Using Geometric Algebra to Understand Pattern Rotations in Multiple Mirror Optical Systems

Jack Hanlon and Hans Ziock
Los Alamos National Laboratory

Geometric Algebra (GA) is a new formulation of Clifford Algebra that includes vector analysis without notation changes. Most applications of GA have been in theoretical physics, but GA is also a very good analysis tool for engineering. As an example, we use GA to study pattern rotation in optical systems with multiple mirror reflections. The common ways to analyze pattern rotations are to use rotation matrices or optical ray trace codes, but these are often inconvenient. We use GA to develop a simple expression for pattern rotation that is useful for designing or tolerancing pattern rotations in a multiple mirror optical system by inspection.

1.0. Introduction

Pattern rotation is used in many optical engineering systems, but it is not normally covered in optical system engineering texts [1]. Pattern rotation is important in optical systems such as: 1.) the 192 beam National Ignition Facility (NIF) [2], which uses square laser beams in close packed arrays to cut costs; 2.) visual optical systems, which use pattern rotation to present the image to the observer in the appropriate orientation, and 3.) the UR90 unstable ring resonator [3], which uses pattern rotation to fill a rectangular laser gain region and provide a filled-in laser output beam.

It is easy to illustrate pattern rotation in simple two mirror layouts, such as those layouts with in-plane or 90° out-of-plane turns. However, when the layout gets more complex, optical ray trace codes are needed and that is not always convenient. In this paper we develop a general principle to conceptualize new designs and tolerance optical system pattern rotation problems by inspection.

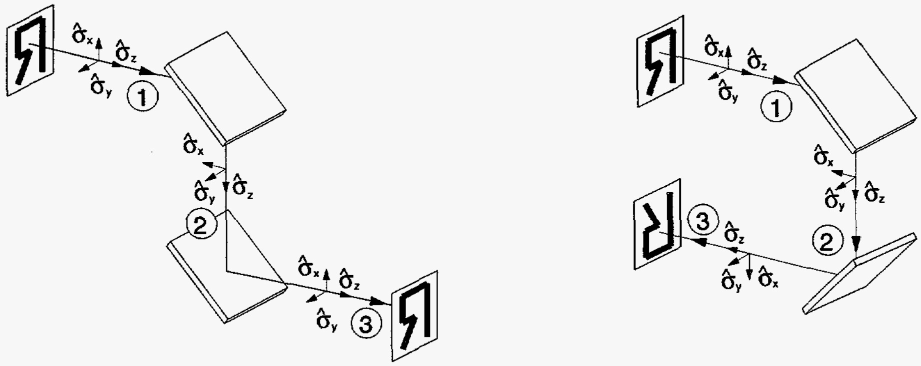
Since rotations are difficult to handle with traditional vector analysis, we choose to use the geometrical algebra (GA) developed by David Hestenes and others [4-11]. GA is a nearly ideal mix of vector algebra and geometry that often leads to conceptual pictures and easily remembered results that are ideal for engineering problems. GA is a non-commuting vector algebra that incorporates traditional vector analysis without notation changes. GA also uses an extended set of vector like objects and additional vector multiplication types, and it handles rotations in a consistent introductory way, which makes it ideal for studying pattern rotations in multiple mirror optical systems.

A brief introduction to the pattern rotation problem is given in Section 2, and the GA needed to solve pattern rotation problems is introduced in Section 3. The general pattern rotation problem is studied in Section 4.

2.0. Simple Pattern Rotations

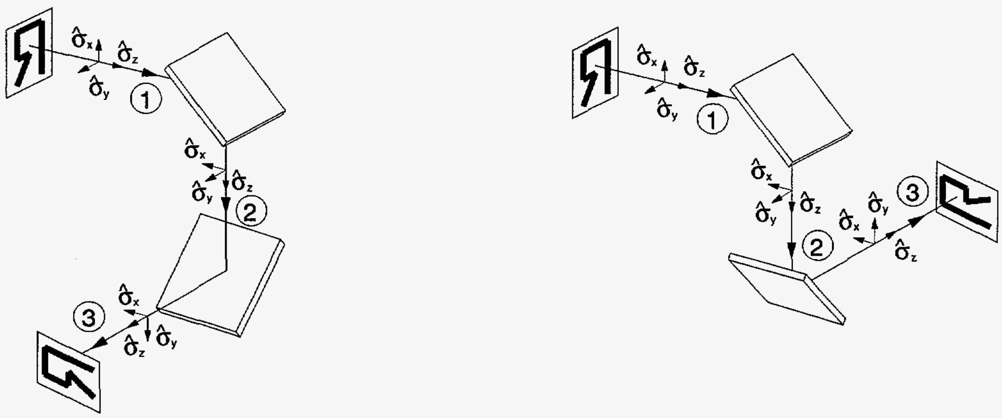
Pattern rotation in simple two mirror optical layouts is shown in Figures 2.1 and 2.2. Figure 2.1 illustrates pattern rotation for in-plane reflections, while Figure 2.2 illustrates pattern rotations for so-called 90° out-of-plane reflections. In Figure 2.2, it is clear that two planes are defined; one plane is defined by ray directions 1 and 2, and the other plane is defined by ray directions 2 and 3. These examples show pattern rotation in the following sense: first, the orientation (let's say upright) of a reference figure (in this case the letter R) is defined as the orientation while looking along the input ray 1 in the input 1-2 plane; then, starting with the same orientation (upright) while looking along the output ray 3 in the output 2-3 plane, rotate the figure by an angle θ about ray 3 to bring it into its final orientation. Looking at the figures closely, we note that the angle θ is the same as the angle between the two planes defined using ray 2 as the rotation axis, *e.g.*, θ is 0° in Figure 2.1(a), 180° in Figure 2.1(b), 90° in Figure 2.2(a) and 270° in Figure 2.2(b). This turns out to be the general result; namely, if all two-mirror reflection problems are

represented by an input and output plane as above, then the input pattern is rotated about ray 3, from the reference position in the output space, by the angle θ between the input and the output planes. Geometric Algebra, which is reviewed in the next section, is an interesting way to derive this general result.



(a) 0° (b) 180°

Figure 2.1. In-plane two mirror reflections.



(a) 90° (b) 270°

Figure 2.2. ± 90° out of plane reflections.

3.0. Geometric Algebra Introduction

In this section we review the two and three dimensional geometric algebra relationships needed to study optical pattern rotation problems. Many basic concepts of geometric algebra are illustrated by a new vector product type called the geometric product. The geometric product contains all the geometric and algebraic information about two vectors; it is written as

$$\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b}. \tag{3.1}$$

The left hand side of (3.1) is the geometric product of vectors **a** and **b**, and the first term on the right side is the scalar dot product of vector algebra. The second term on the right side is called the outer product of vectors **a** and **b**; it is related to, but not the same as the vector cross product.

