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Level-Treewidth Property, Exact Algorithms and Approximation Schemes

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Abstract

Informally, a class of graphs $\mathcal{G}$ is said to have the *level-treewidth* property (LT-property) if for every $G \in \mathcal{G}$ there is a layout (breadth first ordering) $\mathcal{L}_G$ such that the subgraph induced by the vertices in k-consecutive levels in the layout have treewidth $O(f(k))$, for some function f. We show that several important and well known classes of graphs including planar and bounded genus graphs, (r, s)-civilized graphs, etc, satisfy the LT-property.

Building on the recent work of Hunt et al. [HM+94], Khanna and Motwani [KM96] and Eppstein [Ep95], we present two general types of results for the class of graphs obeying the LT-property.

1. All problems in the classes MPSAT, TMAX and TMIN (first considered in [KM96]) have polynomial time approximation schemes.
2. The problems considered in Eppstein [Ep95] (such as subgraph isomorphism, shortest path queries, etc,) have *efficient* polynomial time algorithms.

These results can be extended to obtain polynomial time approximation algorithms and approximation schemes for a number of PSPACE-hard combinatorial problems specified using different kinds of succinct specifications studied in [MH+95, Or82a, LW87a]. Many of the results can also be extended to δ-near genus and δ-near civilized graphs, for any fixed δ.

Our results significantly extend the work in [KM96, MH+95, DTS93, Ep95] and affirmatively answer recent open questions in [KM96, Ep95].

Keywords. Syntactic Characterization, PTAS, NP-hardness, Approximation Algorithms, Succinct Specifications.

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1 Introduction and Motivation

Extensive work has been done on the design of efficient sequential and parallel exact and approximation algorithms for problems restricted to planar instances (See [Ba83, KM96, HM+94, CK95, NC88, LT80], etc,).[3] In particular a substantial progress has been made in the direction of developing a theory of approximability based on the syntactic characterization of optimization problems. (See Papadimitriou and Yannakakis [PY91], Kolaitis and Thakur [KT94], Khanna et al. [KM+95] and Crescenzi and Kann [CK95], Panconesi and Ranjan [PR93], etc.)

In this paper, we further investigate the question of finding general graph theoretic restrictions that can be placed on input instances so as to obtain efficient exact and approximate algorithms for general classes of problems. In [HM+94], we studied the problems MAX SAT(S) and their generalizations when the underlying bipartite graph is planar. The results obtained in [HM+94] directly imply PTAS for most of the natural MAX SNP problems restricted to planar instances. Khanna and Motwani [KM96] generalized this work by presenting PTAS for a large class of problems including those belonging to MAX SNP, MAX NP, RMAX(2), etc when restricted to instances in which the underlying variable predicate interaction graph was planar. In a different direction, Eppstein [Ep95] presented efficient (often linear or quadratic) time algorithms for solving a number of problems such as variants of the subgraph isomorphism problem, shortest path queries, h-clustering, etc, when restricted to planar graphs. He also extended his results to apply to a subclass of all minor closed graphs. All the above papers were based on extending the powerful ideas first presented in Baker [Ba83].

It can be verified that planarity plays a central role in the work of [KM96] and minor closed property plays a central role in the work of [KM96]. Thus in [KM96], Khanna and Motwani remark that "*Our work provides evidence towards this claim raising the possibility that this input restriction may well be a notion related to planarity*". In [Ep95], Eppstein proved a certain key decomposition lemma (Lemma 5.2 in [Ep95]) about planar graphs and remarks "*The proof of Lemma 5.2 relies on the diameter treewidth property, and on another key property of planar graphs: any minor of a planar graph is also planar.*" We show that both these comments are only partially true. In particular, we present a *simple* and natural graph theoretic property underlying the results in [KM96, Ep95, Ba83, DTS93, HM+94]: namely the *level treewidth property.* It can be easily seen that planarity (and its property of being minor closed) is one possible restriction that implies the existence of this property.

2 Summary of Results and Implications

As mentioned earlier, the results presented here build on the earlier work of [Ba83, KM96, HM+94, Ep95, HM85]. We first define the notion of *level-treewidth* property for a graph class $\mathcal{G}$. A **level numbering** of a graph G is the numbering of the vertices of the graph obtained by selecting an arbitrary vertex v_0 in the graph and assigning the level number of each vertex to be the shortest distance from v_0 (i.e., the number of vertices in a shortest path from v_0,

[3]Following earlier work (see [CK95, KM96], we define the class PTAS to consist of all NP-optimization problems that have polynomial time approximation schemes.

including the two endpoints). Define a graph class G to have the *level-treewidth* property (LT-property) if for every $G \in \mathcal{G}$, for all $k \geq 1$ the treewidth of the subgraph induced by k consecutive levels is $O(f(k))$. We show that:

