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Definition of total bootstrap current in tokamaks

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Alternative definitions of the total bootstrap current are compared. An analogous comparison is given for the ohmic and auxiliary currents. It is argued that different definitions than those usually employed lead to simpler analyses of tokamak operating scenarios.

I. INTRODUCTION

In connection with our low-aspect-ratio tokamak studies,[1] I am concerned about the proper definition of the total bootstrap current (and by implication, the ohmic and auxiliary currents as well). In recent literature,[2-4] particularly in discussions of advanced tokamak operating scenarios, the bootstrap current within a poloidal flux surface ψ_0 is defined by

$$I_{bs}(\psi_0) \equiv \int^{\psi_0} d\psi \, 2\pi q \frac{\langle \mathbf{j} \cdot \mathbf{B} \rangle_{bs}}{\langle B^2 \rangle}, \tag{1}$$

where q is the safety factor. In this formulation an additional term, referred to as the 'diamagnetic' current, must be included in the accounting of the total current. (I use quotation marks because this term can be shown to involve both the perpendicular and Pfirsch-Schlüter currents.) The diamagnetic term is sometimes neglected, but it can be large at high beta, and especially at low aspect ratio, and therefore important in determining

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the 'alignment' of the bootstrap current. The various terms contributing to the total current are derived in an unpublished memo by Nevins.[5]

In the following I suggest alternative definitions of the components of the total current, which seem to me to be more useful. In this formulation, the 'diamagnetic' contribution does not enter explicitly but is included implicitly in the ohmic and bootstrap terms. The two sets of definitions are exactly equivalent, but in my version fewer terms are required to test the bootstrap 'alignment.' I also believe my expressions are more directly related to the resistive field penetration equation and are therefore more convenient to calculate in a transport code.

The purpose of this note is to compare the two sets of definitions and ask if any reader can tell me whether the usual convention is indeed as I describe it, and how the accounting is handled in the outputs of various transport codes, *e.g.*, SUPERCODE.[6, 7] My interest is in transport codes and in constructing a spreadsheet similar to the one described by Nevins[8] for fusion studies (TPX and ITER).

II. DEFINITIONS

We write the toroidal and poloidal magnetic fields as

$$B_T = \frac{F(\psi)}{R}, \tag{2a}$$

and

$$B_p = \frac{|\nabla \psi|}{R}, \tag{2b}$$

respectively. We let V and S denote volume and cross-sectional area, respectively, and write the volume derivative and volume integral as

$$V'(\psi) = \frac{dV}{d\psi} = 2\pi \oint \frac{dl_p}{B_p}, \tag{3a}$$

and

$$\int d^3V X = \int d\psi V'(\psi)\langle X\rangle, \tag{3b}$$

where the flux surface average is given by

$$\langle X\rangle = \frac{d}{dV}\int d^3V X = \frac{1}{V'}\int \frac{X\,dS}{|\nabla\psi|} = \frac{2\pi}{V'}\oint \frac{X\,dl_p}{B_p} = \frac{\oint \frac{X\,dl_p}{B_p}}{\oint \frac{dl_p}{B_p}}. \tag{3c}$$

With the toroidal flux denoted by Φ, we also note that the safety factor is given by

$$q = \frac{1}{2\pi}\frac{\partial\Phi}{\partial\psi} = \frac{FV'}{4\pi^2}\left\langle\frac{1}{R^2}\right\rangle. \tag{4}$$

III. THE TOROIDAL CURRENT

A. Standard expression

Denote the toroidal current enclosed within a given flux surface ψ_0 by $I_T(\psi_0)$. Nevins[5] derives the expression

$$\begin{aligned} I_T(\psi_0) &= \int^{\psi_0} J_T\,dS \\ &= \int^{\psi_0} d\psi V'(\psi)\left\langle\frac{J_T}{2\pi R}\right\rangle \\ &= \int^{\psi_0} d\psi \frac{V'(\psi)}{2\pi}\left[\frac{F(\psi)\langle 1/R^2\rangle}{\langle B^2\rangle}\langle \mathbf{j}\cdot\mathbf{B}\rangle + p'(\psi)\left(1 - \frac{F^2(\psi)\langle 1/R^2\rangle}{\langle B^2\rangle}\right)\right]. \end{aligned} \tag{5}$$

That is,

$$I_T(\psi_0) = \int^{\psi_0} d\psi \frac{V'(\psi)}{2\pi}\left[\frac{\langle B_T^2\rangle}{\langle B^2\rangle}\frac{\langle \mathbf{j}\cdot\mathbf{B}\rangle}{F(\psi)} + p'(\psi)\frac{\langle B_p^2\rangle}{\langle B^2\rangle}\right], \tag{6}$$

where $B^2 = B_T^2 + B_p^2$. The second term in square brackets yields the 'diamagnetic' current.