The outer product of two vectors is called a bivector. Bivectors are algebraic representations of planes, and like vectors, bivectors have direction, magnitude and sign. All bivectors in a plane are scalar multiples of a unit bivector that defines the plane, just as all vectors on a line are scalar multiples of a unit vector that defines a line. The outer product of vectors **a** and **b** represents the direction, magnitude and sign of the piece of the plane defined by **a** and **b**. The unit bivector **i** defines the direction of a unit plane, and the scalar bivector magnitude defines the size of the area in the **i** plane. Although the area can take any shape, it is convenient to think of it as a parallelogram formed by two vectors in the **i** plane; then, if the two vectors are $\mathbf{a}$ and $\mathbf{b}$ with an angle θ between them, the bivector is $\mathbf{a} \wedge \mathbf{b} = \mathbf{i}|\mathbf{a}||\mathbf{b}|\sin\theta = \mathbf{i}\text{Area}$. From this expression we see that the conventional vector product and outer products are duals; the vector product produces a vector perpendicular to the area represented by the outer product.

The sign of the bivector associates a circulation direction (orientation) to areas in the $\mathbf{i}$ plane. Circulation direction is defined by attaching the tail of the second vector in a bivector product to the head of the first vector and following the direction of circular flow. According to convention, counterclockwise circulation defines a positive area, clockwise circulation a negative area. Because of this definition, changing the order of vectors in the bivector product changes the circulation direction and also the bivector sign; it follows that

$$\mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a}. \tag{3.2}$$

The sign change in (3.2) is very important because it means that vectors in a geometric product like (3.1) do not normally commute. In fact, vectors in a geometric product commute only if they are collinear (because then the outer product in (3.1) is zero), and they anticommute only if they are orthogonal (because then the dot product in (3.1) is zero).

Geometric products with more than two vectors are defined to follow the distributive and associative rules of algebra, but since vectors in geometric products do not normally commute, the order of vectors in the products must be preserved, *i.e.*,

$$(\mathbf{ab})\mathbf{c} = \mathbf{a}(\mathbf{bc}) \tag{3.3}$$

$$\mathbf{a}(\mathbf{b} + \mathbf{c}) = \mathbf{ab} + \mathbf{ac} \tag{3.4}$$

$$(\mathbf{b} + \mathbf{c})\mathbf{a} = \mathbf{ba} + \mathbf{ca} \tag{3.5}$$

Other geometric algebra rules evolve from the basic definitions (3.1) and (3.2) and rules (3.3) to (3.5). For example, (3.1) and (3.2) are used to express the dot and outer products of two vectors in terms of either the geometric product or the angle θ between the vectors.

$$\mathbf{a} \cdot \mathbf{b} = 1/2\,(\mathbf{ab}+\mathbf{ba}) = |\mathbf{a}||\mathbf{b}|\cos\theta \tag{3.6}$$

$$\mathbf{a} \wedge \mathbf{b} = 1/2\,(\mathbf{ab}-\mathbf{ba}) = \mathbf{i}|\mathbf{a}||\mathbf{b}|\sin\theta \tag{3.7}$$

A very useful GA operation not defined in vector analysis is the reverse operation $(\mathbf{M})^\dagger$ defined by Hestenes. The reverse operation is similar to the complex conjugate in the theory of complex numbers. In the reverse operation the elements of a product are arranged in reverse order and then the elements themselves are reversed; *e.g.*,

$$(\mathbf{Ba})^\dagger = \mathbf{a}^\dagger\mathbf{B}^\dagger, \tag{3.8}$$

where $\mathbf{B}$ is a general bivector and $\mathbf{a}$ is a vector[1]. Some useful reverse operations for vectors and bivectors are:

1.) Reversing a vector reproduces the same vector (by definition);

$$\mathbf{v}^\dagger = \mathbf{v} \tag{3.9}$$

2.) Reversing a bivector changes the bivector sign and circulation direction.

$$\mathbf{B}^\dagger = (\mathbf{a} \wedge \mathbf{b})^\dagger = \mathbf{b}^\dagger \wedge \mathbf{a}^\dagger = -\,\mathbf{a} \wedge \mathbf{b} = -\mathbf{B} \tag{3.10}$$

As a corollary, since $\mathbf{i}$ is a bivector, $\mathbf{i}^\dagger = -\,\mathbf{i}$.

As the order of the geometry increases new products are defined. For example, the geometric product and the dot and outer products of a vector $\mathbf{a}$ and a bivector $\mathbf{B}$ are defined by relationships (3.11) and (3.12), which are similar in structure to (3.1) and (3.2).

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[1] Vectors are denoted by lower case boldface letters, unit vectors in addition have a caret (^), and bivectors and trivectors are denoted by capital boldface letters. The exceptions are the unit bivector, which is denoted by a boldface **i**, and the unit trivector which is denoted by an italic *i*.

$$\mathbf{aB} = \mathbf{a} \cdot \mathbf{B} + \mathbf{a} \wedge \mathbf{B} \quad \text{and} \quad \mathbf{Ba} = \mathbf{B} \cdot \mathbf{a} + \mathbf{B} \wedge \mathbf{a} \tag{3.11}$$

$$\mathbf{a} \cdot \mathbf{B} = 1/2\,(\mathbf{aB} - \mathbf{Ba}) = -\mathbf{B} \cdot \mathbf{a} \quad \text{and} \quad \mathbf{a} \wedge \mathbf{B} = 1/2\,(\mathbf{aB} + \mathbf{Ba}) = \mathbf{B} \wedge \mathbf{a} \tag{3.12}$$

In Appendix 1, we show that $\mathbf{a} \cdot \mathbf{B}$ is a vector. The product $\mathbf{a} \wedge \mathbf{B}$ is either zero (if $\mathbf{a}$ is in the $\mathbf{B}$ plane), or it produces a trivector. If we limit the geometry to two dimensions we can always take $\mathbf{a} \wedge \mathbf{B} = 0$; then, for two dimensional problems, the product $\mathbf{ai} = \mathbf{a} \cdot \mathbf{i}$ is a vector, and using (3.12) we get

$$\mathbf{ai} = -\,\mathbf{ia}. \tag{3.13}$$

The following general vector identities (see Appendix 1 for proofs), are often useful:

$$\mathbf{a} \cdot (\mathbf{b} \wedge \mathbf{c}) = (\mathbf{a} \cdot \mathbf{b})\,\mathbf{c} - (\mathbf{a} \cdot \mathbf{c})\,\mathbf{b} \tag{3.14}$$

$$(\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c} = \mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}). \tag{3.15}$$

The right hand side of (3.14) is the familiar bac – cab rule of traditional vector analysis, so we have another identity that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = -\ \mathbf{a} \cdot (\mathbf{b} \wedge \mathbf{c})$.

The expansion of vector $\mathbf{a}$ relative to vector $\mathbf{b}$ is written as:

$$\mathbf{a}\mathbf{b}\mathbf{b}^{-1} = (\mathbf{a} \cdot \mathbf{b})\,\mathbf{b}^{-1} + (\mathbf{a} \wedge \mathbf{b})\,\mathbf{b}^{-1} = \mathbf{a}, \tag{3.16}$$

where, $\mathbf{b}^{-1}$ is a vector with the same direction but reciprocal magnitude as vector $\mathbf{b}$; *e.g.,* $\mathbf{b}^{-1} = \hat{\mathbf{b}}/\mathrm{b}$. Equation (3.16) expresses vector $\mathbf{a}$ in components along $\mathbf{b}$ and perpendicular to $\mathbf{b}$.