1. Several important and well known classes of graphs including planar and bounded genus graphs and (r, s)-civilized graphs satisfy the LT-property.
2. Let $\mathcal{G}$ be a class of graphs that obey the LT-property. Then every problem in the class MPSAT, TMAX and TMIN has a polynomial time approximation scheme when restricted to instances in $\mathcal{G}$.
3. Each of the problems considered in [Ep95] have efficient algorithms (with running times essentially identical to those in [Ep95]) when restricted to graphs obeying the LT-property.
4. A large subclass of problems in MPSAT, TMAX and TMIN have PTAS when restricted to graphs that are δ-near genus or δ-near (r, s)-civilized.
5. Most of the results in (2), (3) and (4) can be extended to obtain NC-algorithms/approximation schemes.
6. Most of the results in (2), (3) and (4) can be extended to apply to PSPACE-hard optimization problems for succinctly specified instances as in [Or82a, MH+95, HLW92].

These results generalize and extend the previous work in [KM96, HM+94, HM+94a, Ep95] in the following ways

1. In [KM96], Khanna and Motwani pose the following question: "*An interesting direction for future research is: Are there more general classes of graphs to which our results can be extended ?* " One corollary of results stated in (2) above is that all problems in the classes MPSAT, TMAX and TMIN restricted to graphs of bounded genus and (r, s)-civilized graphs have polynomial time approximation schemes. Thus the results in this paper affirmatively answer the question posed by [KM96] and significantly generalize their results.
2. In [HM+94a], we raised the question of designing PTAS for natural NP-hard problems considered in [Ba83, HM+94a] when restricted to a class of graphs that contain both planar graphs and unit disk graphs. Our approximation schemes for graphs obeying LT-property are a first step towards answering this question since the class of graphs satisfying the LT-property contain both planar and λ-precision unit disk graphs.
3. In [Ep95], Eppstein poses the following question: "*Are there natural families of graphs that are not minor closed and that have the diameter-treewidth property ?*" We affirmatively answer this question by presenting large families of graphs that are not minor closed but have the diameter-treewidth property. We illustrate this with two natural classes of graphs (See Section 4). Both these classes obey the LT-property and thus obey the DT-property. However, we can show that both the classes of graphs are not minor closed.

4. The results here provide a large class of PSPACE-hard problems for succinctly specfied instances having PTAS, thus significantly extending our results in [MH+95]. The approximability of PSPACE-hard optimization problems has been a subject of extensive research in the recent past. The results also attempt to provide a syntactic class representation for PSPACE-hard optimization problems for succinct specifications that have a PTAS. Our work is motivated by the question raised in [CF+94] to find general syntactic classes for PSPACE-hard optimization problems for which (non)approximability results can be proven in a uniform fashion.

The rest of the paper consists of basic definitions and discussion of selected results. Details will be given in the complete version of the paper.

3 Basic Definitions

Recall that an approximation algorithm for an optimization problem Π provides a **performance guarantee** of ρ if, for every instance I of Π, the solution value returned by the approximation algorithm is within a factor ρ of the optimal value for I. A **polynomial time approximation scheme** (PTAS) for problem Π is a family of algorithms such that, given an instance I of Π and an $\epsilon > 0$, there is a polynomial time algorithm in the family that returns a solution which is within a factor $(1 + \epsilon)$ of the optimal value for I.

A graph is said to be *planar* if it can be laid out in the plane in such a way that there are no crossovers of edges. A graph is said to have genus g if it can be laid out on a sphere with g handles [Wi79].

Informally, for a fixed $\delta \geq 0$, a δ-near-planar graph is a graph with vertex set V together with a planar layout with $\leq \delta \cdot |V|$ crossovers of edges. The class of δ-near-planar graphs is a generalization of planar graphs. This definition can be easily extended to define δ-near genus graphs for any fixed genus g.

For each pair of reals $r > 0$ and $s > 0$, a graph G can be drawn in R^d in an (r, s)-*civilized manner* if its vertices can be mapped to points in R^d so that

1. the length of each edge is $\leq r$, and
2. the distance between any two points is $\geq s$.

A civilized layout of a graph that can be drawn in a civilized manner in R^d consists of the coordinates of the vertices in R^d and the set of edges in the graph. We assume throughout this paper that the dimension (d) of the Euclidean space considered is at least 2. Define a *planar* (r, s) *civilized graph* to be an (r, s) civilized graph whose vertices can be embedded in the Euclidean plane (i.e., R^2). We discuss our algorithms for planar (r, s) civilized graphs. However, all the algorithms presented here extend directly to civilized graphs drawn in higher dimensions, albeit with slightly worse performance guarantee versus time trade-offs. For the remainder of this section, we use (r, s) civilized graphs to mean planar (r, s) civilized graphs.