B. Alternative expression

An alternative expression for the toroidal current can be obtained, *e.g.*, from Eqs. (48) and (63) of Blum and Le Foll:[9]

$$I_T(\psi_0) = F(\psi_0)\int^{\psi_0} d\psi \frac{V'(\psi)}{2\pi}\frac{\langle \mathbf{j}\cdot\mathbf{B}\rangle}{F^2(\psi)}. \tag{7}$$

Equations (6) and (7) agree exactly, as shown in Appendix A. That is,

$$0 = \int^{\psi_0} d\psi \frac{V'(\psi)}{2\pi}\left[\left(\frac{\langle B_T^2\rangle}{\langle B^2\rangle} - \frac{F(\psi_0)}{F(\psi)}\right)\frac{\langle \mathbf{j}\cdot\mathbf{B}\rangle}{F(\psi)} + p'(\psi)\frac{\langle B_P^2\rangle}{\langle B^2\rangle}\right]. \tag{8}$$

Equations (7) and (8) show that it is not necessary to treat the 'diamagnetic' term separately.

IV. BOOTSTRAP, OHMIC, AND DRIVEN CURRENTS

A. Standard expressions

From the first term in square brackets on the right-hand side of Eq. (6) Nevins defines the bootstrap current to be[5]

$$I_{\mathrm{bs}}(\psi_0) \equiv \int^{\psi_0} d\psi \frac{V'(\psi)}{2\pi}\frac{\langle B_T^2\rangle}{\langle B^2\rangle}\frac{\langle \mathbf{j}\cdot\mathbf{B}\rangle_{\mathrm{bs}}}{F(\psi)}, \tag{9}$$

where $\langle \mathbf{j}\cdot\mathbf{B}\rangle_{\mathrm{bs}}$ is the flux-surface-averaged bootstrap current density, as calculated, *e.g.*, by Hirshman,[10] or with collisional corrections by Kessel.[10] By substituting Eqs. (2a) and (4) into Eq. (9), we obtain the usually quoted formula, Eq. (1), which Nevins also discusses.

Analogously, we can define $I_{\text{non-bs}} = I_{\text{ohmic}} + I_{\text{aux}}$, where 'aux' refers to any non-inductive current drive, *e.g.*, from RF or neutral beams. For example, letting the parallel electrical conductivity be $\sigma_\parallel$, we write $\langle \mathbf{j} \cdot \mathbf{B} \rangle_{\text{ohmic}} = \sigma_\parallel \langle \mathbf{E} \cdot \mathbf{B} \rangle$, and

$$I_{\text{ohmic}}(\psi_0) \equiv \int^{\psi_0} d\psi \, \frac{V'(\psi)}{2\pi} \frac{\langle B_T^2 \rangle}{\langle B^2 \rangle} \frac{\sigma_\parallel \langle \mathbf{E} \cdot \mathbf{B} \rangle}{F(\psi)}. \tag{10}$$

Finally, the term referred to as the 'diamagnetic' current is obtained from the second term in square brackets and given by[4, 5]

$$I_{\text{dia}}(\psi_0) = \int^{\psi_0} d\psi \frac{V'(\psi)}{2\pi} \frac{\langle B_p^2 \rangle}{\langle B^2 \rangle} p'(\psi). \tag{11}$$

Kessel[4] describes this as 'the contribution from perpendicular current, that is the toroidal component of the Pfirsch-Schlüter current.' It is, in fact, the sum of the toroidal components of the perpendicular and Pfirsch-Schlüter currents (the latter being a parallel current) as shown in Appendix B.

B. Alternative expressions

An alternative definition of the bootstrap current, based on Eq. (7), is

$$\tilde{I}_{\text{bs}}(\psi_0) \equiv F(\psi_0) \int^{\psi_0} d\psi \, \frac{V'(\psi)}{2\pi} \frac{\langle \mathbf{j} \cdot \mathbf{B} \rangle_{\text{bs}}}{F^2(\psi)}. \tag{12}$$

Again, we can define $\tilde{I}_{\text{non-bs}} = \tilde{I}_{\text{ohmic}} + \tilde{I}_{\text{aux}}$, *e.g.*,

$$\tilde{I}_{\text{ohmic}}(\psi_0) \equiv F(\psi_0) \int^{\psi_0} d\psi \, \frac{V'(\psi)}{2\pi} \frac{\sigma_\parallel \langle \mathbf{E} \cdot \mathbf{B} \rangle}{F^2(\psi)}. \tag{13}$$