Equations (3.6) and (3.7) provide a way to write the geometric product of unit vectors that leads to a useful expression for rotating a vector in a plane. If the unit bivector for the plane containing unit vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ is $\mathbf{i}$, the geometric product can be written as,

$$\hat{\mathbf{a}}\hat{\mathbf{b}} = \hat{\mathbf{a}} \cdot \hat{\mathbf{b}} + \hat{\mathbf{a}} \wedge \hat{\mathbf{b}} = \cos\theta + \mathbf{i}\sin\theta = \mathrm{e}^{\mathbf{i}\theta}. \tag{3.17}$$

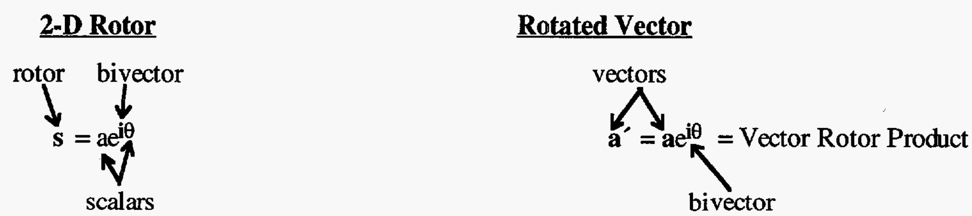
Equation (3.17) defines the exponential $\mathrm{e}^{\mathbf{i}\theta}$, which is also called a two dimensional rotor. Multiplying a vector $\mathbf{a}$ by rotor $\mathrm{e}^{\mathbf{i}\theta}$ produces a new vector $\mathbf{a}'$, which is the old vector $\mathbf{a}$ rotated by the angle θ in the $\mathbf{i}$ plane

$$\mathbf{a}' = \mathbf{a}\,\mathrm{e}^{\mathbf{i}\theta} = \mathbf{a}\cos\theta + \mathbf{a}\,\mathbf{i}\sin\theta. \tag{3.18}$$

If $\theta = 90°$,

$$\mathbf{a}' = \mathbf{a}\mathrm{e}^{\mathbf{i}\pi/2} = \mathbf{ai}, \tag{3.19}$$

indicating that $\mathbf{ai}$ is a vector perpendicular to $\mathbf{a}$. The important result (3.18), shows that it takes a vector-rotor product to create a consistent two dimensional vector algebra expression for rotating a vector[2]. These ideas, which consistently link GA, vector analysis and complex numbers, are summarized in Table 3.1



$s = a\cos\theta + a\mathbf{i}\sin\theta$ | $\mathbf{a}' = \mathbf{a}\cos\theta + \mathbf{a}\mathbf{i}\sin\theta$

rotor = scalar + bivector | vector = vector + vector

Table 3.1. Properties of a two dimensional rotor and a rotated vector.

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[2]Rotors and phasors, which are often used in engineering analysis, are easily confused. In phasor notation a rotated vector is written as $\mathbf{a} = \mathrm{a}\mathrm{e}^{\mathbf{i}\theta}$. This is not a legitimate equality that can be manipulated by the rules of vector algebra because the left hand side is a vector, but the right hand side is not.

Equations (3.20) summarize the common GA relationships used for GA algebra in a plane.

$$\mathbf{ai} = -\,\mathbf{ia} \qquad \mathbf{ii} = -1 \qquad \mathbf{a}' = \mathbf{a}e^{\mathbf{i}\theta} = (\mathbf{a}e^{\mathbf{i}\theta})^{\dagger} = e^{-\mathbf{i}\theta}\mathbf{a} \tag{3.20}$$

In three dimensions, vectors, bivectors and trivectors are allowed. Vectors are written in the traditional orthogonal component form as

$$\mathbf{v} = v_x\hat{\sigma}_x + v_y\hat{\sigma}_y + v_z\hat{\sigma}_z, \tag{3.21}$$

and bivectors are similarly expanded in an orthogonal component form as

$$\mathbf{B} = B_1\hat{\sigma}_y \wedge \hat{\sigma}_z + B_2\hat{\sigma}_z \wedge \hat{\sigma}_x + B_3\hat{\sigma}_x \wedge \hat{\sigma}_y = B_1\hat{\sigma}_y\hat{\sigma}_z + B_2\hat{\sigma}_z\hat{\sigma}_x + B_3\hat{\sigma}_x\hat{\sigma}_y. \tag{3.22}$$

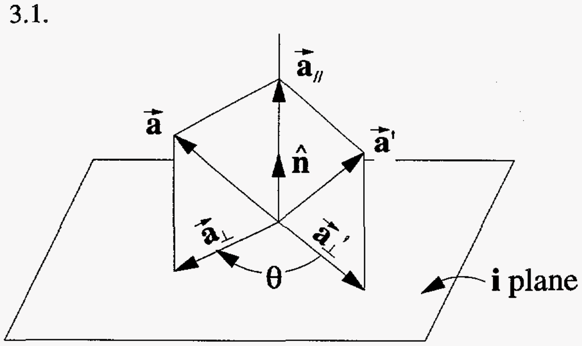
The components of (3.21) are the projections of **v** onto the reference directions ($\hat{\sigma}_x$, $\hat{\sigma}_y$, $\hat{\sigma}_z$), while the components of (3.22) are projections of the bivector **B** onto the reference planes ($\hat{\sigma}_y\hat{\sigma}_z$, $\hat{\sigma}_z\hat{\sigma}_x$, $\hat{\sigma}_x\hat{\sigma}_y$). The last equality in (3.22) holds because the reference vectors are orthogonal and thus the dot product parts of the geometric products are zero.

Three dimensional analysis uses a unit trivector denoted by the symbol i (italics, not bold faced, and different than **i**); i represents a positive unit volume, and any other positive volume in three space is iV, where V is a scalar that represents the magnitude of the volume. The unit trivector i can also be expressed as the geometric product of any right handed set of reference vectors; *e.g.*, $i = \hat{\sigma}_x\hat{\sigma}_y\hat{\sigma}_z$. Negative volumes are defined and written as - iV, and they are represented by a left handed set of reference vectors. More generally, the unit trivector i geometrically represents a unit volume at any position with any shape in three dimensional space. Algebraically, the unit trivector commutes with every scalar, vector, bivector, and trivector. This is easy to prove: consider the vector $\hat{\sigma}_x$; then, $\hat{\sigma}_x i = \hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_y\hat{\sigma}_z = \hat{\sigma}_x\hat{\sigma}_y\hat{\sigma}_z\hat{\sigma}_x = i\hat{\sigma}_x$ (because $\hat{\sigma}_x$ anticommutes with $\hat{\sigma}_y$ and $\hat{\sigma}_z$). Using the same approach we can show that i commutes with any bivector. Of course, i commutes with scalars and with itself. It is also easy to show that the geometric product $ii = -1$ and $i^{\dagger} = -\,i$. The geometric product of i and a bivector produces a vector perpendicular to the bivector plane, and the geometric product of i and a vector produces a bivector perpendicular to the vector. This is also easy to prove: let the vector be $\hat{\sigma}_x$; then,

$$i\hat{\sigma}_x = \hat{\sigma}_x i = \hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_y\hat{\sigma}_z = \hat{\sigma}_y\hat{\sigma}_z = \hat{\sigma}_y \wedge \hat{\sigma}_z. \tag{3.23}$$