We now recall some additional definitions from [KM96]. A *minterm* is any conjunction of literals. A minterm is positive if it has only positive literals and is negative if it has only negative terms. In the definitions of the syntactic classes considered next, we will have weighted first order formulas (FOF) and variables. The class MPSAT consists of all NP-optimization (NPO) problems that are expressible as: Given a collection $\mathcal{C}$ of FOFs over n variables such that each formula $\phi \in \mathcal{C}$ is a disjunction of $O(n^{O(1)})$ minterms, find a truth assignment T of weight at most W, maximizing the total weight of the FOFs in $\mathcal{C}$ that are satisfied. The class TMAX consists of NPO problems expressible as: Given a collection $\mathcal{C}$ of FOFs over n variables such that each formula $\phi \in \mathcal{C}$ is a disjunction of $O(n^{O(1)})$ negative minterms, find a maximum weighted truth assignment T that satisfies all the FOFs in $\mathcal{C}$. The class TMAX consists of NPO problems expressible as: Given a collection $\mathcal{C}$ of FOFs over n variables such that each formula $\phi \in \mathcal{C}$ is a disjunction of $O(n^{O(1)})$ positive minterms, find a minimum weighted truth assignment T that satisfies all the FOFs in $\mathcal{C}$. Given a set $\mathcal{C}$ of FOFs and the associated variables we define the standard *bipartite graph* $I(\mathcal{C}, V)$ as follows: We have one vertex for each variable and each FOF. We join an FOF vertex with a variable vertex if the variable appears in that FOF.

4 Sketch of the idea for the class MPSAT

The basic idea for devising a PTAS for a problem in MPSAT when restricted class of graphs having DT-property is essentially as in [Ba83, HM+94, KM96]. We need one additional result from [KM96].

Theorem 4.1 *The problems in the classes* MPSAT, TMAX *and* TMIN *have pseudo polynomial time algorithms when restricted to instances in which the bipartite graph is bounded treewidth.*

Given an instance $F \in \mathcal{G}$, and a problem $\Pi \in$ MPSAT, ALGORITHM HEU-MPSAT outlines the basic idea.

We now sketch the proof of correctness of ALGORITHM HEU-MPSAT. Given that G belongs to a family $\mathcal{G}$ satisfying the LT-property, the bipartite graph corresponding to each subformula in Step 2a has bounded treewidth. We can now apply the dynamic programming procedure given in [SH95, KM96] to find a table of optimal assignments: one each for weight bound $0 \leq w_i \leq W$. If the weights on variables are in unary the above procedure runs in polynomial time. If not we can use standard scaling and rounding procedures to compute this. The correctness of the algorithm follows by a standard averaging argument yielding the result. The proof of correctness and the performance guarantee of the algorithm follow from the above arguments and the following additional properties.

1. The ability to decompose the given graph into vertex (edge) disjoint subsets such that an (near) optimal solution to the subgraph induced by each subset can be obtained in polynomial time.
2. The ability to obtain a near-optimal solution for the whole graph by merging the optimal solution obtained for each subset.

Theorem 4.2 *Let $\mathcal{G}$ be a class of graphs that obey the* LT-property. *Then every problem in the class* MPSAT, TMAX *and* TMIN *has a polynomial time approximation scheme when restricted to instances in $\mathcal{G}$.*

ALGORITHM HEU-MPSAT:

- **Input:** An instance $G \in \mathcal{G}$ satisfying the LT-property of a problem $\Pi \in$ MPSAT. G is represented as a bipartite graph $BG(f)$.

- 1. Using **NC-BFS**, perform a breadth-first-search (BFS) on the planar graph $BG(f)$ starting at any node v in $BG(f)$. The level number of each node w is the length of the path (i.e., the number of nodes in the path including the end points) from v to w in the BFS tree.

  2. For each i ($0 \leq i \leq k$), obtain an assignment to the variables of f as follows:

  (a) Partition $BG(f)$ into subgraphs $G_1, G_2, \ldots, G_{r_i}$, each of diameter at most $2k$, by deleting clause nodes at levels at levels whose index is congruent to $k = (3i+1) \bmod (3k+1)$. **Remark:** Each G_j ($1 \leq j \leq r_i$), is the bipartite graph of a subformula f_j of f. The variable set V_j for f_j is the set of vertices of G_j and a clause c is included in f_j if and only if each variable appearing in c is also in V_j.

  (b) Using Lemma 4.1, obtain a table of assignment for each value of weight bound w_i, $0 \leq w_i \leq W$ to the variables of the subformula f_j such that the number of satisfied clauses in f_j is a maximum.
  Remark: The graph of f_j has treewidth $O(k)$, which is a constant for a fixed k.

