exterior (escape set). The escaping points can be studied by calculating the escape time, related to the potential of the Julia set. By virtue of the Riemann mapping theorem, the potential of any connected prisoner's set is can be brought into a one-to-one correspondence with the potential of a unit disk. Viewed from this perspective, the different potential values of the escape set can be interpreted as levels of a fuzzy set in two dimensions.

More precisely, let Σ denote a target set consisting of the points in the complex plane whose distance from the origin is larger than a given value R, supplemented by the point at infinity:

$$\Sigma = \{z \in C: |z| > R\} \cup \{\infty\} \ . \tag{17}$$

Define iteratively a sequence of inverse images of Σ, denoted as Σ_n, $n = 0, 1, 2, \ldots$, with $\Sigma_0 = \Sigma$, $\Sigma_1 = F^{-1}(\Sigma_0)$, ..., where the function F is defined by Eq. (12). It can readily be shown [7] that Σ_n can be defined as the set of points whose orbits need at most n iterations to reach Σ. The regions $L_n = \Sigma_{n+1} \setminus \Sigma_n$ have orbits which reach Σ in exactly $n + 1$ iterations.

In the following we restrict our attention to 16 levels. After normalization to unity, the level sets define a fractal fuzzy set; that is, a fuzzy set whose levels are fractals (cf. Sec 4.1). Figure 5 shows the level set of $\alpha = 0.31$ (level 5) whose Hausdorff dimension turns out to be 1.4906. Again, we perturb the iteration process by the additive noise terms scaled by q, resulting for $q = 1.0$, in a level set illustrated in Fig. 6 with Hausdorff dimension of 1.8767.

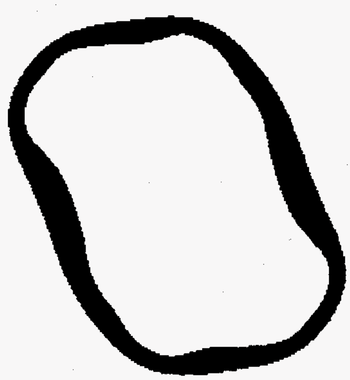

Fig. 5 Level set, identified with the α-cut for α = 0.25, of the Julia set defined as the set of points in the complex plane requiring 4 iterations to reach a large target set. Hausdorff dimension h of this level set is 1.4906.

In Fig 7, we show the Hausdorff dimension as a function of the α-cut size, depicted for different noise levels. It can be seen that, for α close to unity (large escape times) and high noise values, the Hausdorff dimension of the level sets drops to small fractional values: most points are lost due to the random perturbations of the trajectory. Had we retained the points that escape the level sets due to noise, the Hausdorff dimension of the level set would tend to the constant value of 2, typical of two-dimensional geometric objects.

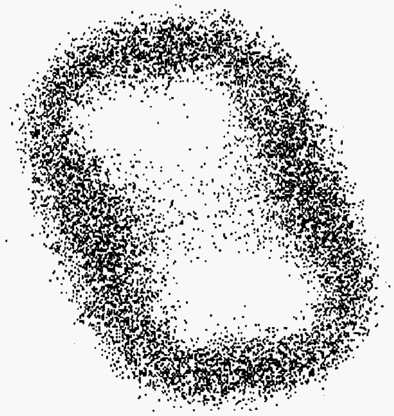

Fig. 6 Noise-perturbed (q = 1.0) α-cut of Fig. 5. Hausdorff dimension h = 1.8767.

As stressed in Ref [6], Fig. 7 demonstrates that the properties of an iterated mapping in two dimensions can be described in terms of a fuzzy set that provides the measure of the Hausdorff dimension for different α-cuts. The fuzzy set is constructed by normalizing the values of the Hausdorff dimension to unity.

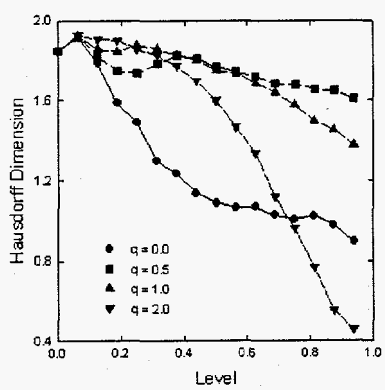

Fig. 7 Hausdorff dimension as a function of level set size for different noise values.

The shape of the membership function of this fuzzy set should tell us the difference between noise-dominated and chaos-dominated iterative mappings.

5. Conclusions

To distinguish between noisy and chaotic mappings, we analyzed both-one and two-dimensional systems. In one-dimension, the application of the fuzzy rule-based system allows us to detect anomalies in an ergodic mapping, thus distinguishing this mapping from a set of numbers drawn from a uniform distribution. In two dimensions, we have focused on the quadratic mappings in the complex plane, having a Julia set as an attractor. The Hausdorff dimension, considered as a function of the a-cut size, can be regarded as a fuzzy set whose functional form encodes the chaotic or noisy features of the mapping.

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