Since $\hat{\sigma}_x = \hat{\sigma}_y \times \hat{\sigma}_z$, (3.23) is a special case of a general relationship between the vector cross product and the outer product of two vectors; *i.e.*, $\mathbf{a} \wedge \mathbf{b} = i\,(\mathbf{a} \times \mathbf{b})$. The vector-trivector product $\mathbf{a} \wedge i = 0$ if the algebra is limited to three dimensions or else it produces a four vector. We often limit the algebra to three dimensions, and then we take $\mathbf{a} \wedge i = 0$.

There are rotor expressions for rotating a three dimensional vector about an axis of rotation $\hat{\mathbf{n}}$ that are similar to (3.18) for rotating a vector in two dimensions. Three dimensional rotation is illustrated in Figure 3.1.



$$\mathbf{a}' = e^{-i\hat{\mathbf{n}}\theta/2}(\mathbf{a})e^{+i\hat{\mathbf{n}}\theta/2}$$

$$\mathbf{a}' = e^{-i\hat{\mathbf{n}}\theta/2}(\mathbf{a}_{/\!/} + \mathbf{a}_{\perp})e^{+i\hat{\mathbf{n}}\theta/2} \tag{3.24}$$

$$\mathbf{a}' = \mathbf{a}_{/\!/} + \mathbf{a}_{\perp}e^{+i\hat{\mathbf{n}}\theta}$$

$$\mathbf{a}' = \mathbf{a}_{/\!/} + \mathbf{a}'_{\perp}$$

Figure 3.1. Geometric Algebra representation for rotating a vector **a** about an axis $\hat{\mathbf{n}}$ by an angle θ.

Equations (3.24) are the algebraic expressions for three dimensional vector rotation. The bivector $i\hat{\mathbf{n}}$ is the unit magnitude plane **i** perpendicular to the vector axis of rotation $\hat{\mathbf{n}}$. The vector **a** has components parallel and perpendicular to $\hat{\mathbf{n}}$. Since $\mathbf{a}_{/\!/}$ is parallel to $\hat{\mathbf{n}}$, it commutes with $e^{-i\hat{\mathbf{n}}\theta/2}$, and since $\mathbf{a}_{\perp}$ is perpendicular to $\hat{\mathbf{n}}$, it anticommutes with $\hat{\mathbf{n}}$, so $e^{-i\hat{\mathbf{n}}\theta/2}\mathbf{a}_{\perp} = \mathbf{a}_{\perp}e^{+i\hat{\mathbf{n}}\theta/2}$. Both results are easy to prove by expanding the exponentials in power series and manipulating expressions using commutation rules already defined. For two dimensional problems the component $\mathbf{a}_{//}$ to $\hat{\mathbf{n}}$ is always absent because the rotation axis $\hat{\mathbf{n}}$ is always perpendicular to the $\mathbf{i}$ plane.

4.0 Using Geometric Algebra to Study Pattern Rotation

4.1 Single Mirror Reflections

For any vector $\mathbf{v}$ the law of reflection is written in GA as (4.1), where $\mathbf{v}'$ is the vector after reflection, and $\hat{\mathbf{n}}$ is a unit vector normal to the mirror.

$$\mathbf{v}' = -\hat{\mathbf{n}}\mathbf{v}\hat{\mathbf{n}} \tag{4.1}$$

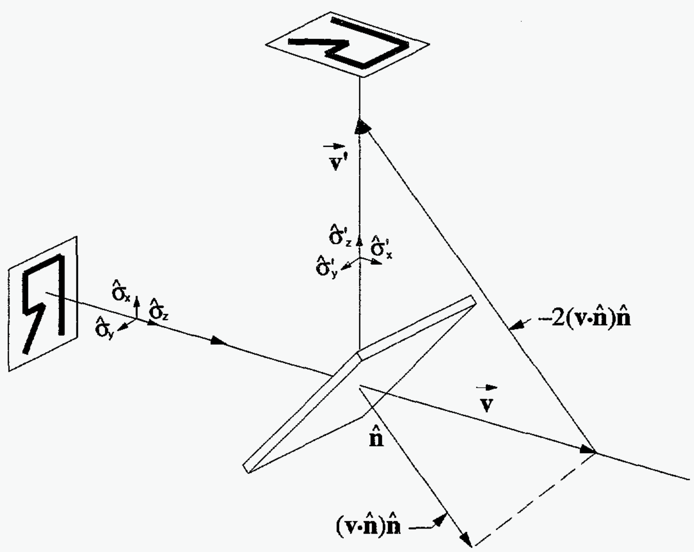


Figure 4.1. There is no pattern rotation about the output ray axis in a single mirror reflection.

Using (3.1), (3.14) and (3.15), (4.1) is expanded as[3],

$$\mathbf{v}' = -\hat{\mathbf{n}}(\mathbf{v}\cdot\hat{\mathbf{n}} + \mathbf{v}\wedge\hat{\mathbf{n}}) = \mathbf{v} - 2(\mathbf{v}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}. \tag{4.2}$$

This expression can be found in most vector analysis texts. The components of (4.2) are shown in Fig 4.1, with $\mathbf{v} = \hat{\sigma}_z$, the ray vector in the direction of light travel, but (4.1) and (4.2) also apply for any vector $\mathbf{v}$. Note that (4.1) holds if $\hat{\mathbf{n}}$ is replaced by $-\hat{\mathbf{n}}$, and that there is no pattern rotation about the output ray in a single mirror reflection for any angle between the input and the output ray directions. This is easily shown by pushing a patterned card through the reflection, but it also comes out of the algebra as expressed by (4.1), as shown in Appendix II.

Note, that because our reference position is looking along the direction of travel, we observe the pattern after reflection from the other side relative to our view in the input space. This is all taken care of by (4.1).