  (c) The assignment to the variables of f is the union of the assignments to the variables of each subformula f_j such that the sum of true clauses is no more than W. This is done by nonserial dynamic programming algorithm in a standard fashion.

- **Output:** The required solution is the assignment which satisfies the maximum number of clauses among the assignments obtained for the different values of i considered in Step 2.

Theorem 4.2 implies the existence of PTAS for problems in the classes MPSAT, TMAX and TMIN when restricted to several well known classes of graphs. To prove this we only need to show that these graph classes obey the LT-property. Following [Ep95] define an *apex graph* G to be a graph such that for some vertex v (the apex) $G/\{v\}$ is planar. We will use the following important theorem proved Eppstein [Ep95].

Theorem 4.3 *(Eppstein [Ep95])*

1. *Let D denote the diameter of a genus g graph G with n vertices, $g > 0$. Then the treewidth of G is $O(f(gD))$, where the function*[4] *f is independent of n.*

2. *Let $\mathcal{G}$ be a family of minor-closed graphs. The $\mathcal{G}$ has the* DT-property *if and only if $\mathcal{G}$ does not contain all apex graphs.*

[4]Epstein [Ep95a] has observed that for graphs of constant genus, the function f is linear.

Proposition 4.4 *Given an n-node graph G that is minor closed and and does not contain all apex graphs. Then G obeys the* LT-property.

Proof: We prove the theorem for the class of genus bounded graphs. The proof for the general case is similar and is omitted. Specifically, we show that given an n-node graph G of genus $g \geq 0$, the subgraph of G induced by the vertices in any k consecutive levels of a BFS layout has treewidth $O(k)$.

Consider any k consecutive levels of G. Let these levels be numbered from i to $i+k-1$. Collapse the subgraph of G induced on the vertices at levels lower than i into a single vertex. This results in there being only one vertex representing levels 1 to $i-1$, and this vertex is connected to all the vertices at level i. Delete all vertices at levels $i+k$ or greater. We now have the subgraph induced by k consecutive levels with one additional node which is connected to all vertices at level i. Let us denote this graph by G'. Further, let us denote the subgraph induced by the nodes in levels i through $i+k-1$ by G''. Since G is genus g-bounded and G' is obtained from G by a sequence of edge contractions, G' is genus g-bounded. Since the shortest distance from any vertex in G' to any other vertex in G' is no more than $2k$, the diameter of G' is no more than $2k$. Hence, by assumptions of the theorem, the treewidth of G' is $O(k)$. Since G'' is a subgraph of G', the treewidth of G'' is also $O(k)$. Thus, the graph induced by any k levels of a genus g graph has treewidth $O(k)$. □

Next we prove a theorem that relates the DT-property and the LT-property.

Theorem 4.5 1. *If a graph G satisfies the* LT-property *then it satisfies the* DT-property.

2. *In contrast, graphs satisfying the* DT-property *do not necessarily satisfy the* LT-property. *This holds even for graphs for which each subgraph satisfies the* DT-property.

Proof: We omit the proof of the first part.

For the converse, we give two examples. They are depicted in Figure 1. The graph in Figures 1(a) is an example in which the graph has diameter $\Omega(n)$ and also has treewidth $O(n)$. But the level treewidth property does not hold. In particular the vertices in the clique all belong to the same level if we do a BFS search from the other end and have treewidth $O(n)$. Figure 1 (b) shows that even if a graph and all its subgraphs satisfy the DT-property, the graph might not satisfy the LT-property. Consider the family of graphs $\mathcal{F}$ with n vertices defined as follows. We have $\frac{n}{2^{\log n - i}}$ vertices at level i of the BF ordering of the vertices of G. Thus level 0 has 1 vertex and level $r = \lceil \log n \rceil$ has $O(n)$ vertices. The vertices at each level i themselves form a grid. The edges are added as follows. The grid at level i consists of 4 subgrids each of which has the size equal to the grid at level $(i-1)$. We now attach edges in a on-one fashion from each subgrid to the grid at level $(i-1)$. Now vertices at k consecutive levels do not necessarily have treewidth $f(k)$. In fact the last k levels have treewidth $O(\sqrt{n})$. But if you look at a subgraph of G that has a diameter D then clearly its treewidth is $O(f(D))$. This is clear by noting that the graph constructed is of bounded degree and hence the number of vertices in a diameter D subgraph are no more than $O(d_{max}^D)$. Here d_{max} denotes the maximum degree in the network.□

Next, we consider (r, s)-civilized graphs.