Equation (8), *i.e.*, the equivalence of Eqs. (6) and (7), implies an expression for the difference between (9) and (12),

$$I_{bs}(\psi_0)-\tilde{I}_{bs}(\psi_0)=-I_{non\text{-}bs}(\psi_0)+\tilde{I}_{non\text{-}bs}(\psi_0)+I_{dia}(\psi_0)$$

$$=\int^{\psi_0} d\psi \frac{V'(\psi)}{2\pi}\left[\left(\frac{F(\psi_0)}{F(\psi)}-\frac{\langle B_T^2\rangle}{\langle B^2\rangle}\right)\frac{\langle \mathbf{j}\cdot\mathbf{B}\rangle_{non\text{-}bs}}{F(\psi)}+\frac{\langle B_p{}^2\rangle}{\langle B^2\rangle}p'(\psi)\right]. \qquad (14)$$

Consider, for example, an ohmic discharge in the cylindrical limit. Since the bootstrap current vanishes as the inverse aspect ratio, ε, goes to zero, we may use either Eq. (8) or Eq. (14) to obtain an expression for the diamagnetic current in that limit,

$$I_{dia}(\psi_0)\equiv\int^{\psi_0} d\psi \frac{V'(\psi)}{2\pi}\frac{\langle B_p{}^2\rangle}{\langle B^2\rangle}p'(\psi) \xrightarrow[\varepsilon\to 0]{} I_{ohmic}(\psi_0)-I(\psi_0)$$

$$=\int^{\psi_0} d\psi \frac{V'(\psi)}{2\pi}\left(\frac{\langle B_T^2\rangle}{\langle B^2\rangle}-\frac{F(\psi_0)}{F(\psi)}\right)\frac{\sigma_\parallel\langle \mathbf{E}\cdot\mathbf{B}\rangle}{F(\psi)}, \qquad (15)$$

where we have noted that $\langle \mathbf{j}\cdot\mathbf{B}\rangle_{non\text{-}bs} \to \langle \mathbf{j}\cdot\mathbf{B}\rangle=\sigma_\parallel\langle \mathbf{E}\cdot\mathbf{B}\rangle$. Thus, I_{dia} persists in the cylindrical limit, where it is indeed purely diamagnetic, but it is not usually treated as distinct from the ohmic current.

V. IMPLICATIONS

In any accounting of the total current, all the terms must be included. Using the usual expression, Eq. (6), with the definitions (9)-(11) and an analogous expression for I_{aux}, we write the total current as

$$I_T=I_{ohmic}+I_{bs}+I_{dia}+I_{aux}, \qquad (16)$$

while Eq. (7), with the definitions (12) and (13) etc., gives

$$I_T=\tilde{I}_{ohmic}+\tilde{I}_{bs}+\tilde{I}_{aux}. \qquad (17)$$

For example, one exercise is to specify a desirable current profile, *e.g.*, for MHD stability, and subtract the ohmic and bootstrap current profiles to determine the needed auxiliary drive. This has been done in many of the TPX and ITER presentations I have seen. Since these graphical presentations do not ordinarily mention $I_{\rm dia}$, I infer that either the diamagnetic term is neglected, or the formulation of Eq. (17) is the one commonly used.

My query: which is the usual interpretation? How is this information handled in SUPERCODE,[6, 7] for example? It seems to me that Eq. (7) and, by implication, (12), (13), and (17), are simpler to calculate and more convenient than Eqs. (6), (9)-(11), and (16). In general, whichever definitions are used, careful treatment of $F(\psi)$ may be required. It is found from the solution to the transport equations, using the flux-surface averaged Grad-Shafranov equation, and then fed back into the equilibrium solver. These equations, *e.g.*, Eqs. (51) and (62) of Blum and LeFoll,[9] involve $\partial p/\partial\psi$ and $\langle B_p^2\rangle$, so considerable complication may remain, either way.

I emphasize that there is an ambiguity in the definitions of the ohmic and auxiliary currents as well as the bootstrap current. In the most general discussions of the transport problem,[9, 11-13] the equation for the resistive evolution of the current (or poloidal flux derivative) is given in a formulation equivalent to the one I advocate, namely Eq. (7) or (17), whether or not neoclassical effects are included. These authors do not separate out the 'diamagnetic' term explicitly, as would have to be done if Eq. (6) or (13) were used. These distinctions may be unimportant at low beta and high aspect ratio, where $I_{\rm dia}$ is small, but they will be important for the spherical tokamak.