4.2. Two Mirror Reflections

Several two mirror reflection problems were shown in Section 2.0. For two mirror reflections there are two mirror normals $\hat{\mathbf{n}}_1$ and $\hat{\mathbf{n}}_2$, and we use (4.1) twice. If we let $\mathbf{v}''$ represent a vector $\mathbf{v}$ after two reflections; then,

$$\mathbf{v}'' = \hat{\mathbf{n}}_2\hat{\mathbf{n}}_1\mathbf{v}\hat{\mathbf{n}}_1\hat{\mathbf{n}}_2. \tag{4.3}$$

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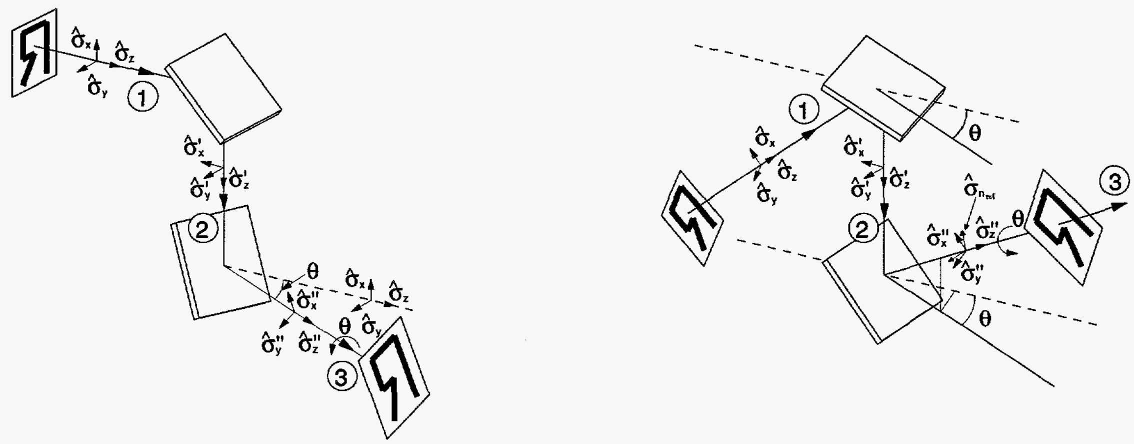
[3]Use the geometric product expansion and identity (3.14) to get $\mathbf{v}' = -\hat{\mathbf{n}}(\mathbf{v}\cdot\hat{\mathbf{n}} + \mathbf{v}\wedge\hat{\mathbf{n}}) = -(\mathbf{v}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}} - (\hat{\mathbf{n}}\cdot\mathbf{v})\hat{\mathbf{n}} + (\hat{\mathbf{n}}\cdot\hat{\mathbf{n}})\mathbf{v} + \hat{\mathbf{n}}\wedge\mathbf{v}\wedge\hat{\mathbf{n}} = \mathbf{v} - 2(\mathbf{v}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}$. The last equality follows since $(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}) = 1$, and $\hat{\mathbf{n}}\wedge\mathbf{v}\wedge\hat{\mathbf{n}} = -(\hat{\mathbf{n}}\wedge\hat{\mathbf{n}})\wedge\mathbf{v} = 0$.



(a) (b)

Figure 4.2. Layouts for two mirror pattern rotation problems.

Two mirror reflection problems define three lines and two planes. For the case shown in Figure 2.1, the two planes $\hat{\sigma}_z \wedge \hat{\sigma}_z'$ and $\hat{\sigma}_z' \wedge \hat{\sigma}_z''$ are the same plane, *i.e.*, the angle between the two planes is either 0° or 180°. For the 0° case, $\hat{\mathbf{n}}_2 = -\hat{\mathbf{n}}_1$ and (4.3) yields $\mathbf{v}'' = \mathbf{v}$ for all vectors $\mathbf{v}$. For this case there is no image rotation about the output ray. For the 180° case, $\hat{\mathbf{n}}_2 = \hat{\mathbf{n}}_1 e^{-i\hat{\sigma}_y\pi/2}$, $\hat{\mathbf{n}}_1\hat{\mathbf{n}}_2 = \hat{\mathbf{n}}_1\hat{\mathbf{n}}_1 e^{-i\hat{\sigma}_y\pi/2} = e^{-i\hat{\sigma}_y\pi/2}$, and (4.3) yields (4.4), which according to (3.24) is a rotation of $\mathbf{v}$ about $\hat{\sigma}_y = \hat{\sigma}_y''$ by $\theta = \pi$.

$$\mathbf{v}'' = \hat{\mathbf{n}}_2\hat{\mathbf{n}}_1\mathbf{v}\hat{\mathbf{n}}_1\hat{\mathbf{n}}_2 = e^{i\hat{\sigma}_y\pi/2}\mathbf{v}e^{-i\hat{\sigma}_y\pi/2} \tag{4.4}$$

Expressions like (4.4) are correct, but for a general two mirror layout, we want to algebraically express the final pattern orientation as a rotation of a reference orientation about the final ray vector $\hat{\sigma}_z''$. From (4.4) we see that the two mirror reflection problem is really a rotation problem, which can be solved in several ways. In fact the result shown in Figure 2.1(b), can also be produced by rotating the pattern by 180° about the $\hat{\sigma}_z''$ axis of rotation.

We set as our goal to write the two mirror reflection problem as a rotation about the final ray direction. To derive this result we define $\hat{\sigma}_z''$ as the unit vector in the direction of light travel in the output space, and we notice that a right handed system always transforms to a right handed system because under a two mirror transformation $i = \hat{\sigma}_x\hat{\sigma}_y\hat{\sigma}_z$ becomes,

$$\hat{\mathbf{n}}_2\hat{\mathbf{n}}_1\hat{\sigma}_x\hat{\sigma}_y\hat{\sigma}_z\hat{\mathbf{n}}_1\hat{\mathbf{n}}_2 = (\hat{\mathbf{n}}_2\hat{\mathbf{n}}_1\hat{\sigma}_x\hat{\mathbf{n}}_1\hat{\mathbf{n}}_2)(\hat{\mathbf{n}}_2\hat{\mathbf{n}}_1\hat{\sigma}_y\hat{\mathbf{n}}_1\hat{\mathbf{n}}_2)(\hat{\mathbf{n}}_2\hat{\mathbf{n}}_1\hat{\sigma}_z\hat{\mathbf{n}}_1\hat{\mathbf{n}}_2) = \hat{\sigma}_x''\hat{\sigma}_y''\hat{\sigma}_z'' = i. \tag{4.5}$$

Consequently, we only need to find one of the transformed input transverse vectors. We define reference orientations by starting with a figure like Figure 2.1a. We define $\hat{\sigma}_x$ as the reference transverse vector in the input $\hat{\sigma}_z \wedge \hat{\sigma}_z'$ plane, and define $\hat{\sigma}_x''$ as the vector in the output $\hat{\sigma}_z' \wedge \hat{\sigma}_z''$ plane perpendicular to the output ray $\hat{\sigma}_z''$. We use the notation shown in Figure 4.2(a). We could use (4.1) to show that for the first reflection, $\hat{\sigma}_x' = -\hat{\sigma}_z$, $\hat{\sigma}_y' = \hat{\sigma}_y$, and $\hat{\sigma}_z' = -\hat{\sigma}_x$, but these are also easy to follow and read from the diagram. For the second reflection we use (4.1), but first we need $\hat{\mathbf{n}}_2$. Using (4.2) in the form (4.6), we get,

$$\hat{\sigma}_z'' = \hat{\sigma}_z' - 2(\hat{\sigma}_z' \cdot \hat{\mathbf{n}}_2)\hat{\mathbf{n}}_2. \tag{4.6}$$