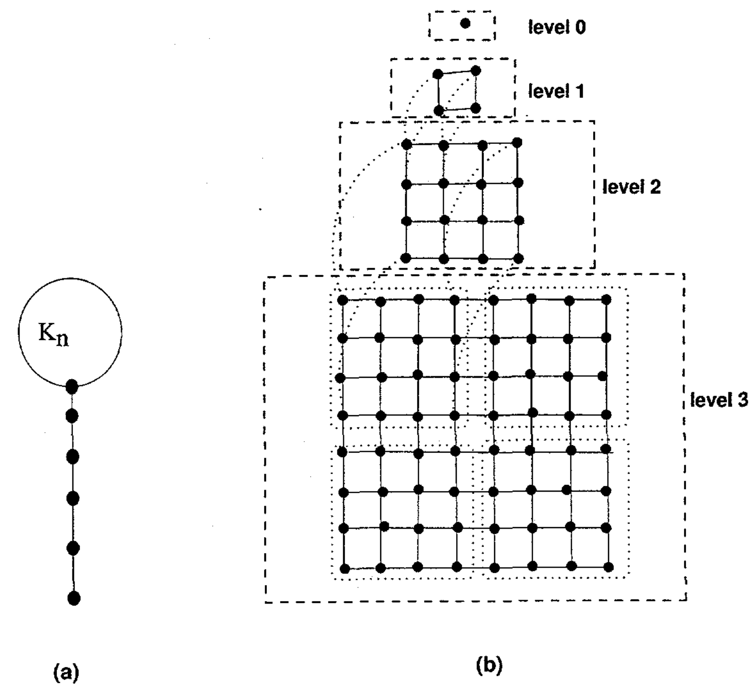

Figure 1: Figures illustrating δ-near-planarity and bounded genus properties. (a) The Lollipop graph. The graph consists of a clique of size $O(n)$ and a chain of $O(n)$ vertices attached to one of the vertices in the clique. (b) A graph that has grids of size $\frac{n}{2^{\log n - i}}$ at level $0 \leq i \leq \log n$. The subgrids at level $(i+1)$ to level i.

Theorem 4.6 *Let G be a n-node (r,s)-civilized graph. The subgraph of G induced by the vertices in any k consecutive levels has treewidth $O(k)$. Thus the set of (r,s)-civilized graphs satisfy the* DT-property.

Proof: The vertices in k consecutive levels of a (r,s) civilized graph lie in a rectangular slice of side height $O(rk)$ and width $O(n)$. Since G is an (r,s) civilized graph, the maximum number of vertices in a rectangular region of dimensions $O(rk) \times O(s)$ is at most krs. Furthermore, removal of the vertices in this square breaks the graph into disjoint pieces. By recursively applying the above idea on each smaller piece, we can construct a tree decomposition of the graph G with treewidth $krs = O(k)$ since r and s are fixed. □

5 Subgraph Isomorphism and Shortest path Queries

We briefly discuss how to extend the idea in [Ep95] to graphs satisfying the LT-property. For this we prove the following decomposition theorem that is analogous to Lemma 5.2 in [Ep95].

Lemma 5.1 *Let G be a graph that satisfies the* LT-property *and $\mathcal{L}_G$ be the corresponding layout. Then in time $O(n)$ we can find a collection $O(n)$ of subgraphs G_i with the following properties:*

1. *For every vertex $v \in G$ the subgraph G' induced by the vertices in G that are at a distance w of v is a subgraph of one of the graphs G_i.*

2. *Each vertex is included in at most 3 different subgraphs G_i.*

3. *Each G_i has treewidth $O(w)$.*

Proof: Given the layout $\mathcal{L}_G$ of G let G_i be the subgraph induced by the vertices in levels iw up until $(i+3)w-1$. Each G_i contains the w-neighborhood of vertices in levels $i+1)w$ to $(i+2)w-1$. Thus all the neighborhoods are covered. Each vertex belongs to three graphs G_i, namely a vertex at level j belongs to the subgraphs $G_{\lfloor j/w \rfloor + k}$, $-2 \leq k \leq 0$. By the fact that G satisfies the LT-property, we get that each G_i has treewidth $O(w)$.□

The proof of the following theorem follows on the lines of the proof of an analogous result for planar graphs in [Ep95].

Theorem 5.2 *Let l be a fixed. Given a graph G obeying the* LT-property, *we can construct a data structure $\mathcal{DT}$ in time $O(l^2 n)$, such that size of $\mathcal{DT}$ is $O(ln)$, we can in time $O(l^2 \log n)$ (per query) tell if distance between any two nodes u and v is no more than l.*

6 NLC-produced near planar graphs

The results in the previous sections motivate looking at other general families of graphs that have the LT-property. Towards this, we define the notion of two level graphs. We will use the terminology in Ehrenfeucht et al. and Wanke [EE+96, Wa94] in the sequel. In *two level graphs*, we are given a *skeleton* graph G. Each node of the skeleton graph is then replaced by a subgraph drawn from a family of fixed subgraphs $\mathcal{F}$. The edges between two subgraphs H and F are given as an explicit relation. If H and F replace two vertices in the skeleton graph that have an edge between them then we add edges between vertices in H and F according to the given relation. The following theorem is an easy but useful consequence of the above definitions and results on graphs satisfying the LT-property.