APPENDIX A - EQUIVALENCE OF EQS. (6) AND (7)

We differentiate Eqs. (6) and (7) in the main text with respect to ψ_0, obtaining

$$I_T{}'(\psi_0)=\left\{\frac{V'(\psi)}{2\pi}\left[\frac{\langle B_T^2\rangle}{\langle B^2\rangle}\frac{\langle \mathbf{j}\cdot\mathbf{B}\rangle}{F(\psi)}+p'(\psi)\frac{\langle B_p^2\rangle}{\langle B^2\rangle}\right]\right\}_{\psi=\psi_0}, \tag{A1}$$

and

$$I_T{}'(\psi_0) = \left[\frac{V'(\psi)}{2\pi F(\psi)}\langle \mathbf{j}\cdot\mathbf{B}\rangle\right]_{\psi=\psi_0} + \frac{F'(\psi_0)}{F(\psi_0)} I_T(\psi_0), \tag{A2}$$

respectively. We also make use of a standard expression for the parallel current [see, *e.g.*, Hinton and Hazeltine,[14] Eqs. (2.70) and (2.89)],

$$\langle \mathbf{j}\cdot\mathbf{B}\rangle = \frac{\langle B^2\rangle}{\mu_0} F' + Fp' \tag{A3}$$

and the integral form of the toroidal Ampere law,

$$\mu_0 I_T = \oint dl_p\, B_p = \oint \frac{dl_p}{B_p} B_p^2 = \frac{V'}{2\pi}\langle B_p^2\rangle. \tag{A4}$$

Substituting (A3) and (A4) into the right-hand side of (A2), we obtain (A1). This proves the desired equivalence.

APPENDIX B - INTERPRETATION OF I_{dia}

The perpendicular or true diamagnetic current density is given by

$$J_\perp = R\frac{B_p}{B}\frac{\partial p}{\partial \psi} \tag{B1}$$

Its contribution to the toroidal current [*cf.* Eq. (5)] is

$$\begin{aligned} I_{\text{dia}}^{\perp}(\psi_0) &\equiv \int^{\psi_0} d\psi \frac{V'(\psi)}{2\pi}\left\langle\left(\frac{J_\perp}{R}\right)_T\right\rangle \\ &= \int^{\psi_0} d\psi \frac{V'(\psi)}{2\pi}\frac{\partial p}{\partial \psi}\left\langle\frac{B_p^2}{B^2}\right\rangle. \end{aligned} \tag{B2}$$

The Pfirsch-Schlüter (parallel) current density is

$$J_{\text{PS}} = \frac{B_T R}{B}\left(1 - \frac{B^2}{\langle B^2\rangle}\right)\frac{\partial p}{\partial \psi}, \tag{B3}$$

as, for example in Eq. (4.101) of Hinton and Hazeltine.[14] Its contribution to the toroidal current is then

$$I_{\text{dia}}^{\text{PS}}(\psi_0) \equiv \int^{\psi_0} d\psi \frac{V'(\psi)}{2\pi}\left\langle\left(\frac{J_{\text{PS}}}{R}\right)_T\right\rangle$$

$$= \int^{\psi_0} d\psi \frac{V'(\psi)}{2\pi}\frac{\partial p}{\partial \psi}\left\langle \frac{B_T^2}{B^2} - \frac{B_T^2}{\langle B^2\rangle}\right\rangle. \tag{B4}$$

Adding Eqs. (B2) and (B4) we obtain

$$I_{\text{dia}}^{\perp}(\psi_0) + I_{\text{dia}}^{\text{PS}}(\psi_0) = \int^{\psi_0} d\psi \frac{V'(\psi)}{2\pi}\frac{\partial p}{\partial \psi}\left\langle \frac{B_p^2}{B^2} + \frac{B_T^2}{B^2} - \frac{B_T^2}{\langle B^2\rangle}\right\rangle$$

$$= \int^{\psi_0} d\psi \frac{V'(\psi)}{2\pi}\frac{\partial p}{\partial \psi}\left\langle 1 - \frac{B_T^2}{\langle B^2\rangle}\right\rangle$$

$$= \int^{\psi_0} d\psi \frac{V'(\psi)}{2\pi}\frac{\partial p}{\partial \psi}\frac{\langle B_p^2\rangle}{\langle B^2\rangle}$$

$$= I_{\text{dia}}(\psi_0). \tag{B5}$$

Thus, there are both parallel and perpendicular contributions to the 'diamagnetic' current.

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