Multiplying both sides of (4.6) by $\hat{\mathbf{n}}_2$ and rewriting the result as (4.7) yields

$$\hat{\mathbf{n}}_2 \cdot (\hat{\sigma}_z'' - \hat{\sigma}_z') + \hat{\mathbf{n}}_2 \wedge (\hat{\sigma}_z'' - \hat{\sigma}_z') = -2(\hat{\sigma}_z' \cdot \hat{\mathbf{n}}_2). \tag{4.7}$$

Equating the bivector parts of both sides of (4.7) shows that $\hat{\mathbf{n}}_2$ and $(\hat{\sigma}_z'' - \hat{\sigma}_z')$ have the same direction because,

$$\hat{\mathbf{n}}_2 \wedge (\hat{\sigma}_z'' - \hat{\sigma}_z') = 0; \tag{4.8}$$

therefore, $\hat{\mathbf{n}}_2$ can be written as (4.9)

$$\hat{\mathbf{n}}_2 = (\hat{\sigma}_z'' - \hat{\sigma}_z') \,/\, |(\hat{\sigma}_z'' - \hat{\sigma}_z')|. \tag{4.9}$$

From the figure, $\hat{\sigma}_z'' = \hat{\sigma}_y \sin\theta + \hat{\sigma}_z \cos\theta$, and $\hat{\sigma}_z' = -\hat{\sigma}_x$; consequently,

$$\hat{\mathbf{n}}_2 = (\hat{\sigma}_x + \hat{\sigma}_y \sin\theta + \hat{\sigma}_z \cos\theta)/\sqrt{2}. \tag{4.10}$$

Using (4.10) in (4.11), which follows from (4.2),

$$\hat{\sigma}_x'' = \hat{\sigma}_x' - 2(\hat{\sigma}_x' \cdot \hat{\mathbf{n}}_2)\hat{\mathbf{n}}_2, \tag{4.11}$$

leads to

$$\hat{\sigma}_x'' = \hat{\sigma}_x \cos\theta - (\hat{\sigma}_z \sin\theta - \hat{\sigma}_y \cos\theta)\sin\theta, \tag{4.12}$$

which we rewrite in terms of $\hat{\sigma}_z''$ and $\hat{\sigma}_x$. Using the fact that $\hat{\sigma}_z = -\hat{\sigma}_x i\hat{\sigma}_y$ and $\hat{\sigma}_y = \hat{\sigma}_x i\hat{\sigma}_z$, (4.12) reduces to (4.13),

$$\hat{\sigma}_x'' = \hat{\sigma}_x \cos\theta + \hat{\sigma}_x i(\hat{\sigma}_y \sin\theta + \hat{\sigma}_z \cos\theta)\sin\theta = \hat{\sigma}_x \cos\theta + \hat{\sigma}_x i\hat{\sigma}_z'' \sin\theta. \tag{4.13}$$

which is written as:

$$\hat{\sigma}_x'' = \hat{\sigma}_x e^{i\hat{\sigma}_z''\theta}. \tag{4.14}$$

The angle θ between the input and the output planes is defined using $\hat{\sigma}_z'$, the middle ray, as a rotation axis.

Equation (4.14) tells us how the pattern is oriented in the output plane, but it is useful to develop an expression for transforming a general vector **v** from the input to the output plane. To do that, we need expressions for $\hat{\sigma}_y''$ and $\hat{\sigma}_z''$. Because $\hat{\sigma}_z''$ is just $\hat{\sigma}_z$ rotated into position using the middle ray $\hat{\sigma}_z'$ as an axis of rotation;

$$\hat{\sigma}_z'' = \hat{\sigma}_z e^{i\hat{\sigma}_z'\theta}. \tag{4.15}$$

Also, since $\hat{\sigma}_y'' = i\hat{\sigma}_z''\hat{\sigma}_x''$ we have

$$\hat{\sigma}_y'' = i\hat{\sigma}_z''\hat{\sigma}_x'' = i\hat{\sigma}_z e^{i\hat{\sigma}_z'\theta}\hat{\sigma}_x e^{i\hat{\sigma}_z''\theta} = i\hat{\sigma}_z\hat{\sigma}_x e^{i\hat{\sigma}_z'\theta} e^{i\hat{\sigma}_z''\theta}, \tag{4.16}$$

because $\hat{\sigma}_x$ is parallel to $\hat{\sigma}_z'$, and therefore the exponential $e^{i\hat{\sigma}_z'\theta}$ commutes with $\hat{\sigma}_x$. Finally, since $i\hat{\sigma}_z\hat{\sigma}_x = \hat{\sigma}_y$,

$$\hat{\sigma}_y'' = \hat{\sigma}_y e^{i\hat{\sigma}_z'\theta} e^{i\hat{\sigma}_z''\theta} \tag{4.17}$$

To put (4.14) and (4.15) into the form (4.17) we need to realize that rotating a vector about an axis along the vector leaves the vector unchanged, so that $\hat{\sigma}_z'' = \hat{\sigma}_z'' e^{i\hat{\sigma}_z''\theta}$ and $\hat{\sigma}_x = \hat{\sigma}_x e^{i\hat{\sigma}_z'\theta}$. When we use these results in (4.14) and (4.15), we get a uniform format for transforming all the input basis vectors to the output space:

$$\hat{\sigma}_x'' = \hat{\sigma}_x e^{i\hat{\sigma}_z'\theta} e^{i\hat{\sigma}_z''\theta}; \quad \hat{\sigma}_y'' = \hat{\sigma}_y e^{i\hat{\sigma}_z'\theta} e^{i\hat{\sigma}_z''\theta}, \quad \text{and} \quad \hat{\sigma}_z'' = \hat{\sigma}_z e^{i\hat{\sigma}_z'\theta} e^{i\hat{\sigma}_z''\theta}; \tag{4.18}$$

consequently, any vector **v** in the input space can be transformed to a vector **v**″ in the output space as

$$\mathbf{v}'' = \mathbf{v} e^{i\hat{\sigma}_z'\theta} e^{i\hat{\sigma}_z''\theta}. \tag{4.19}$$

This is the main result for two mirror reflection problems.

Equation (4.19) is a logical and easily remembered result that would be difficult to derive using ordinary vector analysis, which does not have any consistent expressions for vector rotations except in rectangular or matrix form. It shows how to find the pattern orientation in the output space after two mirror reflections. The geometrical description of (4.19) is that we should move the pattern into the 0° output space, which is a diagram like Figure 2.1a; then rotate the pattern into the correct direction, $\hat{\sigma}_z''$, using $\hat{\sigma}_z'$ as a rotation axis, and finally rotate the pattern into the final orientation using $\hat{\sigma}_z''$ as a rotation axis. With (4.19) all two mirror problems are easy to follow. For example, for Figure 2.1 (a), $\theta = 0$; for Figure 2.1(b), $\theta = \pi$, and for Figure 2.2, $\theta = \pm\pi/2$.