In the literature extensive amount of work has been done on graphs that are defined using NLC-graph grammars [Wa94, EE+96]. Our graphs are precisely a subclass of such graphs, in which the derivation tree is only of depth 2 and the underlying skeleton graph has a special structure.

Theorem 6.1 *Let $\mathcal{F}$ be a family of graphs satisfying the* LT-property *and $\mathcal{S}$ be a fixed collection of subgraphs, and $\mathcal{R}$ a finite set of edge relation between subgraphs of $\mathcal{S}$. Let $G \in \mathcal{G}$ be an arbitrary instance. The graph GS obtained by replacing the nodes of G by subgraphs from $\mathcal{S}$ and adding edges as discussed above yields a graph that obeys the* LT-property.

Proof Idea: The proof of Theorem 6.1 follows ideas similar to the ones given in Section 4. We first decompose the skeleton graph into disjoint subgraphs by using the layout satisfying the LT-property. Observe that due to the way the vertices are replaced the treewidth of the subgraph induced by k consecutive levels is still some function of k (independent of n). Moreover the decomposition of the graph performed on the basis of the skeleton is a valid decomposition for the substituted graph. The rest of the proof is similar and is omitted.□

7 Extension to Succinct Specifications

Next, we discuss the extension to succinct specifications. We recall briefly the periodic specifications. The definition of hierarchical specification can be found in [MH+95, LW87a, HLW92] and is also given in Appendix for completeness. The definition of one dimensional periodic specifications is due to Orlin and Lenguaer et al.[Or82a, HLW92]. Henceforth **N** and **Z** denotes the set of non-negative integers and integers respectively.

Definition 7.1 *Let $G(V, E)$ (referred to as a static graph) be a finite undirected graph such that each edge (u, v) has an associated non-negative integral weight $t_{u,v}$. The one way infinite graph $G^{\infty}(V', E')$ is defined as follows:*

1. $V' = \{v(p) \mid v \in V \text{ and } \quad p \in \mathbf{N}\}$
2. $E' = \{(u(p), v(p + t_{u,v})) \mid (u, v) \in E \quad, t_{u,v}$ *is the weight associated with the edge* (u, v) *and* $p \in \mathbf{N}\}$

A 1-dimensional periodic specification Γ (referred to as 1-P-specification) is given by $\Gamma = (G(V, E))$ and specifies the graph $G^{\infty}(V', E')$ (referred to as 1-P-specified graph).

A 1-P-specification Γ is said to be narrow *or* 1-level restricted *if $\forall(u, v) \in E$, $t_{u,v} \in \{0, 1\}$. This implies that $\forall(u(p), v(q)) \in E'$, $|p - q| \leq 1$. Similarly, a 1-P-specification is k*-narrow *or* k-level restricted *if $\forall(u, v) \in E$, $t_{u,v} \in \{0, 1, \ldots k\}$.*

It is sometimes useful to imagine a narrow periodically specified graph G^{∞} as being obtained by placing a copy of the vertex set V at each integral point (also referred to as lattice point) on the X-axis (or the time line) and joining vertices placed on neighboring lattice points in the manner specified by the edges in E.

G^m is the subgraph of the infinite periodic graph G^{∞} induced by the vertices associated with nonnegative lattice points less than or equal to m. A 1-dimensional finite periodic specification Γ (referred to as 1-FPN-specification) is given by $\Gamma = (G(V, E), m)$ and specifies the graph G^m (referred to as 1-FPN-specified graph).

7.1 Overall idea

The basic idea is simple. We illustrate our ideas by describing our approximation algorithm for the maximum independent set. Given a 1-FPN-specification $\Gamma = (G(V, E), m)$ of a planar graph G^m and an $\epsilon > 0$, we find the corresponding integer l that satisfies the inequality $(\frac{l}{l+1})^2 \geq (1 - \epsilon)$. For $0 \leq i \leq l$, we remove the vertices placed at the lattice points j such that $j = i \ mod(l + 1)$. This partitions the graph G^m into a number of smaller disjoint subgraphs, each induced by l consecutive lattice points.

Specifically, for a given i, let $l_p^i = \max\{0, (p - 1)(l + 1) + (i + 1)\}$ and $r_p^i = \min\{m, p(l + 1) + (i - 1)\}$, where $0 \leq p \leq t_i$. Here $t_i = \lceil \frac{m-(i-1)}{(l+1)} \rceil$. Let the subgraph induced by vertices $v(j_p)$, where $l_p^i \leq j_p \leq r_p^i$, be denoted by $H(l_p^i, r_p^i)$. For a given $\epsilon > 0$, the graphs $H(l_p^i, r_p^i)$ are linear in the size of Γ. Next, we solve the MIS problem near optimally on each of the subgraphs. This can be done by using the linear time algorithm developed in the¡ previous sections. The union of these solutions is a solution for the iteration i. The heuristic simply picks up the best solution obtained over all $l+1$ iterations.