We now have all the pieces needed for understanding pattern rotations in a general two mirror reflection problem such as the one illustrated in Figure 4.2(b). In Figure 4.2(b) the input ray $\hat{\sigma}_z$ is rotated in the input plane, and the output ray $\hat{\sigma}_z''$ is rotated in the output plane compared to Figure 4.2(a). Since we have already shown that these "in plane" rotations do not produce any pattern rotations about $\hat{\sigma}_z''$, (4.19) tells all there is to know about two mirror reflections. For some layouts the orientation of the reference figure before rotation in the output space is not obvious, but a reference orientation can always be determined by starting with a diagram like Figure 2.1a. We summarize the general procedure as follows: 1.) create a single plane reference diagram by positioning both the input and the output rays perpendicular to the middle ray in a plane parallel to the input plane (Figure 2.1(a)); 2.) position the input reference figure in the output space with the same orientation as in the input space; 3.) rotate the output 2-3 plane into position using the middle ray as an axis (Figure 4.2a); 4.) rotate the output pattern about the output ray 3 by the same angle as the final angle between the input and output planes (Figure 4.2a), and finally 5.) rotate the input and output rays into position in their respective planes (this is like Figure 4.2b).

An interesting exercise is to follow the pattern rotation in the four mirror ring shown in Figure 4.3. If $h = 0$, the four mirrors are in the same plane and there is no pattern rotation. If $h > 0$, the angle between planes is α, and the pattern is rotated by 2α about the ray direction in one trip around the ring. If $h < 0$, the angle between planes is $-\alpha$, and the pattern is rotated by -2α in one trip around the ring.

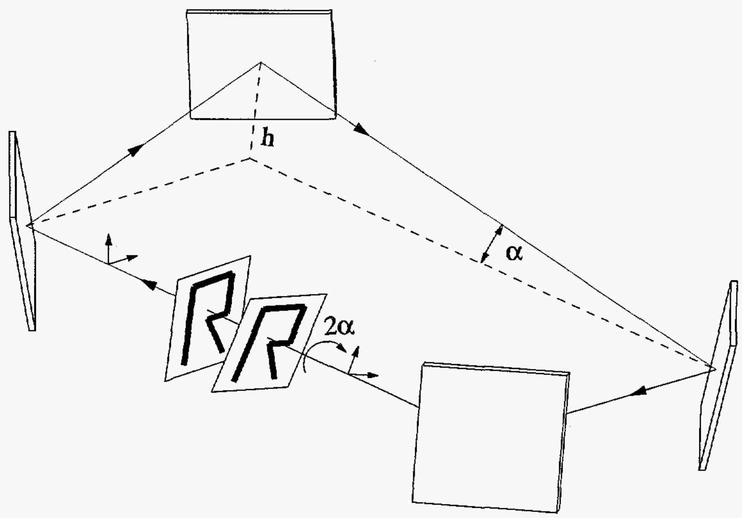


Figure 4.3. Pattern rotation in a four mirror ring.

Pattern rotations in multiple mirror layout are combinations of the cases we have studied. Three mirror problems are two mirror plus one mirror problems, and four mirror problems are two, two mirror problems in sequence, *etc.* The corner cube, however, is one interesting three mirror problem that is especially easy to handle. For a corner cube the mirror normals are orthogonal; therefore, $\hat{\mathbf{n}}_1\hat{\mathbf{n}}_2\hat{\mathbf{n}}_3 = \pm i$, where the sign depends on the order that a ray strikes the mirrors; however, since the product is used twice, the order of reflection does not effect the final result. Since i commutes with everything, after reflection in the corner cube,

$$\mathbf{v}''' = \hat{\mathbf{n}}_3\hat{\mathbf{n}}_2\hat{\mathbf{n}}_1\mathbf{v}\hat{\mathbf{n}}_1\hat{\mathbf{n}}_2\hat{\mathbf{n}}_3 = -i^\dagger\mathbf{v}i = ii\mathbf{v} = -\mathbf{v} \qquad (4.20)$$

This shows that a corner cube reverses the direction of every input vector. This is easily checked by looking into a corner cube.

5.0. Conclusions

Geometric algebra contains vector analysis without notation changes and offers a consistent useful interpretation for pattern rotations in an introductory and convenient way. Our main conclusion is that GA is a very useful analysis tool for all types of engineering problems.

For example, GA is a convenient way to analyze pattern rotations in multiple mirror optical systems. The main result is equation (4.19), which is a simple geometric expression for pattern rotation in a general two mirror system. In two mirror reflection problems two planes are defined. The orientation of the output pattern is produced by positioning the input pattern in a reference orientation in the output space and rotating the pattern about the output ray by the angle between the input and output planes. Pattern rotations in multiple mirror systems are at most a sequence of the two mirror problems followed by a single reflection, and the single reflection only changes the direction of the light; it does not rotate the pattern about the output ray.

We chose to use geometric algebra as defined by Hestenes to study pattern rotations in mirror reflections because traditional vector analysis does not include a consistent way to represent vector rotations. Surely the problems in this paper can be solved using other approaches, but none of them lead to a consistent interpretation like (4.19). Equation (4.2) is available in many vector analysis texts, but the derivation using (4.1) and GA is very satisfying and easily extended to multiple reflections. Expressions for rotations (3.18) and (3.24) do not exist in traditional vector analysis. Phasor notations attempt to represent rotations, but it does not provide the consistent algebraic interpretation that is built into GA.

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Appendix I

In this appendix we prove relationships (3.14) and (3.15). To prove (3.14) we start with definition (A.1).

$$\mathbf{a\,b\,c} = (\mathbf{a\,b})\,\mathbf{c} \tag{A.1}$$

We rewrite the right hand side using (3.1), (3.3), and (3.7) as

$$\begin{aligned}(\mathbf{a\,b})\mathbf{c} &= (\mathbf{a}\cdot\mathbf{b})\mathbf{c} + (\mathbf{a}\wedge\mathbf{b})\mathbf{c} = (\mathbf{a}\cdot\mathbf{b})\mathbf{c} + 1/2\,(\mathbf{a\,b} - \mathbf{b\,a})\mathbf{c} \\ &= 2(\mathbf{a}\cdot\mathbf{b})\mathbf{c} - \mathbf{b\,a\,c} = 2(\mathbf{a}\cdot\mathbf{b})\mathbf{c} - 2(\mathbf{a}\cdot\mathbf{c})\mathbf{b} + \mathbf{b\,c\,a}\end{aligned} \tag{A.2}$$

Collecting terms gives

$$\mathbf{a\,b\,c} - \mathbf{b\,c\,a} = 2(\mathbf{a}\cdot\mathbf{b})\mathbf{c} - 2(\mathbf{a}\cdot\mathbf{c})\mathbf{b} \tag{A.3}$$

The left hand side is then expanded using (3.1), (3.11), and recalling that the outer product of two vectors is a bivector to yield

$$\begin{aligned}\mathbf{a\,b\,c} - \mathbf{b\,c\,a} &= \mathbf{a}\,(\mathbf{b}\cdot\mathbf{c}) + \mathbf{a}\,(\mathbf{b}\wedge\mathbf{c}) - (\mathbf{b}\cdot\mathbf{c})\mathbf{a} - (\mathbf{b}\wedge\mathbf{c})\mathbf{a} \\ &= \mathbf{a}\,(\mathbf{b}\cdot\mathbf{c}) + \mathbf{a}\cdot(\mathbf{b}\wedge\mathbf{c}) + \mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c}) - (\mathbf{b}\cdot\mathbf{c})\mathbf{a} - (\mathbf{b}\wedge\mathbf{c})\cdot\mathbf{a} - (\mathbf{b}\wedge\mathbf{c})\wedge\mathbf{a} \\ &= \mathbf{a}\,(\mathbf{b}\cdot\mathbf{c}) - (\mathbf{b}\cdot\mathbf{c})\mathbf{a} + \mathbf{a}\cdot(\mathbf{b}\wedge\mathbf{c}) - (\mathbf{b}\wedge\mathbf{c})\cdot\mathbf{a} + \mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c}) - (\mathbf{b}\wedge\mathbf{c})\wedge\mathbf{a}\end{aligned} \tag{A.4}$$

Equating the right hand sides of (A.4)and (A.3) and noting that the trivector parts must be equivalent gives

$$\mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c}) - (\mathbf{b}\wedge\mathbf{c})\wedge\mathbf{a} = 0. \tag{A.5}$$

Similarly, noting that the remaining parts must be equivalent gives

$$2(\mathbf{a}\cdot\mathbf{b})\mathbf{c} - 2(\mathbf{a}\cdot\mathbf{c})\mathbf{b} = \mathbf{a}\,(\mathbf{b}\cdot\mathbf{c}) - (\mathbf{b}\cdot\mathbf{c})\mathbf{a} + \mathbf{a}\cdot(\mathbf{b}\wedge\mathbf{c}) - (\mathbf{b}\wedge\mathbf{c})\cdot\mathbf{a}. \tag{A.6}$$

Recalling that $(\mathbf{b}\cdot\mathbf{c})$ is a scalar, and a scalar commutes with a vector [*i.e.* $\mathbf{a}\,(\mathbf{b}\cdot\mathbf{c}) = (\mathbf{b}\cdot\mathbf{c})\,\mathbf{a}$], and using (3.12), we can rewrite (A.6) as

$$2(\mathbf{a}\cdot\mathbf{b})\mathbf{c} - 2(\mathbf{a}\cdot\mathbf{c})\mathbf{b} = 2\mathbf{a}\cdot(\mathbf{b}\wedge\mathbf{c}), \tag{A.7}$$

which is equivalent to the identity (3.14). The result also shows that $\mathbf{a}\cdot\mathbf{B}$ is a vector.

To prove (3.15), we start with

$$(\mathbf{ab})\mathbf{c} = \mathbf{a}(\mathbf{bc}) \tag{A.8}$$

and expand using (3.1) and (3.11) as

$$\begin{aligned}(\mathbf{ab})\mathbf{c} &= (\mathbf{a}\cdot\mathbf{b})\,\mathbf{c} + (\mathbf{a}\wedge\mathbf{b})\cdot\mathbf{c} + (\mathbf{a}\wedge\mathbf{b})\wedge\mathbf{c} \\ \mathbf{a}(\mathbf{bc}) &= \mathbf{a}\,(\mathbf{b}\cdot\mathbf{c}) + \mathbf{a}\cdot(\mathbf{b}\wedge\mathbf{c}) + \mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c})\end{aligned} \tag{A.9}$$

Equating the trivector parts of the right hand sides of (A.9) leads to (3.15), which shows that the parentheses can be moved around at will in an outer product.

$$(\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c} = \mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}) \qquad \text{(A.10)}$$

When we equate the remaining parts of (A.9); apply (A.7) to $\mathbf{a}(\mathbf{b} \cdot \mathbf{c}) - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} = \mathbf{b} \cdot (\mathbf{c} \wedge \mathbf{a})$, and rearrange the terms we get (A.11), which is Jacobi's identity.

$$\mathbf{a} \cdot (\mathbf{b} \wedge \mathbf{c}) + \mathbf{b} \cdot (\mathbf{c} \wedge \mathbf{a}) + \mathbf{c} \cdot (\mathbf{a} \wedge \mathbf{b}) = 0 \qquad \text{(A.11)}$$

Appendix II

In a single mirror reflection there is no rotation about the final ray direction for any angle between the input and the output rays (refer to Figure 4.1). For example, if the direction of $\hat{\sigma}_z'$ is held fixed but the input direction is rotated in-plane so that the angle between the input and output ray directions is changed by θ, then $\hat{\mathbf{n}}$ must be rotated by $\theta/2$ about the axis $\hat{\sigma}_y$, and $\hat{\sigma}_z$ and also $\mathbf{v}$ must be rotated by θ about the axis $\hat{\sigma}_y$. This is expressed by (A.12), which follows from (4.1), where "new" and "old" refer to before and after rotating the mirror. We use the fact that $\hat{\sigma}_y$ and $\hat{\mathbf{n}}$ are perpendicular, which changes the sign of the exponential when it commutes with $\hat{\mathbf{n}}$, so that:

$$\hat{\mathbf{n}}_{new} = \hat{\mathbf{n}}_{old} e^{i\hat{\sigma}_y\theta/2} = e^{-i\hat{\sigma}_y\theta/2}\hat{\mathbf{n}}_{old}; \quad \text{also}$$

$$\mathbf{v}_{new} = e^{-i\hat{\sigma}_y\theta/2}\, \mathbf{v}_{old}\, e^{+i\hat{\sigma}_y\theta/2}, \quad \text{and}$$

$$\mathbf{v}_{new}' = -\hat{\mathbf{n}}_{new}\mathbf{v}_{new}\hat{\mathbf{n}}_{new} = -\hat{\mathbf{n}}_{old} e^{i\hat{\sigma}_y\theta/2}\, e^{-i\hat{\sigma}_y\theta/2}\, \mathbf{v}_{old} e^{i\hat{\sigma}_y\theta/2}\, e^{-i\hat{\sigma}_y\theta/2}\hat{\mathbf{n}}_{old} = -\hat{\mathbf{n}}_{old}\mathbf{v}_{old}\hat{\mathbf{n}}_{old} \qquad \text{(A.12)}$$

Similarly, if we keep the input ray direction fixed and rotate the output direction and pattern in the plane of reflection there is no pattern rotation about the new output ray direction. In this case,

$$\mathbf{v}_{new}' = -\hat{\mathbf{n}}_{new}\mathbf{v}_{new}\hat{\mathbf{n}}_{new} = -\hat{\mathbf{n}}_{old} e^{i\hat{\sigma}_y\theta/2}\mathbf{v}_{old}\hat{\mathbf{n}}_{old}\, e^{i\hat{\sigma}_y\theta/2} = -e^{-i\hat{\sigma}_y\theta/2}\, \hat{\mathbf{n}}_{old}\mathbf{v}_{old}\hat{\mathbf{n}}_{old}\, e^{i\hat{\sigma}_y\theta/2}$$

$$= -e^{-i\hat{\sigma}_y\theta/2}\, \mathbf{v}_{old}'\, e^{i\hat{\sigma}_y\theta/2} \qquad \text{(A.13)}$$

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