8 Conclusions

We presented simple graph theoretic property that helped unify and extend earlier results on the existence of PTAS and efficient algorithms for a large collection of problems studied in the literature. We conclude by mentioning other related remarks and directions for future research.

1. In a recent related work we have been able to extend the ideas presented here to certain optimization problems for quantified formulas and domino games [MHS96]; thus significantly extending the class of problems amenable to such a solution strategy.

2. In Teng's thesis [Te91] and papers following this work, Teng et al. [GMT94, MTV91, ST96] have considered intersection graphs of k-ply neighborhood systems. These graphs are a strict generalization of (r, s)-civilized graphs as well as planar graphs. Teng et al [Te91, GMT94, MTV91, ST96] show that intersection graphs of k-ply neighborhood system have a "good" seperator; thus implying that most of the problems considered here have PTAS in an asymptotic sense when restricted to such graphs (approximation schemes similar to those obtained in [LT80]). An interesting question is investigate if the techniques presented here and in [Ep95, KM96, HM+94] can be extended to obtain PTAS for intersection graphs of k-ply neighborhood systems ?

3. Another interesting direction for future work is to extend the class of problems studied so far. As mentioned in [KM96], problems such as TSP for planar and geometric instances (Recently [GKP95, Ar96] show that these problems have PTAS .) cannot be directly formulated in terms of the predicates considered here. A similar class of problems is bicriteria network design problems considered in [MR+95] when restricted to treewidth bounded graphs. Again these problems have PTAS but cannot be captured directly by the predicates considered here. Defining predicates that capture such problems would significantly extend these results.

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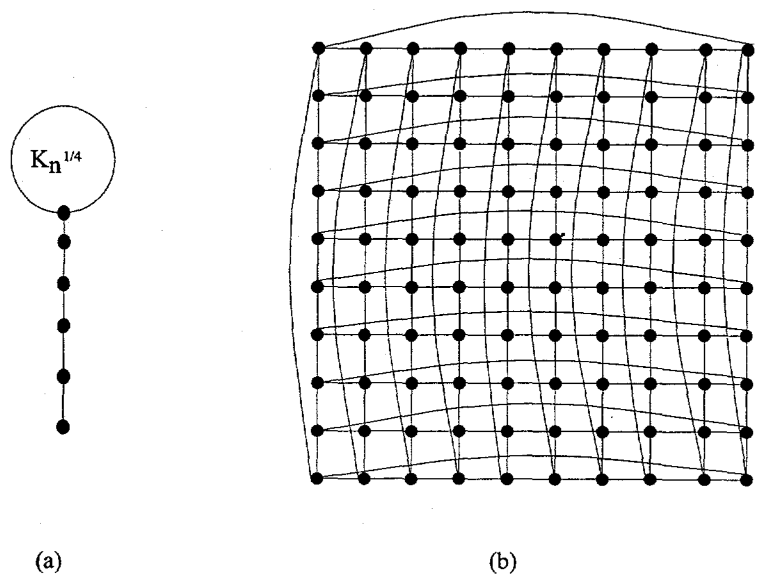

Figure 2: Figures illustrating the orthogonality of δ-near-planarity and bounded genus properties. (a) The Lollipop graph. The graph consists of a clique of size $n^{1/4}$ and a chain of $n - n^{1/4}$ vertices attached to one of the vertices in the clique. (b) A graph which has large number of crossovers but whose genus is 1.

9 Appendix

10 Extension to near-genus and near civilzed graphs

Next, we discuss the extension of results in previous Sections to instances restricted to δ-near-genus g graphs. Let δ be a fixed positive number. A **δ-near-genus g graph** is a graph with vertex set V together with a genus g layout with $\leq \delta \cdot |V|$ crossover nodes. We note that the classes of δ-near-planar graphs and bounded genus graphs are in a sense orthogonal. Specifically, there are classes of graphs which are δ-near-planar for small δ but whose genus is not bounded by any constant. For example, consider a family of graphs that for each positive integer $n > 0$, consist of a clique on $n^{\frac{1}{4}}$ nodes and a simple chain of length $(n - n^{\frac{1}{4}})$ attached to one of the nodes in the clique (Figure 2(a)). These graphs have genus $\Theta(n^{\frac{1}{2}})$ since the genus of an r-clique is $\Theta(r^2)$ [Wi79]. However, the graphs are 1-near-planar, since the number of crossovers is at most n. Similarly, there are graphs of bounded genus that are not δ-near-planar for any fixed δ. To illustrate this point, consider another family of graphs, that for each positive integer $n > 0$ consist of a $\sqrt{n} \times \sqrt{n}$ *torus* as laid out in Figure 2(b). In this layout, the number of crossovers is $\Theta(n^2)$. Thus the graphs with the given layouts are not δ-near-planar graphs for any constant δ. On the other hand, it can be seen that the graphs have genus at most 1; i.e., they can be laid out with non-crossing edges on a sphere with 1 handle. A δ-near (r, s)-civilized graph consists of a (r, s)-civilized graph with some edges of lenth more than s such that the total length of these edges is $O(n)$.

We show following theorem for graphs with bounded genus.

Theorem 10.1 *For all fixed g and $\delta \geq 0$, there are NC-approximation schemes for the following problems restricted to δ-near-genus g graphs and δ-near (r,s)-civilized graphs: bounded degree maximum independent set, bounded degree minimum dominating set, maximum k-colorable subgraph,* MAX SAT(S) *for every fixed set S the problems* MPSAT *and* MAX CUT.

Proof Sketchx: The proof of the theorem consists of giving an L-reduction to an appropriately chosen satisifiability pròblem. To illustrate our ideas we specifiy a reduction for the BOUNDED DEGREE MINIMUM DOMINATING SET problem for δ-near-planar graphs.

Given an instance $I(V,E)$ with $V = \{v_1, \ldots v_n\}$ and $|E| = m$ of the BOUNDED DEGREE MINIMUM DOMINATING SET problem for δ-near-planar graphs, we construct an instance $I'(U,C)$ of MAX δ-near-Pl-SAT(S), where S will be defined subsequently. Let B be the bound on the degree of a vertex. The variables in I' are in one-to-one correspondence with the vertices in I. A variable will be true if its corresponding vertex is in the dominating set. For each vertex v in I with neighbors $v_{i_1}, v_{i_2}, \ldots v_{i_l}$, where $l \leq B$, we create a clause $(u \vee u_{i_1} \vee u_{i_2} \vee \ldots \vee u_{i_l})$. When l is less than B, we can repeat one of the variables so that each clause is of size exactly $B+1$. We also add the clauses, $\overline{u_i}$ for each vertex v_i. Formally, $U = \{u_1, \ldots, u_n\}$ and $C = C_1 \cup C_2$ where $C_1 = \{(u_i \vee u_{i_1} \vee \ldots \vee u_{i_l}) \mid (v_i, v_{i_1}), (v_i, v_{i_2}), \ldots (v_i, v_{i_l}) \in E\}$ and $C_2 = \{\overline{u_i} \mid v_i \in V\}$. Observe that $|C_1| = |C_2| = n$. Here also, the set S contains only two relations, one of arity $B+1$ and the other of arity 1. Further, is easy to see that I' can be laid out in such a way that $\delta(I') \leq B^2 + \delta(I)$. The clauses in C_2 do not contribute any crossovers. Since each vertex can dominate at most $B+1$ vertices (including itself), $|OPT(I)| \geq \frac{n}{B+1}$. The resulting instance of MAX SAT(S) has at most $2n$ clauses and so $|OPT(I')| \leq 2n \leq 2(B+1)|OPT(I)|$. Since B is a constant, this verifies the first condition of an L-reduction. We now verify the second condition in the definition of an L-reduction. Consider a solution $A(I')$ for I'. For each unsatisfied clause in C_1, set the value of one of the variables in the clause to true to obtain another solution which satisfies the same number of clauses as the original solution and also satisfies all the clauses in C_1. The above modification yields another solution which satisfies at least $A(I')$ clauses of I' and also satisfies all the clauses in C_1. Now, the approximate dominating set consists of vertices v_i such that the corresponding variable u_i is set to true. Note that the total number of variables set to false in $A(I')$ is $|A(I')| - |C_1|$. Hence, the size of the approximate solution for I is given by $|A(I)| = n - |A(I')| + |C_1|$. We now have

$$|OPT(I')| = |C_1| + n - |OPT(I)|$$

and

$$|A(I)| - |OPT(I)| = n - |A(I')| + |C_1| - n - |C_1| + |OPT(I')| = |OPT(I')| - A(I).$$

Thus the second condition for an L-reduction is satisfied with $\beta = 1$.□

In contrast, we can show that For any fixed $\delta > 0$, the following hold: Unless NP = QP[5], there is no polynomial time approximation algorithm that guarantees a solution with a performance guarantee better than $\log n$ for the minimum dominating set problem when restricted to δ-near-planar graphs. Similar results hold for other problems for bounded degree δ-near-planar graphs.

[5]Quasi-polynomial time